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Existence criterion for the solutions of fractional order p-Laplacian boundary value problems

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Abstract

The existence criterion has been extensively studied for different classes in fractional differential equations (FDEs) through different mathematical methods. The class of fractional order boundary value problems (FOBVPs) with p-Laplacian operator is one of the most popular class of the FDEs which have been recently considered by many scientists as regards the existence and uniqueness. In this scientific work our focus is on the existence and uniqueness of the FOBVP with p-Laplacian operator of the form: $D^{\gamma}(\phi_p(D^{\theta}z(t)))+a(t)f(z(t))=0$, $3<\theta,\gamma\leq 4$, $t\in[0,1]$, $z(0)=z'''(0)$, $\eta D^{\alpha}z(t)|_{t=1}=z'(0)$, $\xi z''(1)-z''(0)=0$, $0<\alpha<1$, $\phi_p(D^{\theta}z(t))|_{t=0}=0=(\phi_p(D^{\theta}z(t)))'|_{t=0}$, $(\phi_p(D^{\theta}z(t)))''|_{t=1}=\frac{1}{2}(\phi_p(D^{\theta}z(t)))''|_{t=0}$, $(\phi_p(D^{\theta}z(t)))'''|_{t=0}=0$, where $0<\xi,\eta<1$ and D^{θ}, D^{γ}, D^{α} are Caputo's fractional derivatives of orders θ,γ,α, respectively. For this purpose, we apply Schauder's fixed point theorem and the results are checked by illustrative examples.

Keywords: FOBVP with p-Laplacian operator; fixed point theorems; existence and uniqueness

1 Introduction

Fractional calculus has widely been studied by scientists from the era of Leibniz to the present and has drawn the attention of mathematicians, engineers, and physicists in many scientific disciplines based on mathematical modeling, and it was found that the fractional order models are more precise in comparison with integer order models and, therefore, we can see many useful fractional order models in fluid flow, viscoelasticity, signal processing, and many other fields. For instance, see [1–6].

In fractional calculus, the existence of a positive solution for FOBVP with p-Laplacian operator has extensively attracted the attention of the scientific community. This side of the fractional calculus has a wide range of applications in day life problems and these were investigated for the existence of solutions by different mathematical tools. For instance, Lv [7], has studied existence results for m-point FOBVP with p-Laplacian operator with the help of a monotone iterative technique and produced interesting results which were examined by two examples. Prasad and Krushna [8] studied FOBVPs with p-Laplacian operator with the help of the Krasnosel'skii and five functional fixed point theorems and checked their results by examples. Yuan and Yang [9] studied the existence of a positive solution for a q-FOBVP with a p-Laplacian operator using the upper and lower solution method through Schauder's fixed point theorem and the results were examined by examples.

Some of the interesting results in the existence of positive solution for FOBVP with p-Laplacian operator which raised our attention toward this project are, for instance, that Zhang *et al.* [10] has contributed for positive solutions of an FOBVP with p-Laplacian

$$\begin{aligned}&(\phi_p(z'))' = k(\tau, z, z'), \quad \tau \in [0, T],\\ &z(T) = z(0), \qquad z'(T) = z'(0),\end{aligned} \tag{1}$$

with the help of degree theory, and they also used the upper and lower solution method for this work. Xu and Xu in [11] investigated the p-Laplacian equation for sign changing equations of the form

$$\begin{aligned}&(\phi_p(z'(\tau)))' = -f(\tau, z, z'), \quad \tau \in (0,1),\\ &z(0) = 0 = z(1),\end{aligned} \tag{2}$$

using the method of upper and lower solution method with the help of Leray-Schauder degree theory. Wang in [12] considered three solutions of

$$\begin{aligned}&(\phi_p(z'(\tau)))' + m(\tau)f(\tau, z, z') = 0, \quad 0 < \tau < \infty,\\ &z(0) = 0, \qquad \lim_{\tau \to +\infty} z = 0.\end{aligned} \tag{3}$$

In this paper we consider the FOBVP with p-Laplacian of the form

$$\begin{aligned}&D^\gamma(\phi_p(D^\theta z(t))) + a(t)f(z(t)) = 0, \quad 3 < \theta, \gamma \le 4, t \in [0,1],\\ &z(0) = z'''(0), \qquad \eta D^\alpha z(t)|_{t=1} = z'(0), \qquad \xi z''(1) - z''(0) = 0, \quad 0 < \alpha < 1,\\ &\phi_p(D^\theta z(t))|_{t=0} = 0 = (\phi_p(D^\theta z(t)))'|_{t=0},\\ &(\phi_p(D^\theta z(t)))''|_{t=1} = \frac{1}{2}(\phi_p(D^\theta z(t)))''|_{t=0},\\ &(\phi_p(D^\theta z(t)))'''|_{t=0} = 0,\end{aligned} \tag{4}$$

where D^θ, D^γ, and D^α are Caputo's fractional derivatives of fractional order θ, γ, α where $\theta, \gamma \in (3,4]$, and $\alpha \in (0,1)$. $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. We recall some basic definitions and results. For $\alpha > 0$, choose $n = [\alpha] + 1$ in the case α is not an integer and $n = \alpha$ in the case α is an integer. We recall the following definitions of a fractional order integral and a fractional order derivative in Caputo's sense, and some basic results of fractional calculus [2, 13].

Definition 1 [13] For a function $k : (0, \infty) \to R$ and $\gamma > 0$, fractional integral of order γ is defined by

$$I^\gamma k(\tau) = \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau - s)^{\gamma - 1} k(s)\, ds, \tag{5}$$

with the condition that the integral converges.

Definition 2 [13] For $\gamma > 0$ the left Caputo fractional derivative of order γ is defined by

$$\big(D^{\gamma}\big)h\big(z(t)\big) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-x)^{n-\gamma-1}f^{(n)}(x)\,dx, \tag{6}$$

where n is such that $n-1 < \gamma < n$.

Lemma 3 [2] *For $\mu, \beta > 0$, the following relation holds*: $D^{\mu}t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\mu)}t^{\nu-\mu-1}$, $\nu > n$ *and* $D^{\mu}t^{l} = 0$, *for* $l = 0,1,2,\ldots,n-1$.

Lemma 4 *For $\mathcal{H}(t) \in C(0,1)$, the solution of the homogeneous FDE $D^{\omega}_{0^+}\mathcal{H}(t) = 0$ is*

$$\mathcal{H}(t) = k_1 + k_2 t + k_3 t^2 + \cdots + k_n t^{n-1}, \quad k_i \in R, i = 1,2,3,\ldots,n. \tag{7}$$

Definition 5 [14] A cone P is solid in a real Banach space X if its interior is non-empty.

Definition 6 [14] Let P be a solid cone in a real Banach space X, $T : P^0 \to P^0$ be an operator and $0 < \alpha < 1$. Then T is called θ-concave operator if $T(ku) \geq k^{\alpha}T(u)$ for any $0 < k < 1$ and $u \in P^0$.

Lemma 7 [14] *Assume that P is a normal solid cone in a real Banach space X, $0 < \alpha < 1$, and $T : P^0 \to P^0$ is a α-concave increasing operator. Then T has only one fixed point in P^0.*

2 Main results

Lemma 8 *For $z(t) \in C[0,1]$, the FOBVP*

$$D^{\theta}z(t) = h(t), \quad 3 < \theta \leq 4, \tag{8}$$

$$z(0) = z'''(0), \qquad \eta D^{\alpha}z(1) = z'(0), \qquad \xi z''(1) - z''(0) = 0, \quad 0 < \alpha < 1, \tag{9}$$

has a solution of the form

$$z(t) = \int_0^1 G(t,s)h(t)\,ds, \tag{10}$$

where

$$G(t,s) = \begin{cases} \frac{t\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\big[\frac{(1-s)^{\theta-\alpha-1}}{\Gamma(\theta-\alpha)} + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)}\frac{(1-s)^{\theta-3}}{\Gamma(\theta-2)}\big] \\ \quad + \frac{t^2\xi}{2(1-\xi)}\frac{(1-s)^{\theta-3}}{\Gamma(\theta-2)} + \frac{1}{\Gamma(\theta)}(t-s)^{\theta-1}, & 0 < s \leq t < 1, \\ \frac{t\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\big[\frac{(1-s)^{\theta-\alpha-1}}{\Gamma(\theta-\alpha)} + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)}\frac{(1-s)^{\theta-3}}{\Gamma(\theta-2)}\big] \\ \quad + \frac{t^2\xi}{2(1-\xi)}\frac{(1-s)^{\theta-3}}{\Gamma(\theta-2)}, & 0 < t \leq s < 1. \end{cases} \tag{11}$$

Proof Applying the operator I^{α} on the differential equation in (4) and using Lemma 4, we obtain

$$z(t) = I^{\theta}h(t) + c_1 + c_2 t + c_3 t^2 + c_4 t^3. \tag{12}$$

Using boundary conditions (9) for the values of c_1, c_2, c_3, c_4, $z(0) = 0 = z'''(0)$ yield to $c_1 = 0 = c_4$, and by $\xi z''(1) - z''(0) = 0$, we have $c_3 = \frac{\xi}{2(1-\xi)} I^{\theta-2}h(1)$. Using $\eta D^\alpha z(1) = z'(0)$, we obtain $c_2 = \frac{\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}[I^{\theta-\alpha}h(1) + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)} I^{\theta-2}h(1)]$. Substituting the values of c_1, c_2, c_3, c_4, in (12), we get

$$z(t) = I^\theta h(t) + \frac{t\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\left[I^{\theta-\alpha}h(1) + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)} I^{\theta-2}h(1)\right] + \frac{t^2\xi}{2(1-\xi)} I^{\theta-2}h(1). \tag{13}$$

The integral form is given as

$$\begin{aligned} z(t) &= \int_0^t \frac{(t-s)^{\theta-1}h(s)\,ds}{\Gamma(\theta)} + \frac{t^2\xi}{2(1-\xi)} \int_0^1 \frac{(1-s)^{\theta-3}h(s)\,ds}{\Gamma(\theta-2)} \\ &\quad + \frac{t\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\left[\int_0^1 \frac{(1-s)^{\theta-\alpha-1}h(s)\,ds}{\Gamma(\theta-\alpha)}\right. \\ &\quad \left. + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)} \int_0^1 \frac{(1-s)^{\theta-3}h(s)\,ds}{\Gamma(\theta-2)}\right] = \int_0^1 G(t,s)h(s)\,ds. \end{aligned} \tag{14}$$

□

Lemma 9 *Let* $3 < \theta, \gamma \leq 4$. *For* $z(t) \in C[0,1]$, *the FOBVP with p-Laplacian*

$$\begin{aligned} &D^\gamma\left(\phi_p\left(D^\theta z(t)\right)\right) + a(t)f\left(z(t)\right) = 0, \quad 3 < \theta, \gamma \leq 4, t \in [0,1], \\ &z(0) = z'''(0), \qquad \eta D^\alpha z(t)|_{t=1} = z'(0), \qquad \xi z''(1) - z''(0) = 0, \quad 0 < \alpha < 1, \\ &\phi_p\left(D^\theta z(t)\right)|_{t=0} = 0 = \left(\phi_p\left(D^\theta z(t)\right)\right)'|_{t=0}, \\ &\left(\phi_p\left(D^\theta z(t)\right)\right)''|_{t=1} = \frac{1}{2}\left(\phi_p\left(D^\theta z(t)\right)\right)''|_{t=0}, \\ &\left(\phi_p\left(D^\theta z(t)\right)\right)'''|_{t=0} = 0, \end{aligned} \tag{15}$$

has a solution of the form

$$z(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 \mathcal{H}(s,x)a(x)f\left(z(x)\right)dx\right)ds, \tag{16}$$

where

$$\mathcal{H}(t,x) = \begin{cases} -\frac{(t-x)^{\gamma-1}}{\Gamma\gamma} + \frac{t^2}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}, & 0 < x \leq t < 1, \\ \frac{t^2}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}, & 0 < t \leq x < 1, \end{cases} \tag{17}$$

and $G(t,s)$ *is given by* (11).

Applying the integral I^γ on the differential equation in (15) and using Lemma 9, we obtain

$$\phi_p\left(D^\theta z(t)\right) = -I^\gamma a(t)f\left(z(t)\right) + c_1 + c_2 t + c_3 t^2 + c_4 t^3. \tag{18}$$

The boundary conditions $\phi_p(D^\theta z(0)) = (\phi_p(D^\theta z(0)))' = (\phi_p(D^\alpha z(0)))''' = 0$ lead to $c_1 = c_2 = c_4 = 0$. From (18) and $c_1 = c_2 = c_4 = 0$, we deduce

$$\bigl(\phi_p\bigl(D^\theta z(t)\bigr)\bigr)'' = -I^{\gamma-2}a(t)f\bigl(z(t)\bigr) + 2c_3, \tag{19}$$

and the boundary condition $(\phi_p(D^\theta z(1)))'' = \frac{1}{2}(\phi_p(D^\theta z(0)))''$ yield $c_3 = \frac{1}{\Gamma(\gamma-2)}\int_0^1(1-x)^{\gamma-3} \times a(x)f(z(x))\,dx$. Consequently, (18) takes the form

$$\begin{aligned}\phi_p\bigl(D^\theta z(t)\bigr) &= -\frac{1}{\Gamma(\gamma)}\int_0^1 (t-x)^{\gamma-1}a(x)f\bigl(z(x)\bigr)\,dx \\ &\quad + \frac{t^2}{\Gamma(\gamma-2)}\int_0^1 (1-x)^{\gamma-3}a(x)f\bigl(z(x)\bigr)\,dx \\ &= \int_0^1 \mathcal{H}(t,x)a(x)f\bigl(z(x)\bigr)\,dx.\end{aligned} \tag{20}$$

The boundary value problem (15) reduces to the following problem:

$$\begin{aligned}&D^\alpha z(t) = \phi_q\biggl(\int_0^1 \mathcal{H}(t,x)a(x)f\bigl(z(x)\bigr)\,dx\biggr), \\ &z(0) = z'''(0), \qquad \eta D^\alpha z(1) = y'(0), \qquad \xi z''(1) - z''(0) = 0, \quad 0 < \alpha < 1,\end{aligned} \tag{21}$$

which, in view of Lemma 8, yields the required result,

$$z(t) = \int_0^1 G(t,s)\phi_q\biggl(\int_0^1 \mathcal{H}(s,x)a(x)f\bigl(z(x)\bigr)\,dx\biggr)\,ds. \tag{22}$$

Lemma 10 *Let $3 < \gamma \le 4$. The Green's function $\mathcal{H}(t,x)$ is a continuous function and satisfies*

(A) $\mathcal{H}(t,x) \ge 0$, $\mathcal{H}(t,x) \le \mathcal{H}(1,x)$, *for* $t,x \in (0,1]$,
(B) $\mathcal{H}(t,x) \ge t^{\gamma-1}\mathcal{H}(1,x)$ *for* $t,x \in (0,1]$.

The continuity of $\mathcal{H}(t,x)$ is ensured by its definition. For (A), considering the case when $0 < x \le t \le 1$, we have

$$\begin{aligned}\mathcal{H}(t,x) &= -\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{t^2}{\Gamma(\gamma-2)}(1-x)^{\gamma-3} \\ &= -\frac{t^{\gamma-1}(1-\frac{x}{t})^{\gamma-1}}{\Gamma(\gamma)} + \frac{t^2}{\Gamma(\gamma-2)}(1-x)^{\gamma-3} \\ &\ge -\frac{t^{\gamma-1}(1-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{t^{\gamma-1}}{\Gamma(\gamma-2)}(1-x)^{\gamma-3} \\ &\ge \frac{t^{\gamma-1}(1-x)^{\gamma-1}}{\Gamma(\gamma)\Gamma(\gamma-2)}\bigl[\Gamma(\gamma) - \Gamma(\gamma-2)\bigr] > 0.\end{aligned} \tag{23}$$

In the case $0 < t \le x \le 1$, the $\mathcal{H}(t,s) > 0$ is obvious. Now for $\mathcal{H}(t,x) \le \mathcal{H}(1,x)$, for $t,x \in (0,1]$, we have

$$\frac{\partial}{\partial t}\mathcal{H}(t,x) = \begin{cases} -\frac{(t-x)^{\gamma-2}}{\Gamma(\gamma-1)} + 2\frac{t}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}, & 0 < x \le t \le 1, \\ \frac{2t}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}, & 0 < t \le x \le 1. \end{cases} \tag{24}$$

For $x,t \in (0,1]$, such that $0 < x \leq t \leq 1$, from (24), we deduce

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{H}(t,x) &= -\frac{(t-x)^{\gamma-2}}{\Gamma(\gamma-1)} + \frac{2t}{\Gamma(\gamma-2)}(1-x)^{\gamma-3} \\
&= -\frac{t^{\gamma-2}(1-\frac{x}{t})^{\gamma-2}}{\Gamma(\gamma-1)} + \frac{2t}{\Gamma(\gamma-2)}(1-x)^{\gamma-3} \\
&\geq -\frac{t^{\gamma-2}(1-x)^{\gamma-2}}{\Gamma(\gamma-1)} + \frac{2t^{\gamma-2}}{\Gamma(\gamma-2)}(1-x)^{\gamma-3} \\
&\geq \frac{t^{\gamma-2}(1-x)^{\gamma-2}}{\Gamma(\gamma-1)\Gamma(\gamma-2)}\bigl[2\Gamma(\gamma-1)-\Gamma(\gamma-2)\bigr] \geq 0,
\end{aligned} \tag{25}$$

from (25), we have $\frac{\partial}{\partial t}\mathcal{H}(t,x) \geq 0$. In the case $0 < t \leq x \leq 1$, (24) implies that the relation $\frac{\partial}{\partial t}\mathcal{H}(t,x) \geq 0$ is obvious, which implies that $\mathcal{H}(t,x)$ is an increasing function w.r.t. t. Hence $\mathcal{H}(t,x) \leq \mathcal{H}(1,x)$.

Now for part (B), using (17) in the case $0 < x \leq t \leq 1$, we proceed thus:

$$\begin{aligned}
\frac{\mathcal{H}(t,x)}{\mathcal{H}(1,x)} &= \frac{\frac{-(t-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{t^2}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}}{\frac{-(1-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}} \\
&= \frac{\frac{-t^{\gamma-1}(1-\frac{x}{t})^{\gamma-1}}{\Gamma(\gamma)} + \frac{t^2}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}}{\frac{-(1-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}} \\
&\geq \frac{\frac{-t^{\gamma-1}(1-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{t^{\gamma-1}}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}}{\frac{-(1-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}} \\
&= t^{\gamma-1}\left[\frac{\frac{-(1-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}}{\frac{-(1-x)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma-2)}(1-x)^{\gamma-3}}\right] = t^{\gamma-1}.
\end{aligned} \tag{26}$$

We can also prove the result for the case of $0 < t \leq x \leq 1$, this completes the proof.

Assume that the following hold:

(J_1) $0 < \int_0^1 \mathcal{H}(1,x)a(x)\,dx < +\infty$.

(J_2) There exist $0 < \delta < 1$ and $\rho > 0$ such that $f(x) \leq \delta L\phi_p(x)$, for $0 \leq x \leq \rho$, where $0 < L \leq (\phi_p(\varpi)\delta \int_0^1 \mathcal{H}(1,x)a(x)\,dx)^{-1}$.

(J_3) There exists $b > 0$, such that $f(x) \leq M\phi_p(x)$, for $u > b$, $0 < M < (\phi_p(\varpi 2^{q-1}) \int_0^1 \mathcal{H}(1,x) \times a(x)\,dx)^{-1}$.

(J_4) $f(z)$ is non-decreasing with respect to z.

(J_5) There exists $0 \leq \beta < 1$ such that $f(Kz) \geq (\phi_p(K))^\beta f(z)$, for any $0 \leq k < 1$ and $0 < z < +\infty$.

2.1 Existence and uniqueness of solutions

Theorem 11 *Under the assumptions* (J_1) *and* (J_2), *the FOBVP* (4) *has at least one positive solution.*

Proof Define $K_1 = \{z \in C[0,1] : 0 \leq z(t) \leq \rho \text{ on } [0,1]\}$ a closed convex set [15]. Define an operator $\mathcal{T} : K_1 \to C[0,1]$ by

$$\mathcal{T}z(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 \mathcal{H}(s,x)a(x)f\bigl(z(x)\bigr)\,dx\right)ds. \tag{27}$$

By Lemma 9, $z(t)$ is a solution of the FOBVP with p-Laplacian operator (4) if and only if $z(t)$ is a fixed point of $\mathcal{T}$. The compactness of the operator $\mathcal{T}$ can easily be shown. Now consider

$$\begin{aligned}
\int_0^1 G(t,s)\,ds &= \frac{\int_0^t (t-s)^{\theta-1}\,ds}{\Gamma(\theta)} + \frac{t\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\left[\frac{\int_0^1 (1-s)^{\theta-\alpha-1}\,ds}{\Gamma(\theta-\alpha)}\right.\\
&\quad \left. + \frac{\xi\int_0^1 (1-s)^{\theta-3}\,ds}{(1-\xi)\Gamma(3-\alpha)\Gamma(\theta-2)}\right] + \frac{t^2\xi}{2(1-\xi)}\frac{\int_0^1 (1-s)^{\theta-3}\,ds}{\Gamma(\theta-2)}\\
&= \frac{(t-s)^{\theta}|_t^0}{\Gamma(\theta+1)} + \frac{t\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\left[\frac{(1-s)^{\theta-\alpha}|_1^0}{\Gamma(\theta-\alpha+1)}\right.\\
&\quad \left. + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)}\frac{(1-s)^{\theta-2}|_1^0}{\Gamma(\theta-1)}\right] + \frac{t^2\xi}{2(1-\xi)}\frac{(1-s)^{\theta-2}|_1^0}{\Gamma(\theta-1)}\\
&= \frac{t^{\theta}}{\Gamma(\theta+1)} + \frac{t\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\left[\frac{1}{\Gamma(\theta-\alpha+1)}\right.\\
&\quad \left. + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)}\frac{1}{\Gamma(\theta-1)}\right] + \frac{t^2\xi}{2(1-\xi)}\frac{1}{\Gamma(\theta-1)}\\
&\leq \frac{1}{\Gamma(\theta+1)} + \frac{\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}\left[\frac{1}{\Gamma(\theta-\alpha+1)}\right.\\
&\quad \left. + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)}\frac{1}{\Gamma(\theta-1)}\right] + \frac{\xi}{2(1-\xi)}\frac{1}{\Gamma(\theta-1)}\\
&= \varpi,
\end{aligned} \tag{28}$$

where $\varpi = \frac{1}{\Gamma(\theta+1)} + \frac{\eta}{1-\frac{\eta}{\Gamma(2-\alpha)}}[\frac{1}{\Gamma(\theta-\alpha+1)} + \frac{\xi}{(1-\xi)\Gamma(3-\alpha)}\frac{1}{\Gamma(\theta-1)}] + \frac{\xi}{2(1-\xi)}\frac{1}{\Gamma(\theta-1)}$. For any $y \in K_1$, using (J_2) and Lemma 10, we obtain

$$\begin{aligned}
\mathcal{T}\big(z(t)\big) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 \mathcal{H}(s,x)a(x)f\big(z(x)\,dx\big)\right)ds\\
&\leq \int_0^1 G(t,s)\phi_q\left(\int_0^1 \mathcal{H}(1,x)a(x)L\delta\phi_p(\rho)\,dx\right)ds\\
&\leq \varpi\phi_q\left(\int_0^1 \mathcal{H}(1,x)a(x)L\delta\phi_p(\rho)\,dx\right)\\
&= \varpi\phi_q\left(L\delta\int_0^1 \mathcal{H}(1,x)a(x)\,dx\right)\rho \leq \rho,
\end{aligned} \tag{29}$$

which implies $\mathcal{T}(K_1) \subseteq K_1$. By Schauder's fixed point theorem $\mathcal{T}$ has a fixed point in K_1. □

Theorem 12 *Under the assumptions* (J_1), (J_3) *the FOBVP with p-Laplacian operator* (4) *has at least one positive solution.*

Proof Let $b > 0$ as given in (J_3). Define $\mathcal{F}^* = \max_{0\leq x\leq b} f(x)$, then $\mathcal{F}^* \geq f(x)$ for $0 \leq x \leq b$. In view of (J_3), we have

$$\varpi_1 2^{q-1}\phi_q(M)\phi_q\left(\int_0^1 \mathcal{H}(1,x)a(x)\,dx\right) < 1. \tag{30}$$

Choose $b^* > b$ large enough such that

$$\varpi_1 2^{q-1}\big(\phi_q(\mathcal{F}^*) + \phi_q(M)b^*\big)\phi_q\left(\int_0^1 \mathcal{H}(1,x)a(x)\,dx\right) < b^*. \tag{31}$$

Define $K_2 = \{z(t) \in C[0,1] : 0 \leq z(t) \leq b^* \text{ on } [0,1]\}$, $\Omega_1 = \{t \in [0,1] : 0 \leq z(t) \leq b\}$, $\Omega_2 = \{t \in [0,1] : b < z(t) \leq b^*\}$. Then $\Omega_1 \cup \Omega_2 = [0,1]$ and $\Omega_1 \cap \Omega_2 = \varphi$. For $z \in K_1$, (J_3) implies $f(z(t)) \leq M\phi_p(z(t)) \leq M\phi_p(b^*)$ for $t \in \Omega_2$ and

$$\begin{aligned} 0 \leq \mathcal{T}\big(z(t)\big) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 \mathcal{H}(s,x)a(x)f\big(z(x)\big)\,dx\right)ds \\ &\leq \varpi\phi_q\left(\int_{S_1} \mathcal{H}(1,x)a(x)f\big(y(x)\big)\,dx + \int_{S_2} \mathcal{H}(1,x)a(x)f\big(z(x)\big)\,dx\right) \\ &\leq \varpi\phi_q\left(\mathcal{F}^*\int_{S_1} \mathcal{H}(1,x)a(x)\,dx + M\phi_p\big(b^*\big)\int_{S_2} \mathcal{H}(1,x)a(x)\,dx\right) \\ &\leq \varpi\phi_q\big(\mathcal{F}^* + M\phi_p\big(b^*\big)\big)\phi_q\left(\int_0^1 \mathcal{H}(1,x)a(x)\,dx\right). \end{aligned} \tag{32}$$

From (31) and using the inequality $(a+b)^r \leq 2^r(a^r + b^r)$ for any $a,b,r > 0$, with (32) we have

$$0 \leq \mathcal{T}z(t) \leq \varpi_1 2^{q-1}\big(\phi_q(\mathcal{F}^*) + \phi_q(M)b^*\big)\phi_q\left(\int_0^1 \mathcal{H}(1,x)a(x)\,dx\right) \leq b^*, \tag{33}$$

thus $\mathcal{T}(K_2) \subseteq K_2$. Hence by Schauder's fixed point theorem $\mathcal{T}$ has a fixed point $z \in K_2$, therefore, the FOBVP with p-Laplacian operator (4) has at least one positive solution in K_2. □

Theorem 13 *Assume that* (J_1), (J_4), (J_5) *hold. Then the FOBVP with p-Laplacian operator* (4) *has a unique positive solution.*

Proof Consider the set $\Lambda = \{z(t) \in C[0,1] : z(t) \geq 0 \text{ on } [0,1]\}$, where Λ is a normal solid cone in $C[0,1]$ with $\Lambda^0 = \{z(t) \in C[0,1] : z(t) > 0 \text{ on } [0,1]\}$. Let $\mathcal{T} : \Lambda^0 \to C[0,1]$ be defined by (27), we prove that $\mathcal{T}$ is a θ-concave and increasing operator. For $z_1, z_2 \in \Lambda^0$ with $z_1 \geq z_2$ the assumption (J_4) implies that the operator $\mathcal{T}$ is an increasing operator, *i.e.*, $\mathcal{T}(z_1(t)) \geq \mathcal{T}(z_2(t))$ on $t \in [0,1]$. With the help of $f(kz) \geq \phi_p(k^\alpha)f(z)$, we have

$$\begin{aligned} \mathcal{T}\big(kz(t)\big) &\geq \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(t,x)\phi_p\big(k^\alpha\big)a(x)f\big(z(x)\big)\,dx\right)ds \\ &= k^\alpha\int_0^1 G(t,s)\phi_q\left(\int_0^1 H(t,x)a(x)f\big(z(x)\big)\,dx\right)ds \\ &= k^\alpha\mathcal{T}\big(z(t)\big), \end{aligned} \tag{34}$$

which implies that $\mathcal{T}$ is α-concave operator and with the help of Lemma 7, the operator $\mathcal{T}$ has a unique fixed point which is the unique positive solution of the FOBVP with p-Laplacian operator (4) in Λ^0. □

3 Examples

Example 1 Consider the following boundary value problem:

$$\begin{aligned}
&D^{3.5}\big(\phi_p\big(D^{3.5}z(t)\big)\big) + tz(t) = 0,\\
&z(0) = 0 = z'''(0), \qquad \frac{1}{2}D^{0.5}z(t)|_{t=1} - z'(0) = 0, \qquad \frac{1}{2}z''(1) - z''(0) = 0,\\
&\phi_p\big(D^{3.5}z(t)\big)|_{t=0} = 0 = \big(\phi_p\big(D^{3.5}z(t)\big)\big)'|_{t=0},\\
&\big(\phi_p\big(D^{3.5}z(t)\big)\big)''|_{t=1} = \frac{1}{2}\big(\phi_p\big(D^{3.5}z(t)\big)\big)'', \qquad \big(\phi_p\big(D^{3.5}z(t)\big)\big)''|_{t=0} = 0.
\end{aligned} \tag{35}$$

Here we have $\alpha = \beta = 3.5$, $\alpha = 0.5$, $\xi = \gamma = \frac{1}{2}$, $a(t) = t, f(z(t)) = z(t)$. By simple computation, we obtain $0 < L \leq 1.9092$, choose $L = 1.5$, $\delta = 1$, $p = 2$, and then the conditions (J_1) and (J_2) are satisfied. Hence, by Theorem 11, the FOBVP with p-Laplacian operator (35) has at least one positive solution.

Example 2 For the following boundary value problem:

$$\begin{aligned}
&D^{3.5}\big(\phi_p\big(D^{3.5}z(t)\big)\big) + t\sqrt[3]{z}(t) = 0,\\
&z(0) = 0 = z'''(0), \qquad 0.1D^{0.5}z(t)|_{t=1} - z'(0) = 0, \qquad 0.1z''(1) - z''(0) = 0,\\
&\phi_p\big(D^{3.5}z(t)\big)|_{t=0} = 0 = \big(\phi_p\big(D^{3.5}z(t)\big)\big)'|_{t=0},\\
&\big(\phi_p\big(D^{3.5}z(t)\big)\big)''|_{t=1} = \frac{1}{2}\big(\phi_p\big(D^{3.5}z(t)\big)\big)'', \qquad \big(\phi_p\big(D^{3.5}z(t)\big)\big)''|_{t=0} = 0,
\end{aligned} \tag{36}$$

we have $\alpha = \beta = 3.5$, $\xi = \eta = 0.1$, $a(t) = t, f(u(t)) = \sqrt[3]{u}(t)$ and by simple computation we get $M < 4.4792$ and thus by choosing $M = 4.00$, $b = 1$, and $p = q = 2$, we see that (36) satisfy (J_1) and (J_3). Hence by Theorem 12, the FOBVP with p-Laplacian operator (36) has at least one positive solution.

Example 3 The uniqueness of the solution for FOBVP with the p-Laplacian operator; we have

$$\begin{aligned}
&D^{3.5}\big(\phi_p\big(D^{3.5}z(t)\big)\big) + tz(t) = 0,\\
&z(0) = 0 = z'''(0), \qquad \frac{1}{3}D^{0.5}z(t)|_{t=1} - z'(0) = 0, \qquad \frac{1}{3}z''(1) - z''(0) = 0,\\
&\phi_p\big(D^{3.5}z(t)\big)|_{t=0} = 0 = \big(\phi_p\big(D^{3.5}z(t)\big)\big)'|_{t=0},\\
&\big(\phi_p\big(D^{3.5}z(t)\big)\big)''|_{t=1} = \frac{1}{2}\big(\phi_p\big(D^{3.5}z(t)\big)\big)'', \qquad \big(\phi_p\big(D^{3.5}z(t)\big)\big)''|_{t=0} = 0.
\end{aligned} \tag{37}$$

We apply Theorem 13. In (37), we have $\theta = \gamma = 3.5$, $\alpha = 0.5$, $\xi = \gamma = \frac{1}{3}$, $a(t) = t, f(z(t)) = z(t)$ it is clear that (37) satisfies conditions (J_1), (J_4). Also considering $\beta = \frac{1}{2}$, (J_5) is satisfied. Thus by Theorem 13, the fractional differential equation (37) has a unique solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All the authors read and approved the final manuscript.

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