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Monotone positive solution of a fourth-order BVP with integral boundary conditions

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Abstract
In this paper, we investigate the existence of concave and monotone positive solutions for a nonlinear fourth-order differential equation with integral boundary conditions of the form $x^{(4)}(t) = f(t, x(t), x'(t), x''(t))$, $t \in [0,1]$, $x(0) = x'(1) = x'''(1) = 0$, $x''(0) = \int_0^1 g(s)x''(s)\,\mathrm{d}s$, where $f \in C([0,1] \times [0,+\infty)^2 \times (-\infty,0], [0,+\infty))$, $g \in C([0,1],[0,+\infty))$. By using a fixed point theorem of cone expansion and compression of norm type, the existence and nonexistence of concave and monotone positive solutions for the above boundary value problems is obtained. Meanwhile, as applications of our results, some examples are given.

MSC: 34B15; 34B18

Keywords: fourth-order differential equation; integral boundary condition; monotone positive solution; existence; nonexistence

1 Introduction

This paper is the follow-up of [1]. In [1], by using a fixed point theorem for the sum of two operators due to O'Regan [2], we obtained existence of solutions for a fully nonlinear fourth-order equation with integral boundary conditions of type

$$\begin{cases} x^{(4)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)), & t \in [0,1], \\ x(0) = x'(1) = x'''(1) = 0, \\ x''(0) = \int_0^1 h(s, x(s), x'(s), x''(s))\,\mathrm{d}s. \end{cases}$$

In this paper, we study the existence of concave and monotone positive solutions for its simplified form

$$x^{(4)}(t) = f\bigl(t, x(t), x'(t), x''(t)\bigr), \quad t \in [0,1] \tag{1.1}$$

subject to the integral boundary conditions

$$\begin{cases} x(0) = x'(1) = x'''(1) = 0, \\ x''(0) = \int_0^1 g(s)x''(s)\,\mathrm{d}s, \end{cases} \tag{1.2}$$

where $f \in C([0,1] \times [0,+\infty)^2 \times (-\infty,0], [0,+\infty))$, $g \in C([0,1],[0,+\infty))$.

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It is well known that fourth-order boundary value problems models bending equilibria of elastic beams, and have been studied extensively. Among a substantial number of works dealing with fourth-order boundary value problems, we mention [1, 3–31]. We notice that if $g(\cdot) \equiv 0$ in (1.2), the models are known as the one endpoint simply supported and the other one sliding clamped beam. The study of this class of problems was considered by some authors via various methods, we refer the reader to [4, 7, 10, 14, 15, 23, 26].

The aim of this paper is to establish the existence and nonexistence results of concave and monotone positive solutions for the problems (1.1), (1.2). Here, a solution $x(t)$ of the BVP (1.1), (1.2) is said to be monotone and positive if $x'(t) \geq 0$ on $[0,1]$ and $x(t) > 0$ on $t \in (0,1]$. Our main tool is the fixed point theorem of cone expansion and compression of norm type [32]. The paper [33] motivated our study.

2 Preliminary

In this section, we present some lemmas which are needed for our main results.

Throughout this paper, we assume that $f : [0,1] \times [0,+\infty)^2 \times (-\infty,0] \to [0,+\infty)$ and $g : [0,1] \to [0,+\infty)$ are continuous, moreover, $\mu := \int_0^1 g(s)\,\mathrm{d}s < 1$.

Simple computations lead to the following lemma.

Lemma 2.1 *For any $h \in C[0,1]$, the BVP*

$$\begin{cases} x^{(4)}(t) = h(t), \quad t \in [0,1], \\ x(0) = x'(1) = x'''(1) = 0, \\ x''(0) = \int_0^1 g(s)x''(s)\,\mathrm{d}s, \end{cases} \tag{2.1}$$

has a unique solution

$$x(t) = \int_0^1 \left[G_1(t,s) + \frac{2t - t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau \right] h(s)\,\mathrm{d}s,$$

where

$$G_1(t,s) = \begin{cases} t(s - \frac{1}{2}s^2) - \frac{1}{6}t^3, & 0 \leq t \leq s \leq 1, \\ s(t - \frac{1}{2}t^2) - \frac{1}{6}s^3, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1. \end{cases}$$

Lemma 2.2 *Let $G_1(t,s)$ be as in Lemma 2.1. Then*

$$G_1(t,s) \geq \frac{1}{2}\left(t - \frac{1}{2}t^2\right)s, \quad (t,s) \in [0,1] \times [0,1].$$

Proof For $0 \leq t \leq s \leq 1$, one has

$$\begin{aligned} G_1(t,s) &= t\left(s - \frac{1}{2}s^2\right) - \frac{1}{6}t^3 \\ &\geq ts\left(1 - \frac{1}{2}s\right) - \frac{1}{6}t^2 s \end{aligned}$$

$$\begin{aligned}&\geq \frac{1}{2}ts-\frac{1}{6}t^2s\\&\geq \frac{1}{2}\left(t-\frac{1}{2}t^2\right)s.\end{aligned}$$

On the other hand, for $0\leq s\leq t\leq 1$, we have $\frac{1}{6}s^2+\frac{1}{6}t^2\leq\frac{1}{3}t$, and then

$$\begin{aligned}G_1(t,s)&=s\left(t-\frac{1}{2}t^2\right)-\frac{1}{6}s^3\\&\geq s\left(t-\frac{1}{2}t^2\right)-\left(\frac{1}{3}t-\frac{1}{6}t^2\right)s\\&=\frac{2}{3}\left(t-\frac{1}{2}t^2\right)s\\&\geq\frac{1}{2}\left(t-\frac{1}{2}t^2\right)s.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.3 *If $h\in C[0,1]$ with $h(t)\geq 0$ on $[0,1]$, then the unique solution $x=x(t)$ of the BVP* (2.1) *satisfies*:

(1) *$x(t)\geq 0$ for $t\in[0,1]$*;

(2) *$x'(t)\geq 0$, $x''(t)\leq 0$ for $t\in[0,1]$, and*

$$x(t)\geq\frac{1}{2}\left(t-\frac{1}{2}t^2\right)\|x''\|_\infty,\quad t\in[0,1].$$

Proof (1) From Lemma 2.2 and the fact

$$ts\leq G_2(t,s)\leq s,\quad\forall(t,s)\in[0,1]\times[0,1],$$

it follows that

$$x(t)=\int_0^1\left[G_1(t,s)+\frac{2t-t^2}{2(1-\mu)}\int_0^1G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right]h(s)\,\mathrm{d}s\geq 0,\quad t\in[0,1].\tag{2.2}$$

(2) Note that whenever $(t,s)\in[0,1]\times[0,1]$,

$$\frac{\partial}{\partial t}G_1(t,s)\geq 0,\qquad\frac{\partial^2}{\partial t^2}G_1(t,s)=-G_2(t,s)\leq 0,$$

it follows that

$$\begin{aligned}x'(t)&=\int_0^1\left[\frac{\partial}{\partial t}G_1(t,s)+\frac{1-t}{1-\mu}\int_0^1G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right]h(s)\,\mathrm{d}s\geq 0,\quad t\in[0,1],\\x''(t)&=\int_0^1\left[-G_2(t,s)-\frac{1}{1-\mu}\int_0^1G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right]h(s)\,\mathrm{d}s\leq 0,\quad t\in[0,1].\end{aligned}\tag{2.3}$$

On the one hand, by (2.3), we have

$$\|x''\|_\infty\leq\int_0^1\left[s+\frac{1}{1-\mu}\int_0^1G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right]h(s)\,\mathrm{d}s.\tag{2.4}$$

On the other hand, in view of (2.2) and Lemma 2.2, we have

$$\begin{aligned} x(t) &= \int_0^1 \left[G_1(t,s) + \frac{2t-t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s) g(\tau)\,\mathrm{d}\tau \right] h(s)\,\mathrm{d}s \\ &\geq \int_0^1 \left[\frac{1}{2}\left(t - \frac{1}{2}t^2\right)s + \frac{2t-t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s) g(\tau)\,\mathrm{d}\tau \right] h(s)\,\mathrm{d}s \\ &\geq \frac{1}{2}\left(t - \frac{1}{2}t^2\right) \int_0^1 \left[s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau)\,\mathrm{d}\tau \right] h(s)\,\mathrm{d}s, \quad t \in [0,1]. \end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$x(t) \geq \frac{1}{2}\left(t - \frac{1}{2}t^2\right) \|x''\|_\infty, \quad t \in [0,1].$$

This completes the proof of the lemma. □

Let

$$E = \{ x \in C^2[0,1] : x(0) = x'(1) = 0 \}$$

be endowed with the norm $\|x\| = \max_{t\in[0,1]} |x''(t)| =: \|x''\|_\infty$. Then E is a Banach space. If we denote

$$K = \left\{ x \in E : x(t) \geq \frac{1}{2}\left(t - \frac{1}{2}t^2\right)\|x\|, x'(t) \geq 0, x''(t) \leq 0, t \in [0,1] \right\},$$

then it is easy to see that K is a cone in E.

Now, we define an operator T on K as follows: for $x \in K$,

$$(Tx)(t) = \int_0^1 \left[G_1(t,s) + \frac{2t-t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s) g(\tau)\,\mathrm{d}\tau \right] f\big(s, x(s), x'(s), x''(s)\big)\,\mathrm{d}s.$$

By Lemma 2.3, we know that $T(K) \subset K$ and if x is a fixed point of T, then x is a concave and monotone positive solution of the BVP (1.1), (1.2).

Lemma 2.4 *$T : K \to K$ is completely continuous.*

Proof First, we show that T is continuous. To do this, suppose $x_n, x_0 \in K$ and $\|x_n - x_0\| \to 0$ $(n \to \infty)$. Then there exists $M_1 > 0$ such that $\|x_0\|, \|x_n\| \leq M_1$ for all $n \in \mathbb{N} = \{1, 2, \ldots\}$. Hence from the continuity of f on $[0,1] \times [0, M_1]^2 \times [-M_1, 0]$, we have

$$f\big(t, x_n(t), x_n'(t), x_n''(t)\big) \to f\big(t, x_0(t), x_0'(t), x_0''(t)\big) \quad (n \to \infty)$$

uniformly on $[0,1]$. Also, since

$$\begin{aligned} 0 &\leq G_2(t,s) + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau)\,\mathrm{d}\tau \\ &\leq 1 + \frac{1}{1-\mu} \int_0^1 g(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{1-\mu}, \quad (t,s) \in [0,1] \times [0,1], \end{aligned}$$

we have

$$\begin{aligned}(Tx_n)''(t) &= \int_0^1 \left[-G_2(t,s) - \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right] f\bigl(s,x_n(s),x_n'(s),x_n''(s)\bigr)\,\mathrm{d}s \\ &\to \int_0^1 \left[-G_2(t,s) - \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right] f\bigl(s,x_0(s),x_0'(s),x_0''(s)\bigr)\,\mathrm{d}s \\ &= (Tx_0)''(t)(n\to\infty) \quad \text{uniformly on } [0,1],\end{aligned}$$

i.e.,

$$\bigl\|(Tx_n)'' - (Tx_0)''\bigr\|_\infty = \|Tx_n - Tx_0\| \to 0 \quad (n\to\infty).$$

Therefore $T : K \to K$ is continuous.

Next, we prove that T is relatively compact. With this aim, let $D \subset K$ be a bounded set, then there exists a constant $M_2 > 0$ such that $\|x\| \le M_2$ for all $x \in D$. Suppose that $\{y_n\} \subset T(D)$, there exist $\{x_n\} \subset D$ such that $Tx_n = y_n$. Let

$$M_3 = \sup\bigl\{f(t,x_0,x_1,x_2) : (t,x_0,x_1,x_2) \in [0,1]\times[0,M_2]^2\times[-M_2,0]\bigr\}.$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}\bigl|y_n''(t)\bigr| &= \bigl|(Tx_n)''(t)\bigr| \\ &= \int_0^1 \left[G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right] f\bigl(s,x_n(s),x_n'(s),x_n''(s)\bigr)\,\mathrm{d}s \\ &\le M_3 \int_0^1 \left[s + \frac{1}{1-\mu}\int_0^1 sg(\tau)\,\mathrm{d}\tau\right]\mathrm{d}s \\ &= \frac{M_3}{2(1-\mu)}, \quad t \in [0,1]\end{aligned}$$

and

$$\begin{aligned}\bigl|y_n^{(4)}(t)\bigr| &= \bigl|(Tx_n)^{(4)}(t)\bigr| \\ &= f\bigl(t,x_n(t),x_n'(t),x_n''(t)\bigr) \\ &\le M_3, \quad t\in[0,1].\end{aligned}$$

Consequently there exists a constant $M_4 > 0$ such that, for all $n \in \mathbb{N}$,

$$\bigl|y_n'''(t)\bigr| \le M_4, \quad t \in [0,1].$$

By the Arzela-Ascoli theorem, we know that $\{y_n''\}$ has a convergent subsequence in supremum norm, *i.e.*, $\{y_n\}$ has a convergent subsequence in E, which indicates that $T(D) \subset K$ is relatively compact in E. This completes the proof of the lemma. □

The following fixed point theorem of cone expansion and compression of norm type plays a crucial role in our paper.

Lemma 2.5 ([32]) *Let E be a Banach space and let K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tx\| \le \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$ *for* $x \in K \cap \partial\Omega_2$, *or*

(ii) $\|Tx\| \ge \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \le \|x\|$ *for* $x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

For convenience, firstly we introduce some notations:

$$f^0 = \limsup_{x_0+x_1-x_2\to 0^+} \max_{t\in[0,1]} \frac{f(t,x_0,x_1,x_2)}{x_0+x_1-x_2}, \qquad f_0 = \liminf_{x_0+x_1-x_2\to 0^+} \min_{t\in[0,1]} \frac{f(t,x_0,x_1,x_2)}{x_0+x_1-x_2},$$

$$f^\infty = \limsup_{x_0+x_1-x_2\to +\infty} \max_{t\in[0,1]} \frac{f(t,x_0,x_1,x_2)}{x_0+x_1-x_2}, \qquad f_\infty = \liminf_{x_0+x_1-x_2\to +\infty} \min_{t\in[0,1]} \frac{f(t,x_0,x_1,x_2)}{x_0+x_1-x_2},$$

$$H_1 = \frac{3}{2(1-\mu)}, \qquad H_2 = \frac{1}{4}\int_0^1 s^2\left(1-\frac{1}{2}s\right)\left[\frac{1}{2} + \frac{1}{1-\mu}\int_0^1 \tau g(\tau)\,\mathrm{d}\tau\right]\mathrm{d}s.$$

Theorem 3.1 *If $H_1 f^0 < 1 < H_2 f_\infty$, then the BVP* (1.1), (1.2) *has at least one concave and monotone positive solution.*

Proof Since $H_1 f^0 < 1$, there exists $\varepsilon_1 > 0$ such that

$$H_1\big(f^0 + \varepsilon_1\big) < 1. \tag{3.1}$$

By the definition of f^0 and the continuity of f, there exists $\rho_1 > 0$ such that, for $t \in [0,1]$, $x_0 + x_1 - x_2 \in [0, \rho_1]$,

$$f(t,x_0,x_1,x_2) < \big(f^0 + \varepsilon_1\big)(x_0 + x_1 - x_2). \tag{3.2}$$

Let $\Omega_1 = \{x \in E : \|x\| < \rho/3\}$. For all $x \in K \cap \partial\Omega_1$, from (3.1) and (3.2), we have

$$\begin{aligned} \big|(Tx)''(t)\big| &= \int_0^1 \left[G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right] f\big(s,x(s),x'(s),x''(s)\big)\,\mathrm{d}s \\ &\le \int_0^1 \left[s + \frac{1}{1-\mu}\int_0^1 sg(\tau)\,\mathrm{d}\tau\right]\big(f^0+\varepsilon_1\big)\big(x(s)+x'(s)-x''(s)\big)\,\mathrm{d}s \\ &\le H_1\big(f^0+\varepsilon_1\big)\|x\| \\ &< \|x\|, \quad t \in [0,1], \end{aligned}$$

which implies that

$$\|Tx\| = \big\|(Tx)''\big\|_\infty < \|x\|, \quad \forall x \in K \cap \partial\Omega_1. \tag{3.3}$$

On the other hand, in view of $H_2 f_\infty > 1$, there exists $\varepsilon_2 > 0$ such that

$$H_2(f_\infty - \varepsilon_2) > 1. \tag{3.4}$$

By the definition of f_∞, there exists $\rho_2 > \rho_1$ such that, for $t \in [0,1]$, $x_0 + x_1 - x_2 \in [\rho_2, +\infty)$,

$$f(t,x_0,x_1,x_2) > (f_\infty - \varepsilon_1)(x_0 + x_1 - x_2). \tag{3.5}$$

Let $\Omega_2 = \{x \in E : \|x\| < \rho_2\}$. Then for all $x \in K \cap \partial\Omega_2$, from Lemma 2.2, (3.4), and (3.5) it follows that

$$\begin{aligned}
(Tx)(1) &= \int_0^1 \left[G_1(1,s) + \frac{1}{2(1-\mu)} \int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau \right] f\bigl(s,x(s),x'(s),x''(s)\bigr)\,\mathrm{d}s \\
&\geq \int_0^1 \left[\frac{1}{4}s + \frac{1}{2(1-\mu)} \int_0^1 \tau s g(\tau)\,\mathrm{d}\tau \right] (f_\infty - \varepsilon_2)\bigl(x(s) + x'(s) - x''(s)\bigr)\,\mathrm{d}s \\
&\geq \frac{1}{2} \int_0^1 \left[\frac{1}{2}s + \frac{s}{1-\mu} \int_0^1 \tau g(\tau)\,\mathrm{d}\tau \right] (f_\infty - \varepsilon_2) \frac{1}{2} \left(s - \frac{1}{2}s^2 \right) \|x\|\,\mathrm{d}s \\
&= H_2(f_\infty - \varepsilon_2)\|x\| \\
&> \|x\|, \quad t \in [0,1],
\end{aligned}$$

which implies that

$$\|Tx\| \geq \|Tx\|_\infty \geq (Tx)(1) > \|x\|, \quad \forall x \in K \cap \partial\Omega_2. \tag{3.6}$$

Therefore, it follows from (3.3), (3.6), and Lemma 2.5 that the operator T has one fixed point $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a concave and monotone positive solution of the BVP (1.1), (1.2). This completes the proof of the theorem. □

Corollary 3.1 *Suppose that f is superlinear, i.e.,*

$$f^0 = 0, \qquad f_\infty = +\infty.$$

Then the BVP (1.1), (1.2) *has at least one concave and monotone positive solution.*

Theorem 3.2 *If $H_1 f^\infty < 1 < H_2 f_0$, then the BVP* (1.1), (1.2) *has at least one concave and monotone positive solution.*

Proof Since $H_2 f_0 > 1$, there exists $\varepsilon_1 > 0$ such that

$$H_2(f_0 - \varepsilon_1) > 1. \tag{3.7}$$

By the definition of f_0, there exists $\rho_1 > 0$ such that, for $t \in [0,1]$, $x_0 + x_1 - x_2 \in [0, \rho_1]$,

$$f(t,x_0,x_1,x_2) > (f_0 - \varepsilon_1)(x_0 + x_1 - x_2). \tag{3.8}$$

Let $\Omega_1 = \{x \in E : \|x\| < \rho_1\}$. Then, for all $x \in K \cap \partial\Omega_1$, from Lemma 2.2, (3.7), and (3.8) it follows that

$$\begin{aligned}
(Tx)(1) &= \int_0^1 \left[G_1(1,s) + \frac{1}{2(1-\mu)} \int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau \right] f\bigl(s,x(s),x'(s),x''(s)\bigr)\,\mathrm{d}s \\
&\geq \int_0^1 \left[\frac{1}{4}s + \frac{1}{2(1-\mu)} \int_0^1 \tau s g(\tau)\,\mathrm{d}\tau \right] (f_0 - \varepsilon_1)\bigl(x(s) + x'(s) - x''(s)\bigr)\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\ge \frac{1}{2}\int_0^1\left(\frac{1}{2}s+\frac{s}{1-\mu}\int_0^1 \tau g(\tau)\,\mathrm{d}\tau\right)(f_0-\varepsilon_1)\frac{1}{2}\left(s-\frac{1}{2}s^2\right)\|x\|\,\mathrm{d}s \\
&= H_2(f_\infty-\varepsilon_1)\|x\| \\
&> \|x\|, \quad t\in[0,1],
\end{aligned}$$

which implies that

$$\|Tx\| > \|x\|, \quad \forall x\in K\cap\partial\Omega_1. \tag{3.9}$$

On the other hand, in view of $H_1 f^\infty < 1$, there exists $\varepsilon_2 > 0$ such that

$$H_2\left(f^\infty+\varepsilon_2\right) < 1. \tag{3.10}$$

By the definition of f^∞, there exists $\rho^* > 3\rho_1$ such that, for $t\in[0,1]$, $x_0+x_1-x_2\in[\rho^*,+\infty)$,

$$f(t,x_0,x_1,x_2) < \left(f^\infty+\varepsilon_2\right)(x_0+x_1-x_2).$$

Let

$$\beta = \max\left\{f(t,x_0,x_1,x_2) : (t,x_0,x_1,x_2)\in[0,1]\times\left[0,\rho^*\right]^2\times\left[-\rho^*,0\right]\right\}.$$

Then for $(t,x_0,x_1,x_2)\in[0,1]\times[0,+\infty)^2\times(-\infty,0]$ one has

$$f(t,x_0,x_1,x_2) < \left(f^\infty+\varepsilon_2\right)(x_0+x_1-x_2)+\beta. \tag{3.11}$$

Now, we choose $\rho_2 > \frac{1}{3}\max\{\rho^*, \frac{\beta H_1}{1-H_1(f^\infty+\varepsilon_2)}\}$ and let

$$\Omega_2 = \left\{x\in E : \|x\| < \rho_2\right\}.$$

For all $x\in K\cap\partial\Omega_2$, from (3.10) and (3.11) it follows that

$$\begin{aligned}
\left|(Tx)''(t)\right| &\le \left\|\int_0^1\left[G_2(t,s)+\frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right]f\left(s,x(s),x'(s),x''(s)\right)\mathrm{d}s\right\|_\infty \\
&\le \max_{0\le t\le 1}\int_0^1\left[G_2(t,s)+\frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right]\mathrm{d}s\left\|f\left(s,x(s),x'(s),x''(s)\right)\right\|_\infty \\
&\le \int_0^1\left(s+\frac{s}{1-\mu}\int_0^1 g(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\left[\left(f^\infty+\varepsilon_2\right)\left\|x(s)+x'(s)-x''(s)\right\|_\infty+\beta\right] \\
&\le H_1\left(f^\infty+\varepsilon_2\right)\|x\|+\frac{\beta}{3}H_1 \\
&< \rho_2 = \|x\|, \quad t\in[0,1],
\end{aligned}$$

which implies that

$$\|Tx\| < \|x\|, \quad \forall x\in K\cap\partial\Omega_2. \tag{3.12}$$

Therefore, it follows from (3.9), (3.12), and Lemma 2.5 that the operator T has one fixed point $x\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$, which is a concave and monotone positive solution of the BVP (1.1), (1.2). This completes the proof of the theorem. □

Corollary 3.2 *Suppose that f is sublinear, i.e.,*

$$f_0 = +\infty, \qquad f^\infty = 0.$$

Then the BVP (1.1), (1.2) *has at least one concave and monotone positive solution.*

Theorem 3.3 *Suppose that*

$$H_1 f(t, x_0, x_1, x_2) < x_0 + x_1 - x_2$$

for $(t, x_0, x_1, x_2) \in [0,1] \times [0,+\infty)^2 \times (-\infty, 0]$ *with* $x_0 + x_1 - x_2 > 0$. *Then the BVP* (1.1), (1.2) *has no concave and monotone positive solution.*

Proof By contradiction, assume that x is a concave and monotone positive solution of the BVP (1.1), (1.2). Then

$$x(t) \geq 0, \qquad x'(t) \geq 0, \qquad x''(t) \leq 0, \quad t \in [0,1]$$

and

$$x(t) + x'(t) - x''(t) > 0, \quad t \in (0,1].$$

Hence

$$\begin{aligned} \left|x''(t)\right| &= \int_0^1 \left[G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)\,\mathrm{d}\tau\right] f\big(s, x(s), x'(s), x''(s)\big)\,\mathrm{d}s \\ &\leq \int_0^1 \left(s + \frac{1}{1-\mu}\int_0^1 sg(\tau)\,\mathrm{d}\tau\right) f\big(s, x(s), x'(s), x''(s)\big)\,\mathrm{d}s \\ &< \int_0^1 s\left(1 + \frac{\mu}{1-\mu}\right)\frac{1}{H_1}\big(x(s) + x'(s) - x''(s)\big)\,\mathrm{d}s \\ &\leq \|x\|, \quad t \in [0,1], \end{aligned}$$

which implies that

$$\|x\| = \left\|x''\right\|_\infty < \|x\|.$$

This is a contradiction. Therefore the BVP (1.1), (1.2) has no concave and monotone positive solution. This completes the proof of the theorem. □

Theorem 3.4 *Suppose that*

$$H_2 f(t, x_0, x_1, x_2) > x_0 + x_1 - x_2$$

for $(t, x_0, x_1, x_2) \in [0,1] \times [0,+\infty)^2 \times (-\infty, 0]$ *with* $x_0 + x_1 - x_2 > 0$. *Then the BVP* (1.1), (1.2) *has no concave and monotone positive solution.*

Proof Suppose on the contrary that x is a concave and monotone positive solution of the BVP (1.1), (1.2). Then from Lemma 2.2 we have

$$\begin{aligned} x(1) &= \int_0^1 \left[G_1(1,s) + \frac{1}{2(1-\mu)} \int_0^1 G_2(\tau,s) g(\tau)\,\mathrm{d}\tau \right] f\bigl(s, x(s), x'(s), x''(s)\bigr)\,\mathrm{d}s \\ &> \int_0^1 \left[\frac{1}{4}s + \frac{1}{2(1-\mu)} \int_0^1 \tau s g(\tau)\,\mathrm{d}\tau \right] \frac{1}{H_2}\bigl(x(s) + x'(s) - x''(s)\bigr)\,\mathrm{d}s \\ &\geq \frac{1}{2H_2} \int_0^1 \left(\frac{1}{2}s + \frac{s}{1-\mu} \int_0^1 \tau g(\tau)\,\mathrm{d}\tau \right) \frac{1}{2}\left(s - \frac{1}{2}s^2 \right) \|x\|\,\mathrm{d}s \\ &= \|x\|, \end{aligned}$$

which is a contradiction. This completes the proof of the theorem. □

Finally, we give some examples to demonstrate applications of our results.

Example 3.1 Consider the fourth-order boundary value problem

$$x^{(4)}(t) = \frac{1}{1+t} \left[\frac{x + x' - x''}{4e^{x+x'-x''}} + \frac{34(x + x' - x'')^2}{1 + x + x' - x''} \right], \quad t \in [0,1], \tag{3.13}$$

$$x(0) = x'(1) = x'''(1) = 0, \qquad x''(0) = \int_0^1 s x''(s)\,\mathrm{d}s. \tag{3.14}$$

Let

$$f(t, x_0, x_1, x_2) = \frac{1}{1+t} \left[\frac{x_0 + x_1 - x_2}{4e^{x_0+x_1-x_2}} + \frac{34(x_0 + x_1 - x_2)^2}{1 + x_0 + x_1 - x_2} \right], \qquad g(t) = t.$$

Then $f \in C([0,1] \times [0,+\infty)^2 \times (-\infty,0], [0,+\infty))$, $g \in C([0,1],[0,+\infty))$, and $\mu = \int_0^1 g(s)\,\mathrm{d}s = \frac{1}{2} < 1$. It is easy to compute that

$$f^0 = \frac{1}{4}, \qquad f_\infty = 17, \qquad H_1 = 3, \qquad H_2 = \frac{35}{576},$$

and hence

$$H_1 f^0 < 1 < H_2 f_\infty.$$

So, it follows from Theorem 3.1 that the BVP (3.13), (3.14) has at least one concave and monotone positive solution.

Example 3.2 Consider the fourth-order boundary value problem

$$x^{(4)}(t) = \frac{1}{1+t} \left[\frac{14(x + x' - x'')}{1 + \ln(1 + x + x' - x'')} + \frac{(x + x' - x'')^2}{8(1 + x + x' - x'')} \right], \quad t \in [0,1], \tag{3.15}$$

$$x(0) = x'(1) = x'''(1) = 0, \qquad x''(0) = 3\int_0^1 s^3 x''(s)\,\mathrm{d}s. \tag{3.16}$$

Let

$$f(t, x_0, x_1, x_2) = \frac{1}{1+t} \left[\frac{14(x_0 + x_1 - x_2)}{1 + \ln(1 + x_0 + x_1 - x_2)} + \frac{(x_0 + x_1 - x_2)^2}{8(1 + x_0 + x_1 - x_2)} \right], \qquad g(t) = 3t^3.$$

Then $f \in C([0,1] \times [0,+\infty)^2 \times (-\infty,0],[0,+\infty))$, $g \in C([0,1],[0,+\infty))$, and $\mu = \int_0^1 g(s)\,\mathrm{d}s = \frac{3}{4} < 1$. It is easy to compute that

$$f^\infty = \frac{1}{8}, \qquad f_0 = 7, \qquad H_1 = 6, \qquad H_2 = \frac{29}{192},$$

and hence

$$H_1 f^\infty < 1 < H_2 f_0.$$

So, it follows from Theorem 3.2 that the BVP (3.15), (3.16) has at least one concave and monotone positive solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Li, H, Wang, L, Pei, M: Solvability of a fourth-order boundary value problem with integral boundary conditions. J. Appl. Math. **2013**, Article ID 782363 (2013)
2. O'Regan, D: Fixed-point theory for the sum of two operators. Appl. Math. Lett. **9**, 1-8 (1996)
3. Agarwal, RP, Chow, YM: Iterative methods for a fourth order boundary value problem. J. Comput. Appl. Math. **10**, 203-217 (1984)
4. Bai, Z: The upper and lower solution method for some fourth-order boundary value problems. Nonlinear Anal. **67**, 1704-1709 (2007)
5. Cabada, A, Minhós, FM: Fully nonlinear fourth-order equations with functional boundary conditions. J. Math. Anal. Appl. **340**, 239-251 (2008)
6. Del Pino, MA, Manasevich, RF: Existence for a fourth-order boundary value problem under a two parameter nonresonance condition. Proc. Am. Math. Soc. **112**, 81-86 (1991)
7. Du, J, Cui, M: Constructive proof of existence for a class of fourth-order nonlinear BVPs. Comput. Math. Appl. **59**, 903-911 (2010)
8. Ehme, J, Eloe, PW, Henderson, J: Upper and lower solution methods for fully nonlinear boundary value problems. J. Differ. Equ. **180**, 51-64 (2002)
9. Franco, D, O'Regan, D, Perán, J: Fourth-order problems with nonlinear boundary conditions. J. Comput. Appl. Math. **174**, 315-327 (2005)
10. Feng, H, Ji, D, Ge, W: Existence and uniqueness of solutions for a fourth-order boundary value problem. Nonlinear Anal. **70**, 3561-3566 (2009)
11. Graef, JR, Kong, L: A necessary and sufficient condition for existence of positive solutions of nonlinear boundary value problems. Nonlinear Anal. **66**, 2389-2412 (2007)
12. Graef, JR, Qian, CX, Yang, B: A three point boundary value problem for nonlinear fourth order differential equations. J. Math. Anal. Appl. **287**, 217-233 (2003)
13. Grossinho, MR, Tersian, SA: The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation. Nonlinear Anal. **41**, 417-431 (2000)
14. Gupta, CP: Existence and uniqueness theorems for a bending of an elastic beam equation. Appl. Anal. **26**, 289-304 (1988)
15. Jankowski, T: Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions. Nonlinear Anal. **73**, 1289-1299 (2010)
16. Jiang, DQ, Gao, WJ, Wan, AY: A monotone method for constructing extremal solutions to fourth-order periodic boundary value problems. Appl. Math. Comput. **132**, 411-421 (2002)
17. Kang, P, Wei, Z, Xu, J: Positive solutions to fourth-order singular boundary value problems with integral boundary conditions in abstract spaces. Appl. Math. Comput. **206**, 245-256 (2008)
18. Korman, P: Computation of displacements for nonlinear elastic beam models using monotone iterations. Int. J. Math. Math. Sci. **11**, 121-128 (1988)
19. Ma, TF: Positive solutions for a beam equation on a nonlinear elastic foundation. Math. Comput. Model. **39**, 1195-1201 (2004)
20. Ma, R, Xu, J: Bifurcation from interval and positive solutions of a nonlinear fourth-order boundary value problem. Nonlinear Anal. **72**, 113-122 (2010)
21. Minhós, F, Gyulov, T, Santos, AI: Lower and upper solutions for a fully nonlinear beam equation. Nonlinear Anal. **71**, 281-292 (2009)

22. O'Regan, D: Solvability of some fourth (and higher) order singular boundary value problems. J. Math. Anal. Appl. **161**, 78-116 (1991)
23. Pietramala, P: A note on a beam equation with nonlinear boundary conditions. Bound. Value Probl. **2011**, Article ID 376782 (2011)
24. Pei, M, Chang, SK: Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem. Math. Comput. Model. **51**, 1260-1267 (2010)
25. Shanthi, V, Ramanujam, N: A numerical method for boundary value problems for singularly perturbed fourth-order ordinary differential equations. Appl. Math. Comput. **129**, 269-294 (2002)
26. Sun, JP, Wang, XQ: Existence and iteration of monotone positive solution of BVP for an elastic beam equation. Math. Probl. Eng. **2011**, Article ID 705740 (2011)
27. Webb, JRL, Infante, G: Positive solutions of nonlocal boundary value problems: a unified approach. J. Lond. Math. Soc. (2) **74**, 673-693 (2006)
28. Webb, JRL, Infante, G, Franco, D: Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions. Proc. R. Soc. Edinb., Sect. A **138**, 427-446 (2008)
29. Wei, Z: A class of fourth order singular boundary value problems. Appl. Math. Comput. **153**, 865-884 (2004)
30. Yao, QL: Positive solutions for eigenvalue problems of fourth-order elastic beam equations. Appl. Math. Lett. **17**, 237-243 (2004)
31. Zhang, X, Ge, W: Positive solutions for a class of boundary value problems with integral boundary conditions. Comput. Math. Appl. **58**, 203-215 (2009)
32. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, Boston (1988)
33. Sun, JP, Li, HB: Monotone positive solution of nonlinear third-order BVP with integral boundary conditions. Bound. Value Probl. **2010**, Article ID 874959 (2010)

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