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Triple positive solutions for a second order m-point boundary value problem with a delayed argument

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Abstract

In this paper, we establish the new expression and properties of Green's function for an m-point boundary value problem with a delayed argument. Furthermore, using Hölder's inequality and a fixed point theorem due to Leggett and Williams, the existence of at least three positive solutions is also given. We discuss our problem with a delayed argument. In this case, our results cover m-point boundary value problems without delayed arguments and are compared with some recent results. An example is included to illustrate our main results.

Keywords: triple positive solutions; delayed argument; Leggett-Williams' fixed point theorem; Hölder's inequality

1 Introduction

It is well known that multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section composed of N parts of different densities can be set up as a multi-point boundary value problem. Many problems in the theory of elastic stability can be handled as multi-point boundary value problems too. Recently, the existence and multiplicity of positive solutions for multi-point boundary value problems of ordinary differential equations have received a great deal of attention. To identify a few, we refer the reader to [1–18] and the references therein. Recently, Sun and Liu [19] applied the Leray-Schauder nonlinear alternative to study the existence of a nontrivial solution for the problem given by

$$\begin{cases} u''(t) + f(t,u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \alpha u(\eta), \end{cases}$$

where $\eta \in (0,1)$, $\alpha \in R$ and $\alpha \neq 1$.

At the same time, a type of boundary value problems with deviating arguments has also received much attention. For example, in [20], Yang *et al.* studied the existence and multiplicity of positive solutions to a three-point boundary value problem with an advanced argument

$$\begin{cases} u''(t) + a(t)f(u(\alpha(t))) = 0, & t \in (0,1), \\ u(0) = 0, \quad bu(\eta) = u(1), \end{cases} \tag{1.1}$$

where $0 < \eta < 1$, $b > 0$ and $1 - b\eta > 0$. The main tool is the fixed point index theory.

It is easy to see that the solution of problem (1.1) is concave when $a(t) \geq 0$ on $[0,1]$ and $f(u) \geq 0$ on $[0,\infty)$. However, few papers have reported the same problems where the solution is without concavity; for example, see some recent excellent results and applications of the case of ordinary differential equations with deviating arguments to a variety of problems from Jankowski [21–23], Jiang and Wei [24], Wang [25], Wang *et al.* [26] and Hu *et al.* [27].

In the present paper, we shall investigate the existence of triple positive solutions for the following m-point boundary value problem with a delayed argument:

$$\begin{cases} Lx = \omega(t)f(t, x(\alpha(t))), & 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1.2}$$

where $\xi_i \in (0,1)$, $\beta_i \in (0,+\infty)$ $(i = 1,2,\ldots,m-2)$ are given constants and L denotes the linear operator

$$Lx := -x'' - ax' + bx,$$

here $a \in C([0,1],[0,+\infty))$ and $b \in C([0,1],(0,+\infty))$.

Throughout this paper, we assume that $\alpha(t) \not\equiv t$ on $J = [0,1]$. In addition, ω, f and β_i $(i = 1,2,\ldots,m-2)$ satisfy:

(H_1) $\omega \in L^p[0,1]$ for some $p \in [1,+\infty)$, and there exists $n > 0$ such that $\omega(t) \geq n$ a.e. on J;
(H_2) $f \in C([0,1] \times [0,+\infty),[0,+\infty))$, $\alpha \in C(J,J)$ with $\alpha(t) \leq t$ on J;
(H_3) $\sum_{i=1}^{m-2} \beta_i\phi(\xi_i) < 1$, where ϕ satisfies

$$L\phi = 0, \qquad \phi'(0) = 0, \qquad \phi(1) = 1. \tag{1.3}$$

Remark 1.1 By a positive solution of problem (1.2) we mean a function $x \in C^2(0,1) \cap C[0,1]$ with $x(t) > 0$ on $(0,1)$ that satisfies (1.2).

Remark 1.2 Generally, when $y(t) \geq 0$ on J, the solution x is not concave for the linear equation

$$Lx - y(t) = 0.$$

This means that the method depending on concavity is no longer valid, and we need to introduce a new method to study this kind of problems.

For the case $\alpha(t) \equiv t$ on J, problem (1.2) reduces to the problem studied by Feng and Ge in [11]. By using the fixed point theorem in a cone, the authors obtained some sufficient conditions for the existence, nonexistence and multiplicity of positive solutions for problem (1.2) when $\alpha(t) \equiv t$ on J. However, Feng and Ge did not obtain any results of triple solutions on problem (1.2). This paper will resolve this problem.

In this paper, we present several new and more general results for the existence of triple positive solutions for problem (1.2) by using Leggett-Williams' fixed point theorem. Another contribution of this paper is to study the expression and properties of Green's function associated with problem (1.2). The expression of the integral equation is simpler than that of [11].

The organization of this paper is as follows. In Section 2, we present the expression and properties of Green's function associated with problem (1.2). In Section 3, we present some definitions and lemmas which are useful to obtain our main results. In Section 4, we formulate sufficient conditions under which delayed problem (1.2) has at least three positive solutions. In Section 5, we provide an example to illustrate our main results.

2 Expression and properties of Green's function

Lemma 2.1 *Assume that* $\sum_{i=1}^{m-2} \beta_i\phi(\xi_i) \neq 1$. *Then, for any* $y \in C[0,1]$, *the boundary value problem*

$$\begin{cases} Lx = y(t), & 0 < t < 1, \\ x'(0) = 0, & x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) \end{cases} \tag{2.1}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s)q(s)y(s)\,ds, \tag{2.2}$$

where

$$\begin{aligned} &q(t) = \exp\left(\int_0^t a(s)\,ds\right), \qquad H(t,s) = G(t,s) + G_1(t,s), \\ &G(t,s) = \frac{1}{\Delta}\begin{cases} \phi(s)\psi(t), & \text{if } 0 \le s \le t \le 1, \\ \phi(t)\psi(s), & \text{if } 0 \le t \le s \le 1, \end{cases} \\ &G_1(t,s) = \frac{\phi(t)\sum_{i=1}^{m-2} \beta_i G(\xi_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i\phi(\xi_i)}, \\ &\Delta := -\phi(0)\psi'(0), \end{aligned} \tag{2.3}$$

here ϕ *and* ψ *satisfy* (1.3) *and*

$$L\psi = 0, \qquad \psi(0) = 1, \qquad \psi(1) = 0, \tag{2.4}$$

respectively.

Proof First suppose that x is a solution of problem (2.1). Similar to the proof of Lemma 2.3 in [10], we can get

$$x(t) = \int_0^1 G(t,s)q(s)y(s)\,ds + A\phi(t),$$

where

$$A = \frac{\sum_{i=1}^{m-2} \beta_i \int_0^1 G(\xi_i, s)q(s)y(s)\,ds}{1 - \sum_{i=1}^{m-2} \beta_i \phi(\xi_i)}.$$

So

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)q(s)y(s)\,ds + A\phi(t) \\ &= \int_0^1 G(t,s)q(s)y(s)\,ds + \phi(t)\int_0^1 \frac{\sum_{i=1}^{m-2} \beta_i G(\xi_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \phi(\xi_i)} q(s)y(s)\,ds \\ &= \int_0^1 \left[G(t,s) + \phi(t)\frac{\sum_{i=1}^{m-2} \beta_i G(\xi_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \phi(\xi_i)} \right] q(s)y(s)\,ds \\ &= \int_0^1 \left[G(t,s) + G_1(t,s) \right] q(s)y(s)\,ds \\ &= \int_0^1 H(t,s)q(s)y(s)\,ds. \end{aligned}$$

Then the proof is completed. □

Remark 2.1 The proof of Lemma 2.1 is supplementary of Theorem 3 in [28], which helps the readers to understand that (2.2) holds.

Remark 2.2 The expression of the integral equation (2.2) is different from that of (2.10) in [10] and that of (2.9) in [11], which shows that we can use a completely different technique from that of [10] and [11] to study problem (1.2).

Remark 2.3 It is not difficult from [10, 11] to show that $\Delta > 0$ and that (i) ϕ is nondecreasing on J and $\phi > 0$ on J; (ii) ψ is strictly decreasing on J.

Remark 2.4 Noticing $a(t) \in C([0,1], [0, +\infty))$, it follows from the definition of $q(t)$ that

$$1 \le q(t) \le e^M \quad \text{for } t \in J, \tag{2.5}$$

where

$$M = \max_{t\in J} a(t).$$

Lemma 2.2 (See [28]) *Let* $\xi \in (0,1)$, $G(t,s)$, $G_1(t,s)$ *and* $H(t,s)$ *be given as in Lemma* 2.1. *Then we have the following results*:

$$G(t,s) \ge 0, \qquad G_1(t,s) \ge 0, \qquad H(t,s) \ge 0, \quad \forall t,s \in J, \tag{2.6}$$

$$G(t,s) \le G(s,s), \qquad G_1(t,s) \le G_1(1,s), \qquad H(t,s) \le H(s) \le H^0, \quad \forall t,s \in J, \tag{2.7}$$

$$\begin{aligned} &G(t,s) \ge \sigma G(s,s), \qquad G_1(t,s) \ge \phi(0)G_1(1,s), \\ &H(t,s) \ge \sigma H(s) \ge \sigma H_0, \quad \forall t \in [0,\xi], s \in J, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &H(s) = G(s,s) + G_1(1,s), \qquad H^0 = \max_{s\in J} H(s), \\ &H_0 = \min_{s\in J} H(s), \qquad \sigma = \min\{\psi(\xi), \phi(0)\}. \end{aligned} \tag{2.9}$$

Remark 2.5 From (2.8) it follows that

$$H(t,s) > \frac{1}{2}\sigma H_0.$$

Remark 2.6 By (1.3), (2.4) and the definition of σ, we obtain

$$0 < \sigma < 1.$$

3 Preliminaries

In this section, we provide some background material from the theory of cones in Banach spaces, and then we state Hölder's inequality and Legget-Williams' fixed point theorem. The following definitions can be found in the book by Deimling [29] as well as in the book by Guo and Lakshmikantham [30].

Definition 3.1 Let E be a real Banach space over R. A nonempty closed set $K \subset E$ is said to be a cone provided that the following two conditions are satisfied:

(i) $au + bv \in K$ for all $u, v \in K$ and all $a \geq 0$, $b \geq 0$;

(ii) $u, -u \in K$ implies $u = 0$.

Note that every cone $K \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in K$.

Definition 3.2 A map Λ is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E if $\Lambda : K \to R_+$ is continuous and

$$\Lambda\big(tx + (1-t)y\big) \geq t\Lambda(x) + (1-t)\Lambda(y)$$

for all $x, y \in K$ and $t \in J$.

Definition 3.3 An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Lemma 3.1 (Arzelà-Ascoli) *A set $M \subset C(J,R)$ is said to be a pre-compact set provided that the following two conditions are satisfied*:

(i) *All the functions in the set M are uniformly bounded. It means that there exists a constant $r > 0$ such that $|u(t)| \leq r$, $\forall t \in J$, $u \in M$*;

(ii) *All the functions in the set M are equicontinuous. It means that for every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, which is independent of the function u, such that*

$$\big|u(t_1) - u(t_2)\big| < \varepsilon$$

whenever $|t_1 - t_2| < \delta$, $t_1, t_2 \in J$.

Lemma 3.2 (Hölder) *Let $u \in L^p[a,b]$ and $v \in L^q[a,b]$, where $p,q \in (0,+\infty)$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $uv \in L^1[a,b]$ and*

$$\|uv\|_1 \le \|u\|_p \|v\|_q.$$

Let $u \in L^1[a,b]$, $v \in L^\infty[a,b]$. Then $uv \in L^1[a,b]$ and

$$\|uv\|_1 \le \|u\|_1 \|v\|_\infty.$$

The basic space used in this paper is $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by

$$\|x\| = \max_{t\in J} \bigl|x(t)\bigr|.$$

Define a cone K in E by

$$K = \Bigl\{ x \in E : x(t) \ge 0, \min_{t\in[0,\xi]} x(t) \ge \sigma \|x\| \Bigr\}. \tag{3.1}$$

Define an operator $T : K \to K$ by

$$(Tx)(t) = \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds. \tag{3.2}$$

Lemma 3.3 *Assume that* (H_1)-(H_3) *hold. Then $T(K) \subset K$ and $T : K \to K$ is completely continuous.*

Proof For $x \in K$, it follows from (2.7) and (3.2) that

$$\begin{aligned} \|Tx\| &= \max_{t\in J} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\le \int_0^1 H(s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds. \end{aligned} \tag{3.3}$$

It follows from (2.8), (3.2) and (3.3) that

$$\begin{aligned} \min_{t\in[0,\xi]} (Tx)(t) &= \min_{t\in[0,\xi]} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\ge \sigma \int_0^1 H(s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\ge \sigma \|Tx\|. \end{aligned}$$

Thus, $T(K) \subset K$.

Next we shall show that operator T is completely continuous. We break the proof into several steps.

Step 1. Operator T is continuous. Since the function $f(t,x)$ is continuous on $J \times [0,+\infty)$, this conclusion can be easily obtained.

Step 2. For each constant $l > 0$, let $B_l = \{x \in K : \|x\| \leq l\}$. Then B_l is a bounded closed convex set in K. $\forall x \in B_l$, from (3.2), we have

$$\begin{aligned}
\|Tx\| &= \max_{t\in J} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leq \int_0^1 H(s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leq e^M \int_0^1 H(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leq e^M \|H\|_q \|\omega\|_p L,
\end{aligned}$$

where $L = \sup_{t\in J,\|x\|\leq l} f(s,x(\alpha(s)))$. This proves that $T(B_r)$ is uniformly bounded.

Step 3. The family $\{Tx : x \in B_l\}$ is a family of equicontinuous functions. Since $H(t,s)$ is continuous on $J \times J$, and noticing $J = [0,1]$, $H(t,s)$ is uniformly continuous on $J \times J$. Therefore, for all $\varepsilon > 0$, there exists $l > 0$, when $|t_1 - t_2| < l$, such that

$$\bigl|H(t_1,s) - H(t_2,s)\bigr| < \frac{\varepsilon}{e^M \|\omega\|_1 L}.$$

Then, for all $x \in B_l$, when $|t_1 - t_2| < \delta$, we get

$$\begin{aligned}
\bigl|Tx(t_1) - Tx(t_2)\bigr| &= \biggl|\int_0^1 H(t_1,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\quad - \int_0^1 H(t_2,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds\biggr| \\
&= \biggl|\int_0^1 \bigl[H(t_1,s) - H(t_2,s)\bigr]q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds\biggr| \\
&\leq e^M \|\omega\|_1 L \int_0^1 \bigl|H(t_1,s) - H(t_2,s)\bigr|\,ds \\
&< \varepsilon.
\end{aligned}$$

Thus, the set $\{Tx : x \in B_l\}$ is equicontinuous.

As a consequence of Step 1 to Step 3 together with Lemma 3.1, we can prove that $T : K \to K$ is completely continuous. □

Remark 3.1 From Lemma 2.1 and (3.1), we know that $x \in E$ is a solution of problem (1.2) if and only if x is a fixed point of operator T.

Let $0 < r_1 < r_2$ be given and let β be a nonnegative continuous concave functional on the cone K. Define the convex sets K_{r_1}, $K(\beta,r_1,r_2)$ by

$$\begin{aligned}
K_{r_1} &= \bigl\{x \in K : \|x\| < r_1\bigr\}, \\
K(\beta,r_1,r_2) &= \bigl\{x \in K : r_1 \leq \beta(x), \|x\| \leq r_2\bigr\}.
\end{aligned}$$

Finally we state Leggett-Williams' fixed point theorem [31].

Lemma 3.4 *Let K be a cone in a real Banach space E, $A:\bar{K}_c \to \bar{K}_c$ be completely continuous and β be a nonnegative continuous concave functional on K with $\beta(x) \le \|x\|$ $(\forall x \in \bar{K}_c)$. Suppose that there exist $0 < d < a < b \le c$ such that*

(i) $\{x \in K(\beta, a, b) : \beta(x) > a\} \neq \emptyset$ *and* $\beta(Ax) > a$ *for* $x \in K(\beta, a, b)$;
(ii) $\|Ax\| < d$ *for* $\|x\| \le d$;
(iii) $\beta(Ax) > a$ *for* $x \in K(\beta, a, c)$ *with* $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\|x_1\| < d, \qquad a < \beta(x_2), \qquad \|x_3\| > d \quad \text{and} \quad \beta(x_3) < a.$$

4 Existence of triple positive solutions

In this section, we apply Lemma 3.2 and Lemma 3.4 to establish the existence of three positive solutions for problem (1.2). We consider the following three cases for $\omega \in L^p[0,1]$: $p > 1$, $p = 1$, and $p = \infty$. Case $p > 1$ is treated in the following theorem.

For convenience, we write

$$\Gamma = e^M \|H\|_q \|\omega\|_p, \qquad \Psi = \frac{1}{2} n\sigma H_0.$$

Let the nonnegative continuous concave functional Λ on the cone K be defined by

$$\Lambda(x) = \min_{[0,\xi]} |x(t)|.$$

Note that for $x \in K$, $\Lambda(x) \le \|x\|$.

Theorem 4.1 *Assume that* (H_1)-(H_3) *hold. In addition, there exist constants* $0 < d < l < \frac{l}{\sigma} \le c$ *such that*

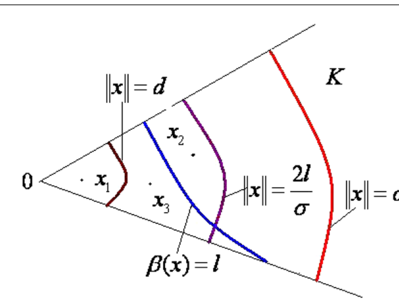
(H_4) $f(t,x) \le \frac{c}{\Gamma}$ *for* $(t,x) \in J \times [0,c]$;
(H_5) $f(t,x) \ge \frac{l}{\Psi}$ *for* $(t,x) \in [0,\xi] \times [l, \frac{l}{\sigma}]$;
(H_6) $f(t,x) \le \frac{d}{\Gamma}$ *for* $(t,x) \in J \times \in [0,d]$.

Then problem (1.2) *has at least three positive solutions x_1, x_2 and x_3 satisfying*

$$\|x_1\| < d, \qquad l < \Lambda(x_2), \qquad d < \|x_3\| \quad \text{and} \quad \Lambda(x_3) < l. \tag{4.1}$$

For details, see Figure 1.

Figure 1 The relationship of x_1, x_2 and x_3.



Proof By the definition of operator T and its properties, it suffices to show that the conditions of Lemma 3.4 hold with respect to T.

Let $x \in \bar{P}_c$. Then $0 \le x(t) \le c$ on J. Since $0 \le \alpha(t) \le t \le 1$ on J, it follows from $0 \le x(t) \le c$ on J that $0 \le x(\alpha(t)) \le c$ on J.

Consequently, for $t \in J$ and $x \in P_c$, it follows from (H_4), (2.8) and (3.2) that

$$\begin{aligned}
\|Tx\| &= \max_{t\in J} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le \int_0^1 H(s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M \int_0^1 H(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M \|H\|_q\|\omega\|_p \int_0^1 f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M \|H\|_q\|\omega\|_p \frac{c}{\Gamma} \\
&\le c,
\end{aligned} \tag{4.2}$$

which implies $Tx \in P_c$. This proves that $T : \bar{P}_c \to \bar{P}_c$ is completely continuous.

We first show that the condition (i) of Lemma 3.4 holds.

Take $x(t) = \frac{1}{2}(l + \frac{l}{\sigma})$, $\forall t \in J$. Then

$$\|x\| = \frac{1}{2}\left(l + \frac{l}{\sigma}\right) < \frac{l}{\sigma}, \qquad \Lambda(x) = \min_{t\in[0,\xi]} x(t) = \frac{1}{2}\left(l + \frac{l}{\sigma}\right) > l.$$

This shows that

$$\left\{x \in K\left(\Lambda, l, \frac{l}{\sigma}\right) : \Lambda(x) > l\right\} \neq \emptyset.$$

Therefore, for all $\{x \in K(\Lambda, l, \frac{l}{\sigma}) : \Lambda(x) > l\}$ and $t \in J$, we have

$$l \le x(t) \le \frac{l}{\sigma}.$$

Since $0 \le \alpha(t) \le t \le \xi$ on $[0,\xi]$, it follows from $l \le x(t) \le \frac{l}{\sigma}$ on $[0,\xi]$ that $l \le x(\alpha(t)) \le \frac{l}{\sigma}$ for $t \in [0,\xi]$.

Therefore, it follows from Remark 2.1 and (H_5) that

$$\begin{aligned}
\Lambda(Tx) &= \min_{t\in[0,\xi]} \bigl|(Tx)(t)\bigr| \\
&\ge \sigma \int_0^1 H(s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\ge n\sigma \int_0^1 H(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\ge n\sigma \int_0^\xi H(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds
\end{aligned}$$

$$> \frac{1}{2} n\sigma H_0 \frac{l}{\Psi}$$

$$= l. \tag{4.3}$$

Therefore, we have

$$\Lambda(Tx) > l, \quad \forall x \in K\left(\Lambda, l, \frac{l}{\sigma}\right).$$

This implies that condition (i) of Lemma 3.4 is satisfied.

Secondly, we prove that condition (ii) of Lemma 3.4 is satisfied. If $x \in K_d$, then $0 \le x(t) \le d$ on J.

Since $0 \le \alpha(t) \le t \le 1$ on J, it follows from $0 \le x(t) \le d$ on J that $0 \le x(\alpha(t)) \le d$ on J.

Thus it follows from (H_6) that

$$\begin{aligned}
\|Tx\| &= \max_{t\in J} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le \int_0^1 H(s)q(s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M \int_0^1 H(s)\omega(s)\,ds\bigl|f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\bigr| \\
&\le e^M \|H\|_q \|\omega\|_p \frac{d}{\Gamma} \\
&= d.
\end{aligned} \tag{4.4}$$

Hence, the condition (ii) of Lemma 3.4 is satisfied.

Finally, we prove that the condition (iii) of Lemma 3.4 is satisfied.

In fact, for all $x \in K(\Lambda, l, c)$ and $\|Tx\| > \frac{l}{\sigma}$, it follows from (2.5), (2.9), (3.2) and (3.3) that

$$\begin{aligned}
\Lambda(Tx) &= \min_{t\in[0,\xi]} \bigl|(Tx)(t)\bigr| \\
&= \min_{t\in[0,\xi]} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\ge \sigma \int_0^1 H(s)q(s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\ge \sigma \|Tx\| \\
&> \sigma \frac{l}{\sigma} \\
&= l.
\end{aligned} \tag{4.5}$$

This gives the proof of the condition (iii) of Lemma 3.4.

To sum up, the hypotheses of Lemma 3.4 hold. Therefore, an application of Lemma 3.4 implies that problem (1.2) has at least three positive solutions x_1, x_2 and x_3 such that

$$\|x_1\| < d, \qquad l < \Lambda(x_2), \quad \text{and} \quad x_3 > d \quad \text{with } \Lambda(x_3) < l. \qquad \square$$

The following corollary deals with the case $p = \infty$.

Corollary 4.1 *Assume that* (H_1)-(H_6) *hold. Then problem* (1.2) *has at least three positive solutions* x_1, x_2 *and* x_3 *satisfying* (4.1).

Proof Let $\|H\|_1\|\omega\|_\infty$ replace $\|H\|_q\|\omega\|_p$ and repeat the argument above, we can get the corollary. □

Finally we consider the case of $p = 1$. Let

$(\mathrm{H}_4)'$ $f(t,x) \le \frac{c}{\Gamma_1}$ for $(t,x) \in J \times [0,c]$;
$(\mathrm{H}_6)'$ $f(t,x) \le \frac{d}{\Gamma_1}$ for $(t,x) \in J \times \in [0,d]$,

where

$$\Gamma_1 = e^M H^0 \|\omega\|_1.$$

Corollary 4.2 *Assume that* (H_1)-(H_3), $(\mathrm{H}_4)'$, (H_5) *and* $(\mathrm{H}_6)'$ *hold. Then problem* (1.2) *has at least three positive solutions* x_1, x_2 *and* x_3 *satisfying* (4.1).

Proof Similar to the proof of (4.1), it follows from (2.3) and $(\mathrm{H}_4)'$ that

$$\begin{aligned}
\|Tx\| &= \max_{t\in J} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le \int_0^1 H(s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M \int_0^1 H(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M H^0 \int_0^1 \omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M H^0 \|\omega\|_1 \int_0^1 f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M H^0 \|\omega\|_1 \frac{c}{\Gamma_1} \\
&\le c,
\end{aligned} \tag{4.6}$$

which shows that $Tx \in \bar{K}_c$, $\forall x \in \bar{K}_c$.

Next turning to $(\mathrm{H}_6)'$, we have

$$\begin{aligned}
\|Tx\| &= \max_{t\in J} \int_0^1 H(t,s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le \int_0^1 H(s)q(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M \int_0^1 H(s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M H^0 \int_0^1 \omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\le e^M H^0 \|\omega\|_1 \int_0^1 f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds
\end{aligned}$$

$$\begin{aligned} &\le e^M H^0 \|\omega\|_1 \frac{d}{\Gamma_1} \\ &\le c, \quad \forall x \in \bar{K}_d. \end{aligned} \tag{4.7}$$

Similar to the proof of Theorem 4.1, we can get Corollary 4.2. □

Remark 4.1 Comparing with Feng and Ge [11], the main features of this paper are as follows.

(i) Three positive solutions are available.
(ii) $\alpha(t) \not\equiv t$ is considered throughout this paper.
(iii) $\omega(t)$ is L^p-integrable, not only $\omega(t) \in C(0,1)$ on $t \in J$.

5 An example

In this section, we present an example. Let

$$p = 2, \qquad \beta_1 = e^{-\frac{1}{2}}, \qquad \xi_1 = \frac{1}{2}, \qquad a(t) \equiv 0, \qquad b(t) \equiv 1.$$

Example 5.1 Consider the following three-point boundary value problem:

$$\begin{cases} -x''(t) + x(t) = \omega(t) f(t, x(\alpha(t))), & 0 < t < 1, \\ x'(0) = 0, \quad x(1) = e^{-\frac{1}{2}} x(\frac{1}{2}), \end{cases} \tag{5.1}$$

where $\alpha \in C(J,J)$, $\alpha(t) \le t$ on J and

$$\omega(t) = \frac{1}{|t - \frac{1}{2}|^{\frac{1}{3}}},$$

$$f(t,x) = \begin{cases} \frac{d}{\Gamma}, & t \in J, x \in [0,d], \\ \frac{d}{\Gamma} \times \frac{l-x}{l-d} + \frac{l}{\Psi} \times \frac{x-d}{l-d}, & t \in J, x \in [d,l], \\ \frac{l}{\Psi}, & t \in J, x \in [l, \frac{l}{\sigma}], \\ \frac{l}{\Psi} \times \frac{c-x}{c-\frac{l}{\sigma}} + \frac{c}{\Gamma} \times \frac{x-\frac{l}{\sigma}}{c-\frac{l}{\sigma}}, & t \in J, x \in [\frac{l}{\sigma}, c], \\ \frac{c}{\Gamma}, & t \in J, x \in [c, +\infty). \end{cases} \tag{5.2}$$

This means that problem (5.1) involves the delayed argument α. For example, we can take $\alpha(t) = t^3$. It is clear that ω is nonnegative and $\omega \in L^2[0,1]$.

Conclusion 5.1 Problem (5.1) has at least three positive solutions x_1, x_2 and x_3 satisfying (4.1).

Proof It follows from (1.3) and (2.4) that ϕ and ψ satisfy

$$\begin{aligned} &L\phi = 0, \qquad \phi'(0) = 0, \qquad \phi(1) = 1, \\ &L\psi = 0, \qquad \psi(0) = 1, \qquad \phi(1) = 0, \end{aligned} \tag{5.3}$$

where $Lx = -x''(t) + x(t)$ and

$$\phi(t) = \frac{e^{1-t} + e^{1+t}}{1 + e^2}, \qquad \phi(0) = \frac{2e}{1 + e^2},$$

$$\psi(t) = \frac{-e^{2-t} + e^t}{1-e^2}, \qquad \psi'(0) = \frac{1+e^2}{1-e^2}, \tag{5.4}$$

$$q(t) = 1, \qquad \Delta := -\phi(0)\psi'(0) = \frac{2e}{e^2-1} > 0, \qquad \beta_1\phi(\xi_1) = e^{-\frac{1}{2}}\phi\left(\frac{1}{2}\right) < 1.$$

On the other hand, it follows from $a(t) = 0$, $m = 3$, $\alpha_1 = e^{-\frac{1}{2}}$ and $\omega(t) = \frac{1}{|t-\frac{1}{2}|^{\frac{1}{3}}}$ that

$$e^M = 1, \qquad n = \sqrt[3]{2}, \qquad \|\omega\|_2 = \left[\int_0^1 \left(t - \frac{1}{2}\right)^{-\frac{2}{3}} dt\right]^{\frac{1}{2}} = \sqrt{\frac{6}{\sqrt[3]{2}}}.$$

Choosing $0 < d < l < \frac{l}{\sigma} \le c$, we have

$$f(t,x) \le \frac{c}{\Gamma} \quad \text{for } (t,x) \in J \times [0,c];$$

$$f(t,x) \ge \frac{l}{\Psi} \quad \text{for } (t,x) \in \left[0, \frac{1}{2}\right] \times \left[l, \frac{2l}{\sigma}\right];$$

$$f(t,x) = \frac{d}{\Gamma} \quad \text{for } (t,x) \in J \times [0,d],$$

which shows that (H_4)-(H_6) hold.

By Theorem 4.1, problem (5.1) has least three positive solutions x_1, x_2 and x_3 satisfying (4.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JZ checked the proofs and verified the calculation. MF completed the main study and carried out the results of this article. All the authors read and approved the final manuscript.

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