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Quasilinear elliptic equations with Hardy terms and Hardy-Sobolev critical exponents: nontrivial solutions

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Abstract

In this paper, we obtain one positive solution and two nontrivial solutions of a quasilinear elliptic equation with p-Laplacian, Hardy term and Hardy-Sobolev critical exponent by using variational methods and some analysis techniques. In particular, our results extend some existing ones.

MSC: 35B33; 35J60

Keywords: quasilinear elliptic equations; Hardy terms; Hardy-Sobolev critical exponents; $(PS)_c$ condition; p-Laplacian

1 Introduction and main results

We shall study the following quasilinear elliptic equation:

$$\begin{cases} -\triangle_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(s)-2}}{|x|^s} u + f(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian differential operator, Ω is an open bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$ and $0 \in \Omega$, $0 \leq \mu < \mu_1 := (\frac{N-p}{p})^p$, $0 \leq s < p$, $1 < p < N$, $p^*(s) = \frac{p(N-s)}{N-p}$ is the Hardy-Sobolev critical exponent and $p^* = p^*(0) = \frac{Np}{N-p}$ is the Sobolev critical exponent. The conditions of f will be given later.

On the Sobolev space $W_0^{1,p}(\Omega)$, we set

$$\|u\| := \left(\int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx \right)^{\frac{1}{p}}, \tag{1.2}$$

which is well defined on $W_0^{1,p}(\Omega)$ by the Hardy inequality [1]. It is comparable with the standard Sobolev norm of $W_0^{1,p}(\Omega)$, but it is not a norm since the triangle inequality or subadditivity may fail, which has been clarified in [2]. The following minimization problem will be useful in what follows:

$$A_{\mu,s} := A_{\mu,s}(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} dx)^{\frac{p}{p^*(s)}}}, \tag{1.3}$$

which is the best Hardy-Sobolev constant.

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For some special f and $p = 2$, some authors ([3–8], $s = 0$) ([1, 9–12], $s \neq 0$) have studied the existence of solutions for (1.1). If $p = 2$, then (1.1) becomes

$$\begin{cases} -\triangle u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u + f(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.4}$$

Ding and Tang [13] obtained the existence and multiplicity of solutions for (1.4) if $0 \leq \mu < (\frac{N-2}{2})^2$, $0 \leq s < 2$ and f satisfies some suitable conditions. Kang [14] considered another special case of (1.1) with $f(x,u) = \lambda \frac{|u|^{q-2}u}{|x|^t}$; for details, we refer the readers to see Remark 1.1.

Let $F(x,u) := \int_0^u f(x,s)\,ds$, $x \in \Omega$, $u \in \mathbb{R}$. In order to state our results, we make the following assumptions:

(A_1) $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$, $\lim_{u\to 0^+} \frac{f(x,u)}{u^{p-1}} = 0$ and $\lim_{u\to\infty} \frac{f(x,u)}{u^{p^*-1}} = 0$ uniformly for $x \in \overline{\Omega}$.
(A_2) There exists a constant ρ_0 with $\rho_0 > p$ such that

$$0 < \rho_0 F(x,u) \leq f(x,u)u, \quad \forall x \in \overline{\Omega}, \forall u \in \mathbb{R}^+ \setminus \{0\}.$$

(A_3) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $\lim_{u\to 0} \frac{f(x,u)}{|u|^{p-1}} = 0$ and $\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p^*-1}} = 0$ uniformly for $x \in \overline{\Omega}$.
(A_4) There exists a constant ρ_0 with $\rho_0 > p$ such that

$$0 < \rho_0 F(x,u) \leq f(x,u)u, \quad \forall x \in \overline{\Omega}, \forall u \in \mathbb{R} \setminus \{0\}.$$

Let $b(\mu)$ be one of zeroes of the function $g(t) = (p-1)t^p - (N-p)t^{p-1} + \mu$, where $t \geq 0$ and $0 \leq \mu < \mu_1$. Now, our main results read as follows.

Theorem 1.1 *Suppose that* $N \geq 3$, $0 \leq \mu < \mu_1$, $0 \leq s < p$, $1 < p < N$. *If* (A_1), (A_2) *and*

$$\rho_0 > \max\left\{p, \frac{N}{b(\mu)}, \frac{p[2N - p - b(\mu)p]}{N-p}\right\} \tag{1.5}$$

hold, then problem (1.1) *has at least one positive solution.*

Theorem 1.2 *Suppose that* $N \geq 3$, $0 \leq \mu < \mu_1$, $0 \leq s < p$, $1 < p < N$. *If* (A_3), (A_4) *and* (1.5) *hold, then problem* (1.1) *has at least two distinct nontrivial solutions.*

Remark 1.1 We extend the special case $p = 2$ in [13] to a more general situation $1 < p < N$. The author [14] obtained one positive solution for a special case of (1.1) with $f(x,u) = \lambda \frac{|u|^{q-2}u}{|x|^t}$, where $\lambda > 0$, $0 \leq t < p$, $\overline{q} < q < p^*(t)$ and

$$\overline{q} = \max\left\{p, \frac{N-t}{b(\mu)}, \frac{p[2N - t - b(\mu)p - p]}{N-p}\right\}.$$

Note that the function f in [14] has to be a homogeneous function, but in the present paper it is not the case. Besides, we also obtain multiple solutions of (1.1) (see our Theorem 1.2).

Remark 1.2 We prove Theorems 1.1 and 1.2 by critical point theory. Due to the lack of compactness of the embeddings in $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p}\,dx)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*(s)}(\Omega, |x|^{-s}\,dx)$, we cannot use the standard variational argument directly.

The corresponding energy functional fails to satisfy the classical Palais-Smale ((PS) for short) condition in $W_0^{1,p}(\Omega)$. However, a local (PS) condition can be established in a suitable range. Then the existence result is obtained via constructing a minimax level within this range and the mountain pass lemma due to Ambrosetti and Rabinowitz (see also [15]).

Notations For the functional $I : X \to \mathbb{R}$ (X is a Banach space), we say that I satisfies the classical Palais-Smale ((PS) for short) condition if every sequence $\{u_n\}$ in X such that $I(u_n)$ is bounded in X and $I'(u_n) \to 0$ as $n \to \infty$ contains a convergent subsequence. We say that I satisfies $(\mathrm{PS})_c$ condition (a local Palais-Smale condition) if every sequence $\{u_n\}$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$ ($c \in \mathbb{R}$) contains a convergent subsequence.

The rest of this paper is organized as follows. In Section 2, we establish some preliminary lemmas, which are useful in the proofs of our main results. In Section 3, we give detailed proofs of our main results.

2 Preliminaries

In what follows, we let $\|\cdot\|_p$ denote the norm in $L^p(\Omega)$. It is obvious that the values of $f(x,u)$ for $u < 0$ are irrelevant in Theorem 1.1, and we may define

$$f(x,u) \equiv 0 \quad \text{if } u \le 0, \forall x \in \Omega.$$

We shall firstly consider the existence of nontrivial solutions for the following problem:

$$\begin{cases} -\triangle_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{(u^+)^{p^*(s)-1}}{|x|^s} + f(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.1}$$

The energy functional corresponding to (2.1) is given by

$$\begin{aligned} I(u) = {} & \frac{1}{p} \int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx - \frac{1}{p^*(s)} \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s} \, dx \\ & - \int_\Omega F(x,u) \, dx, \quad u \in W_0^{1,p}(\Omega). \end{aligned}$$

By the Hardy and Hardy-Sobolev inequalities (see [1, 16]) and (A_1), $I \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Now it is well known that there exists a one-to-one correspondence between the weak solutions of problem (2.1) and the critical points of I on $W_0^{1,p}(\Omega)$. More precisely, we say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (2.1) if for any $v \in W_0^{1,p}(\Omega)$, there holds

$$\begin{aligned} \langle I'(u), v \rangle = {} & \int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla v - \mu \frac{|u|^{p-2} u v}{|x|^p} \right) dx \\ & - \int_\Omega \frac{(u^+)^{p^*(s)-1}}{|x|^s} v \, dx - \int_\Omega f(x,u) v \, dx = 0. \end{aligned}$$

Next, we shall give some lemmas which are needed in proving our main results.

Lemma 2.1 ([17]) *If $f_n \to f$ a.e. in Ω and $\|f_n\|_p \le C < \infty$ for all n and some $0 < p < \infty$, then*

$$\lim_{n\to\infty} \left(\|f_n\|_p^p - \|f_n - f\|_p^p \right) = \|f\|_p^p.$$

Lemma 2.2 ([14]) *Suppose* $1 < p < N$, $0 \leq s < p$ *and* $0 \leq \mu < \mu_1$. *Then the limiting problem*

$$\begin{cases} -\triangle_p u - \mu \frac{u^{p-1}}{|x|^p} = \frac{u^{p^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u > 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\ u \in D^{1,p}(\mathbb{R}^N) \end{cases} \tag{2.2}$$

has radially symmetric ground states

$$\widetilde{V}_\varepsilon(x) := \varepsilon^{-\frac{N-p}{p}} \widetilde{U}_{p,\mu}\left(\frac{x}{\varepsilon}\right) = \varepsilon^{-\frac{N-p}{p}} \widetilde{U}_{p,\mu}\left(\frac{|x|}{\varepsilon}\right), \quad \forall \varepsilon > 0,$$

and satisfies

$$\int_{\mathbb{R}^N} \left(\left|\nabla \widetilde{V}_\varepsilon(x)\right|^p - \mu \frac{|\widetilde{V}_\varepsilon(x)|^p}{|x|^p} \right) dx = \int_{\mathbb{R}^N} \frac{|\widetilde{V}_\varepsilon(x)|^{p^*(s)}}{|x|^s}\, dx = A_{\mu,s}^{\frac{N-s}{p-s}},$$

where $\widetilde{U}_{p,\mu}(x) = \widetilde{U}_{p,\mu}(|x|)$ *is the unique radial solution of* (2.2) *satisfying*

$$\widetilde{U}_{p,\mu}(1) = \left(\frac{(N-s)(\mu_1 - \mu)}{N-p} \right)^{\frac{1}{p^*(s)-p}}.$$

Moreover, $\widetilde{U}_{p,\mu}$ *has the following properties*:

$$\begin{aligned} &\lim_{r\to 0} r^{a(\mu)} \widetilde{U}_{p,\mu}(r) = c_1 > 0, \qquad \lim_{r\to +\infty} r^{b(\mu)} \widetilde{U}_{p,\mu}(r) = c_2 > 0, \\ &\lim_{r\to 0} r^{a(\mu)+1} \widetilde{U}'_{p,\mu}(r) = c_1 a(\mu) \geq 0, \qquad \lim_{r\to +\infty} r^{b(\mu)+1} \widetilde{U}'_{p,\mu}(r) = c_2 b(\mu) > 0, \end{aligned}$$

where c_1 *and* c_2 *are positive constants depending on* p *and* N; $a(\mu)$ *and* $b(\mu)$ *are zeroes of the function* $g(t) = (p-1)t^p - (N-p)t^{p-1} + \mu$ $(t \geq 0, 0 \leq \mu < \mu_1)$ *satisfying* $0 \leq a(\mu) < \frac{N-p}{p} < b(\mu) < \frac{N-p}{p-1}$.

Lemma 2.3 *Assume that* (A_1) *and* (A_2) *hold. If* $c \in (0, \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}})$, *then* I *satisfies* $(PS)_c$ *condition.*

Proof Suppose that $\{u_n\}$ is a $(PS)_c$ sequence of I in $W_0^{1,p}(\Omega)$, that is,

$$I'(u_n) \to 0, \qquad I(u_n) \to c, \quad n \to \infty.$$

By (A_2), we have

$$\begin{aligned} &c + 1 + o(1)\|u_n\| \\ &\quad \geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\ &\quad = \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + \left(\frac{1}{\theta} - \frac{1}{p^*(s)}\right) \int_\Omega \frac{(u_n^+)^{p^*(s)}}{|x|^s}\, dx - \int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n \right) dx \\ &\quad \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p, \end{aligned}$$

where $\theta = \min\{\rho_0, p^*(s)\}$. Hence we conclude that $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. So there exists $u \in W_0^{1,p}(\Omega)$ such that (going if necessary to a subsequence)

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \qquad u_n \to u \quad \text{in } L^\gamma(\Omega), 1 < \gamma < p^* \quad \text{and}$$
$$u_n \to u \quad \text{a.e. in } \Omega \text{ as } n \to \infty.$$

By the continuity of embedding, we have $\|u_n\|_{p^*}^{p^*} \le C_1 < \infty$. Going if necessary to a subsequence, from [1] one can get that

$$\begin{cases} \nabla u_n \to \nabla u \quad \text{a.e. in } \Omega, \\ \frac{|u_n|^{p-2}u_n}{|x|^{p-1}} \rightharpoonup \frac{|u|^{p-2}u}{|x|^{p-1}} \quad \text{weakly in } L^{p'}(\Omega), p' = \frac{p}{p-1}, \\ \int_\Omega \frac{|u_n|^{p^*(s)-2}u_n}{|x|^s} v\,dx \to \int_\Omega \frac{|u|^{p^*(s)-2}u}{|x|^s} v\,dx, \quad \forall v \in W_0^{1,p}(\Omega) \end{cases} \tag{2.3}$$

as $n \to \infty$. By (A_1), for any $\varepsilon > 0$, there exists $a(\varepsilon) > 0$ such that

$$\left|f(x,t)t\right| \le a(\varepsilon) + \frac{1}{2C_1}\varepsilon|t|^{p^*} \quad \text{for } (x,t) \in \overline{\Omega} \times (0,+\infty).$$

Set $\delta := \frac{\varepsilon}{2a(\varepsilon)} > 0$. Let $E \subset \Omega$ with $\operatorname{meas}(E) < \delta = \frac{\varepsilon}{2a(\varepsilon)}$, it follows from the fact $\|u_n\|_{p^*}^{p^*} \le C_1$ that

$$\left|\int_E f(x,u_n)u_n\,dx\right| \le \int_E a(\varepsilon)\,dx + \frac{\varepsilon}{2C_1}\int_E |u_n|^{p^*} < \varepsilon \to 0 \quad \text{as } \operatorname{meas}(E) \to 0.$$

It follows from the fact that $f(x,u_n)u_n \to f(x,u)u$ as $n \to \infty$ a.e. in Ω and Vitali's theorem that

$$\int_\Omega f(x,u_n)u_n\,dx \to \int_\Omega f(x,u)u\,dx \quad \text{as } n \to \infty.$$

Similarly, we can also get

$$\int_\Omega F(x,u_n)\,dx \to \int_\Omega F(x,u)\,dx \quad \text{as } n \to \infty.$$

Let $v_n = u_n - u$. By the definition of $\|\cdot\|$, we get

$$\|u\|^p = \|\nabla u\|_p^p - \mu\|u/x\|_p^p, \qquad \|v_n\|^p = \|\nabla u_n - \nabla u\|_p^p - \mu\left\|(u_n - u)/x\right\|_p^p,$$

it follows from $u_n \to u$ a.e. in Ω, $\nabla u_n \to \nabla u$ a.e. in Ω (see (2.3)) and Lemma 2.1 that

$$\lim_{n\to\infty}\left(\|u_n\|^p - \|v_n\|^p\right) = \|u\|^p.$$

It follows from $I'(u_n) \to 0$, $\int_\Omega f(x,u_n)u_n\,dx \to \int_\Omega f(x,u)u\,dx$ and the Brezis-Lieb lemma [17] that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\langle I'(u_n), u_n\rangle \\ &= \lim_{n\to\infty}\left(\|u_n\|^p - \int_\Omega \frac{(u_n^+)^{p^*(s)}}{|x|^s}\,dx - \int_\Omega f(x,u_n)u_n\,dx\right) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left[\|u_n\|^p - \|v_n\|^p - \int_\Omega \frac{(u_n^+)^{p^*(s)} - (v_n^+)^{p^*(s)}}{|x|^s}\,dx\right.\\
&\quad\left. - \int_\Omega f(x,u_n)u_n\,dx + \|v_n\|^p - \int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right]\\
&= \|u\|^p - \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx - \int_\Omega f(x,u)u\,dx + \lim_{n\to\infty}\left[\|v_n\|^p - \int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right]. \qquad (2.4)
\end{aligned}
$$

From (2.3) and $I'(u_n) \to 0$, we get

$$
0 = \lim_{n\to\infty}\langle I'(u_n), u\rangle = \|u\|^p - \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx - \int_\Omega f(x,u)u\,dx. \qquad (2.5)
$$

By $I(u_n) \to c$, $\int_\Omega F(x,u_n)\,dx \to \int_\Omega F(x,u)\,dx$, $\lim_{n\to\infty}(\|u_n\|^p - \|v_n\|^p) = \|u\|^p$ and the Brezis-Lieb lemma, we have

$$
\begin{aligned}
c &= \lim_{n\to\infty} I(u_n)\\
&= \lim_{n\to\infty}\left(\frac{1}{p}\|v_n\|^p + \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right.\\
&\quad\left. - \frac{1}{p^*(s)}\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx - \int_\Omega F(x,u_n)\,dx\right)\\
&= I(u) + \lim_{n\to\infty}\left(\frac{1}{p}\|v_n\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right).
\end{aligned}
$$

That is,

$$
I(u) + \lim_{n\to\infty}\left(\frac{1}{p}\|v_n\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right) = c. \qquad (2.6)
$$

Obviously, (2.4) and (2.5) imply

$$
\lim_{n\to\infty}\left[\|v_n\|^p - \int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right] = 0.
$$

We claim that $\|v_n\|^p \to 0$ as $n \to \infty$. Otherwise, there exists a subsequence (still denoted by v_n) such that

$$
\lim_{n\to\infty}\|v_n\|^p = k, \qquad \lim_{n\to\infty}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx = k, \quad k > 0. \qquad (2.7)
$$

By the definition of $A_{\mu,s}$ in (1.3), we have

$$
\|v_n\|^p \geq A_{\mu,s}\left(\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right)^{\frac{p}{p^*(s)}}, \quad \forall n \in \mathbb{N}.
$$

It follows from (2.7) that $k \geq A_{\mu,s}k^{\frac{p}{p^*(s)}}$, so we have $k \geq A_{\mu,s}^{\frac{N-s}{p-s}}$, which together with (2.6) and $c < \frac{p-s}{p(N-s)}A_{\mu,s}^{\frac{N-s}{p-s}}$ (see the assumption in Lemma 2.3) implies that

$$
I(u) < 0.
$$

However, (2.5) implies that

$$\|u\|^p = \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx + \int_\Omega f(x,u)u\,dx,$$

it follows from the definition of I, (1.2) and (A_2) that

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx - \int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx - \frac{1}{\rho_0}\int_\Omega f(x,u)u\,dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx + \left(\frac{1}{p} - \frac{1}{\rho_0}\right)\int_\Omega f(x,u)u\,dx \quad \bigl(\rho_0 > p \text{ in } (A_2)\bigr) \\ &\geq 0. \end{aligned}$$

So we get a contradiction. Therefore, we can obtain

$$\|v_n\|^p \to 0 \quad \text{as } n \to \infty.$$

From the discussion above, I satisfies $(PS)_c$ condition. □

In the following, we shall give some estimates for the extremal functions. Define a function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $|x| \leq R$, $\varphi(x) = 0$ for $|x| \geq 2R$, $0 \leq \varphi(x) \leq 1$, where $B_{2R}(0) \subset \Omega$. Set $v_\varepsilon(x) = \varphi(x)\widetilde{V}_\varepsilon(x)$, $\varepsilon > 0$, where $\widetilde{V}_\varepsilon(x)$ see the definition in Lemma 2.2. Then we can get the following results by the method used in [18]:

$$\|v_\varepsilon\|^p = A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p+p-N}\bigr), \tag{2.8}$$

$$\int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx = A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p^*(s)+s-N}\bigr) \tag{2.9}$$

and

$$\begin{cases} C'\varepsilon^{r(b(\mu)+1-\frac{N}{p})} \leq \int_\Omega |v_\varepsilon|^r\,dx \leq C\varepsilon^{r(b(\mu)+1-\frac{N}{p})}, & r < \frac{N}{b(\mu)}, \\ C'\varepsilon^{N+r(1-\frac{N}{p})}|\ln\varepsilon| \leq \int_\Omega |v_\varepsilon|^r\,dx \leq C\varepsilon^{N+r(1-\frac{N}{p})}|\ln\varepsilon|, & r = \frac{N}{b(\mu)}, \\ C'\varepsilon^{N+r(1-\frac{N}{p})} \leq \int_\Omega |v_\varepsilon|^r\,dx \leq C\varepsilon^{N+r(1-\frac{N}{p})}, & r > \frac{N}{b(\mu)}. \end{cases} \tag{2.10}$$

Lemma 2.4 *Suppose that* $0 \leq s < p$ *and* $0 \leq \mu < \mu_1$. *If* (A_1), (A_2) *and* (1.5) *hold, then there exists* $u_0 \in W_0^{1,p}(\Omega)$ *with* $u_0 \not\equiv 0$ *such that*

$$\sup_{t\geq 0} I(tu_0) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}}.$$

Proof We consider the functions

$$\begin{aligned} g(t) &= I(tv_\varepsilon) = \frac{t^p}{p}\|v_\varepsilon\|^p - \frac{t^{p^*(s)}}{p^*(s)}\int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx - \int_\Omega F(x,tv_\varepsilon)\,dx, \\ \overline{g}(t) &= \frac{t^p}{p}\|v_\varepsilon\|^p - \frac{t^{p^*(s)}}{p^*(s)}\int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx. \end{aligned}$$

Since $\lim_{t\to\infty} g(t) = -\infty$, $g(0) = 0$ and $g(t) > 0$ for t small enough, $\sup_{t\geq 0} g(t)$ is attained for some $t_\varepsilon > 0$. Therefore, we have

$$0 = g'(t_\varepsilon) = t_\varepsilon^{p-1}\left(\|v_\varepsilon\|^p - t_\varepsilon^{p^*(s)-p} \int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx - \frac{1}{t_\varepsilon^{p-1}} \int_\Omega f(x, t_\varepsilon v_\varepsilon) v_\varepsilon \,dx \right),$$

hence

$$\|v_\varepsilon\|^p = t_\varepsilon^{p^*(s)-p} \int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx + \frac{1}{t_\varepsilon^{p-1}} \int_\Omega f(x, t_\varepsilon v_\varepsilon) v_\varepsilon \,dx \geq t_\varepsilon^{p^*(s)-p} \int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx.$$

It follows from (2.8) and (2.9) that $t_\varepsilon \leq C$ for ε small enough. From the above inequality, we obtain

$$t_\varepsilon \leq \left(\|v_\varepsilon\|^p \Big/ \int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx \right)^{\frac{1}{p^*(s)-p}} := t_\varepsilon^0.$$

By (A_1), we can easily get

$$\bigl|f(x,t)\bigr| \leq \varepsilon t^{p^*-1} + d(\varepsilon) t^{p-1} \quad \text{for some } d(\varepsilon) > 0.$$

Hence, together with $t_\varepsilon \leq C$, we can get

$$\|v_\varepsilon\|^p \leq t_\varepsilon^{p^*(s)-p} \int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx + \varepsilon C \int_\Omega |v_\varepsilon|^{p^*}\,dx + d(\varepsilon) \int_\Omega |v_\varepsilon|^p\,dx.$$

By (2.8)-(2.10), when ε is small enough, we conclude that

$$t_\varepsilon^{p^*(s)-p} \geq \frac{1}{2}. \tag{2.11}$$

On the other hand, the function $\overline{g}(t)$ attains its maximum at t_ε^0 and is increasing in the interval $[0, t_\varepsilon^0]$, together with (2.8), (2.9) and (2.11) and $F(x,t) \geq C_5|t|^{\rho_0}$ which is directly got from (A_2), we deduce

$$\begin{aligned}
g(t_\varepsilon) &\leq \overline{g}\bigl(t_\varepsilon^0\bigr) - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\,dx \\
&= \frac{p-s}{p(N-s)} \|v_\varepsilon\|^{\frac{p(N-s)}{p-s}} \left(\int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx \right)^{-\frac{N-p}{p-s}} - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\,dx \\
&= \frac{p-s}{p(N-s)} \bigl[A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p+p-N}\bigr) \bigr]^{\frac{N-s}{p-s}} \bigl[A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p^*(s)+s-N}\bigr) \bigr]^{-\frac{N-p}{p-s}} \\
&\quad - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\,dx \\
&= \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p+p-N}\bigr) + O\bigl(\varepsilon^{b(\mu)p^*(s)+s-N}\bigr) - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\,dx \\
&\leq \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p+p-N}\bigr) + O\bigl(\varepsilon^{b(\mu)p^*(s)+s-N}\bigr) - C_6 \int_\Omega |v_\varepsilon|^\rho\,dx.
\end{aligned}$$

Furthermore, from (1.5) and (2.10), we get

$$\int_\Omega |v_\varepsilon|^\rho\,dx \ge C_7\varepsilon^{N+\rho(1-\frac{N}{p})}.$$

Note that $b(\mu) > \frac{N-p}{p}$ implies

$$\frac{p[2N-p-b(\mu)p]}{N-p} > \frac{p[2N-s-p^*(s)b(\mu)]}{N-p}.$$

By (1.5), we have $\rho_0 > \frac{p[2N-p-b(\mu)p]}{N-p}$, which implies

$$b(\mu)p+p-N > N+\rho_0\left(1-\frac{N}{p}\right)$$

and

$$b(\mu)p^*(s)+s-N > N+\rho_0\left(1-\frac{N}{p}\right).$$

Therefore, by choosing ε small enough, we have

$$\sup_{t\ge 0} I(tv_\varepsilon) = g(t_\varepsilon) < \frac{p-s}{p(N-s)}A_{\mu,s}^{\frac{N-s}{p-s}}.$$

Hence the proof of this lemma is then completed by taking $u_0 = v_\varepsilon$. □

3 Proofs of our main results

Proof of Theorem 1.1 From the Sobolev and Hardy-Sobolev inequalities, we can easily get

$$\begin{aligned} &\|u\|_p^p \le C\|u\|^p, \qquad \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\,dx \le C\|u\|^{p^*(s)}, \\ &\|u\|_{p^*}^{p^*} \le C\|u\|^{p^*}, \quad \forall u\in W_0^{1,p}(\Omega). \end{aligned} \tag{3.1}$$

The condition (A_1) implies that for any $\varepsilon > 0$ there exist $\delta_2 > \delta_1 > 0$ such that

$$\left|f(x,u)\right| < \varepsilon u^{p-1}, \quad \forall u\in(0,\delta_1)$$

and

$$\left|f(x,u)\right| < u^{p^*-1}, \quad \forall u > \delta_2.$$

Therefore, there exists a constant $C_\varepsilon > 0$ such that

$$\left|f(x,u)\right| \le \varepsilon u^{p-1} + C_\varepsilon u^{p^*-1}, \quad \forall u\in\mathbb{R}^+, \forall x\in\overline{\Omega}.$$

Then one gets

$$\left|F(x,u)\right| \le \frac{1}{p}\varepsilon|u|^p + C_8|u|^{p^*}, \quad \forall u\in\mathbb{R}^+, \forall x\in\overline{\Omega}. \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx - \int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{p}\|u\|^p - \frac{C}{p^*(s)}\|u^+\|^{p^*(s)} - \frac{C}{p}\varepsilon\|u\|^p - CC_8\|u\|^{p^*} \end{aligned}$$

for $\varepsilon > 0$ small enough. So there exists $\beta > 0$ such that

$$I(u) \geq \beta \quad \text{for all } u \in \partial B_r = \{u \in W_0^{1,p}(\Omega), \|u\| = r\}, r > 0 \text{ small enough.}$$

By Lemma 2.4, there exists $u_0 \in W_0^{1,p}(\Omega)$ with $u_0 \not\equiv 0$ such that

$$\sup_{t\geq 0} I(tu_0) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}}.$$

It follows from the nonnegativity of $F(x,t)$ that

$$\begin{aligned} I(tu_0) &= \frac{1}{p}t^p\|u_0\|^p - \frac{1}{p^*(s)}t^{p^*(s)}\int_\Omega \frac{(u_0^+)^{p^*(s)}}{|x|^s}\,dx - \int_\Omega F(x,tu_0)\,dx \\ &\leq \frac{1}{p}t^p\|u_0\|^p - \frac{1}{p^*(s)}t^{p^*(s)}\int_\Omega \frac{(u_0^+)^{p^*(s)}}{|x|^s}\,dx, \end{aligned}$$

therefore, $\lim_{t\to+\infty} I(tu_0) \to -\infty$. Hence, we can choose $t_0 > 0$ such that

$$\|t_0u_0\| > r \quad \text{and} \quad I(t_0u_0) \leq 0.$$

By virtue of the mountain pass lemma in [19], there is a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ satisfying

$$I(u_n) \to c \geq \beta \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]} I(h(t))$$

and

$$\Gamma = \{h \in C([0,1], W_0^{1,p}(\Omega)) \mid h(0) = 0, h(1) = t_0u_0\}.$$

Note that

$$0 < \beta \leq c = \inf_{h\in\Gamma}\max_{t\in[0,1]} I(h(t)) \leq \max_{t\in[0,1]} I(tt_0u_0) \leq \sup_{t\geq 0} I(tu_0) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}}.$$

By Lemma 2.3, we can assume that $u_n \to u$ in $W_0^{1,p}(\Omega)$. From the continuity of I', we know that u is a weak solution of problem (2.1). Then $\langle I'(u), u^-\rangle = 0$, where $u^- = \min\{u, 0\}$. Thus $u \geq 0$. Therefore u is a nonnegative solution of (1.1). By the strong maximum principle, u is a positive solution of problem (1.1), so Theorem 1.1 holds. □

Proof of Theorem 1.2 By Theorem 1.1, problem (1.1) has a positive solution u_1. Set $g(x,t) = -f(x,-t)$ for $t \in \mathbb{R}$. It follows from Theorem 1.1 that the equation

$$-\triangle_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u^+|^{p^*(s)-2}}{|x|^s} u + g(x,u)$$

has at least one positive solution v. Let $u_2 = -v$, then u_2 is a solution of

$$-\triangle_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u^+|^{p^*(s)-2}}{|x|^s} u + f(x,u).$$

It is obvious that $u_1 \neq 0$, $u_2 \neq 0$ and $u_1 \neq u_2$. So problem (1.1) has at least two nontrivial solutions. Therefore, Theorem 1.2 holds. □

Competing interests
The author declares that he has no competing interests.

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