Cianciaruso and Pietramala *Boundary Value Problems* (2015) 2015:163
DOI 10.1186/s13661-015-0415-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Multiple positive solutions of a (p_1, p_2)-Laplacian system with nonlinear BCs

Filomena Cianciaruso and Paolamaria Pietramala*

*Correspondence:
pietramala@unical.it
Dipartimento di Matematica e Informatica, Università della Calabria, Arcavacata di Rende (Cosenza), 87036, Italy

Abstract

Using the theory of fixed point index, we discuss existence, non-existence, localization and multiplicity of positive solutions for a (p_1, p_2)-Laplacian system with nonlinear Robin and/or Dirichlet type boundary conditions. We give an example to illustrate our theory.

MSC: Primary 45G15; secondary 34B18

Keywords: fixed point index; cone; positive solution; p-Laplacian system; nonlinear boundary conditions

1 Introduction

In the remarkable paper [1], Wang proved the existence of one positive solution of the following one-dimensional p-Laplacian equation:

$$\big(\varphi_p\big(u'\big)\big)'(t) + g(t)f\big(u(t)\big) = 0, \quad t \in (0,1), \tag{1.1}$$

subject to one of the following three pairs of nonlinear boundary conditions (BCs)

$$\begin{aligned}
&u'(0) = 0, \qquad u(1) + B_1\big(u'(1)\big) = 0,\\
&u(0) = B_2\big(u'(0)\big), \qquad u'(1) = 0,\\
&u(0) = B_2\big(u'(0)\big), \qquad u(1) + B_1\big(u'(1)\big) = 0,
\end{aligned}$$

where $B_1, B_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying some suitable growth conditions. The results of [1] were extended by Karakostas [2] to the context of deviated arguments. In both cases, the existence results are obtained via a careful study of an associated integral operator combined with the use of the Krasnosel'skiĭ-Guo theorem on cone compressions and cone expansions.

The Krasnosel'skiĭ-Guo theorem and other topological methods are commonly used tools in the study of existence of positive solutions for the p-Laplacian equation (1.1) subject to different BCs. This is an active area of research, for example, homogeneous Dirichlet BCs were studied in [3–10], homogeneous Robin BCs in [7, 9, 10], nonlocal BCs of Dirichlet type in [1, 11–19] and nonlocal BCs of Robin type in [16, 19–23].

Here we study the one-dimensional (p_1,p_2)-Laplacian system

$$\begin{aligned} &\big(\varphi_{p_1}\big(u'\big)\big)'(t)+g_1(t)f_1\big(t,u(t),v(t)\big)=0, \quad t\in(0,1),\\ &\big(\varphi_{p_2}\big(v'\big)\big)'(t)+g_2(t)f_2\big(t,u(t),v(t)\big)=0, \quad t\in(0,1), \end{aligned} \tag{1.2}$$

with $\varphi_{p_i}(w)=|w|^{p_i-2}w$, subject to the nonlinear boundary conditions

$$u'(0)=0, \qquad u(1)+B_1\big(u'(1)\big)=0, \qquad v(0)=B_2\big(v'(0)\big), \qquad v(1)=0. \tag{1.3}$$

The existence of *positive* solutions for systems of equations of the type (1.2) has been widely studied; see, for example, [24–27] under homogeneous Dirichlet BCs and [5, 28–31] with homogeneous Robin or Neumann BCs. For earlier contributions on problems with nonlinear BCs, we refer to [1, 2, 20, 32–39] and the references therein.

We improve and complement the previous results in several directions: we obtain *multiplicity* results for the (p_1,p_2)-Laplacian *system* subject to *nonlinear* BCs, we allow different growths in the nonlinearities f_1 and f_2, and also we discuss non-existence results. Finally we illustrate in an example that all the constants that occur in our results can be computed.

Our approach is to seek solutions of system (1.2)-(1.3) as fixed points of a suitable integral operator. We make use of the classical fixed point index theory and benefit from ideas of the papers [1, 2, 37, 40].

2 The system of integral equations

We recall that a *cone* K in a Banach space X is a closed convex set such that $\lambda x\in K$ for $x\in K$ and $\lambda\geq 0$ and $K\cap(-K)=\{0\}$.

If Ω is an open bounded subset of a cone K (in the relative topology), we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When Ω is an open bounded subset of X, we write $\Omega_K=\Omega\cap K$, an open subset of K.

The following lemma summarizes some classical results regarding the fixed point index; for more details, see [41, 42].

Lemma 2.1 *Let Ω be an open bounded set with $0\in\Omega_K$ and $\overline{\Omega}_K\neq K$. Assume that $F:\overline{\Omega}_K\to K$ is a compact map such that $x\neq Fx$ for all $x\in\partial\Omega_K$. Then the fixed point index $i_K(F,\Omega_K)$ has the following properties.*

1. *If there exists $e\in K\setminus\{0\}$ such that $x\neq Fx+\lambda e$ for all $x\in\partial\Omega_K$ and all $\lambda>0$, then $i_K(F,\Omega_K)=0$.*
2. *If $\mu x\neq Fx$ for all $x\in\partial\Omega_K$ and for every $\mu\geq 1$, then $i_K(F,\Omega_K)=1$.*
3. *If $i_K(F,\Omega_K)\neq 0$, then F has a fixed point in Ω_K.*
4. *Let Ω^1 be open in X with $\overline{\Omega^1}\subset\Omega_K$. If $i_K(F,\Omega_K)=1$ and $i_K(F,\Omega_K^1)=0$, then F has a fixed point in $\Omega_K\setminus\overline{\Omega_K^1}$. The same result holds if $i_K(F,\Omega_K)=0$ and $i_K(F,\Omega_K^1)=1$.*

To system (1.2)-(1.3) we associate the following system of integral equations, which is constructed in a similar manner as in [1] where the case of a single equation is studied:

$$
\begin{aligned}
u(t) &= \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + B_1\left(\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)\right), \quad 0\le t\le 1, \\
v(t) &= \begin{cases} \int_0^t \varphi_{p_2}^{-1}(\int_s^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau)\,ds \\ \quad + B_2(\varphi_{p_2}^{-1}(\int_0^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau)), & 0\le t\le \sigma_{u,v}, \\ \int_t^1 \varphi_{p_2}^{-1}(\int_{\sigma_{u,v}}^s g_2(\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau)\,ds, & \sigma_{u,v}\le t\le 1, \end{cases}
\end{aligned}
\tag{2.1}
$$

where $\varphi_{p_i}^{-1}(w) = |w|^{\frac{1}{p_i-1}} \operatorname{sgn} w$ and $\sigma_{u,v}$ is the smallest solution $x \in [0,1]$ of the equation

$$
\begin{aligned}
&\int_0^x \varphi_{p_2}^{-1}\left(\int_s^x g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds + B_2\left(\varphi_{p_2}^{-1}\left(\int_0^x g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)\right) \\
&\quad = \int_x^1 \varphi_{p_2}^{-1}\left(\int_x^s g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds.
\end{aligned}
$$

By a *solution* of (1.2)-(1.3), we mean a solution of system (2.1).

In order to utilize the fixed point index theory, we state the following assumptions on the terms that occur in system (2.1):

(C1) For every $i = 1,2$, $f_i : [0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty)$ satisfies Carathéodory conditions, that is, $f_i(\cdot,u,v)$ is measurable for each fixed (u,v) and $f_i(t,\cdot,\cdot)$ is continuous for almost every (a.e.) $t\in[0,1]$, and for each $r>0$ there exists $\phi_{i,r}\in L^\infty[0,1]$ such that

$$f_i(t,u,v)\le \phi_{i,r}(t) \quad \text{for } u,v\in[0,r] \text{ and a.e. } t\in[0,1].$$

(C2) $g_1\in L^1[0,1]$, $g_1\ge 0$ and

$$0 < \int_0^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\,d\tau\right)ds < +\infty.$$

(C3) $g_2\in L^1[0,1]$, $g_2\ge 0$ and

$$0 < \int_0^{1/2} \varphi_{p_2}^{-1}\left(\int_s^{1/2} g_2(\tau)\,d\tau\right)ds + \int_{1/2}^1 \varphi_{p_2}^{-1}\left(\int_{1/2}^s g_2(\tau)\,d\tau\right)ds < +\infty. \tag{2.2}$$

(C4) For every $i=1,2$, $B_i:\mathbb{R}\to\mathbb{R}$ is a continuous function, and there exist $h_{i1},h_{i2}\ge 0$ such that

$$h_{i1}v\le B_i(v)\le h_{i2}v \quad \text{for any } v\ge 0.$$

Remark 2.2 Condition (2.2) is weaker than the condition

$$0 < \int_0^1 \varphi_{p_2}^{-1}\left(\int_s^1 g_2(\tau)\,d\tau\right)ds < +\infty. \tag{2.3}$$

In fact, for example, the function

$$g_2(t) = \begin{cases} \frac{1}{(t-1)^2}, & t \in [0,1/2], \\ \frac{1}{t^2}, & t \in (1/2,1], \end{cases}$$

satisfies (2.2) but not (2.3).

Remark 2.3 From (C2) and (C3) it follows that there exists $[a_1,b_1] \subset [0,1)$ such that $\int_{a_1}^{b_1} g_1(s)\,ds > 0$ and there exists $[a_2,b_2] \subset (0,1)$ such that $\int_{a_2}^{b_2} g_2(s)\,ds > 0$.

We work in the space $C[0,1] \times C[0,1]$ endowed with the norm

$$\|(u,v)\| := \max\{\|u\|_\infty, \|v\|_\infty\},$$

where $\|w\|_\infty := \max\{|w(t)|, t \in [0,1]\}$.

Take the cones

$$K_1 := \{w \in C[0,1] : w \geq 0, \text{ concave and nonincreasing}\},$$
$$K_2 := \{w \in C[0,1] : w \geq 0, \text{ concave}\}.$$

It is known (see, *e.g.*, [1]) that

- for $w \in K_1$, we have $w(t) \geq (1-t)\|w\|_\infty$ for $t \in [0,1]$;
- for $w \in K_2$, we have $w(t) \geq \min\{t, 1-t\}\|w\|_\infty$ for $t \in [0,1]$.

It follows that the functions in K_i are strictly positive on the sub-interval $[a_i, b_i]$ and in particular

- for $w \in K_1$, we have $\min_{t\in[a_1,b_1]} w(t) \geq (1-b_1)\|w\|_\infty$;
- for $w \in K_2$, we have $\min_{t\in[a_2,b_2]} w(t) \geq \min\{a_2, 1-b_2\}\|w\|_\infty$.

In the following we assume $a_1 = 0$ and we make use of the notations

$$c_1 := 1 - b_1, \qquad c_2 := \min\{a_2, 1-b_2\}.$$

Consider now the cone K in $C[0,1] \times C[0,1]$ defined by

$$K := \{(u,v) \in K_1 \times K_2\}.$$

For a *positive* solution of system (2.1) we mean a solution $(u,v) \in K$ of (2.1) such that $\|(u,v)\| > 0$. We seek such solution as a fixed point of the following operator T.

Consider the integral operator

$$T(u,v)(t) := \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix}, \tag{2.4}$$

where

$$\begin{aligned} T_1(u,v)(t) := {} & \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1\big(\tau, u(\tau), v(\tau)\big)\,d\tau\right) ds \\ & + B_1\left(\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1\big(\tau, u(\tau), v(\tau)\big)\,d\tau\right)\right) \end{aligned}$$

and

$$T_2(u,v)(t) := \begin{cases} \int_0^t \varphi_{p_2}^{-1}(\int_s^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau)\,ds \\ \quad + B_2(\varphi_{p_2}^{-1}(\int_0^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau)), & 0 \le t \le \sigma_{u,v}, \\ \int_t^1 \varphi_{p_2}^{-1}(\int_{\sigma_{u,v}}^s g_2(\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau)\,ds, & \sigma_{u,v} \le t \le 1. \end{cases}$$

From the definitions, for every $(u,v) \in K$, we have

$$\max_{t\in[0,1]} T_2(u,v)(t) = T_2(u,v)(\sigma_{u,v}).$$

Under our assumptions, we can show that the integral operator T leaves the cone K invariant and is compact.

Lemma 2.4 *The operator* (2.4) *maps K into K and is compact.*

Proof Take $(u,v) \in K$. From the definition we have that the function $T_1(u,v)$ is nonincreasing. The fact that $T_1(u,v)$ and $T_2(u,v)$ are convex functions is known, see Section 2, p.2279 of [1]. Thus $T(u,v) \in K$. Now, we show that the map T is compact. Firstly, we show that T sends bounded sets into bounded sets. Take $(u,v) \in K$ such that $\|(u,v)\| \le r$. Then, for all $t \in [0,1]$, we have

$$\begin{aligned} T_1(u,v)(t) &= \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right) ds \\ &\quad + B_1\left(\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)\right) \\ &\le \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\phi_{1,r}(\tau)\,d\tau\right) ds \\ &\quad + h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right) \\ &\le \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\phi_{1,r}(\tau)\,d\tau\right) ds + h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\phi_{1,r}(\tau)\,d\tau\right) \\ &\le \int_0^1 \varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\phi_{1,r}(\tau)\,d\tau\right) ds + h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\phi_{1,r}(\tau)\,d\tau\right) < +\infty. \end{aligned}$$

We prove now that T_1 sends bounded sets into equicontinuous sets. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $(u,v) \in K$ such that $\|(u,v)\| \le r$. Then we have

$$\begin{aligned} \bigl|T_1(u,v)(t_1) - T_1(u,v)(t_2)\bigr| &= \left|\int_{t_1}^{t_2} \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right) ds\right| \\ &\le \left|\int_{t_1}^{t_2} \varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\phi_{1,r}(\tau)\,d\tau\right) ds\right| = C_r|t_1 - t_2|. \end{aligned}$$

Therefore we obtain $|T_1(u,v)(t_1) - T_1(u,v)(t_2)| \to 0$ when $t_1 \to t_2$. By the Ascoli-Arzelà theorem we can conclude that T_1 is a compact map.

For the sake of completeness, we sketch the proof of the fact that T_2 sends bounded sets into equicontinuous sets. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $(u,v) \in K$ such that $\|(u,v)\| \le r$. The cases $0 \le t_1 < t_2 \le \sigma_{u,v}$ or $\sigma_{u,v} \le t_1 < t_2 \le 1$ can be handled as in the case of the operator T_1. If $0 \le t_1 < \sigma_{u,v} < t_2 \le 1$, we observe that

$$\begin{aligned}
&\bigl|T_2(u,v)(t_1) - T_2(u,v)(t_2)\bigr| \\
&\quad = \bigl|T_2(u,v)(t_1) - T_2(u,v)(\sigma_{u,v}) + T_2(u,v)(\sigma_{u,v}) - T_2(u,v)(t_2)\bigr| \\
&\quad \le \bigl|T_2(u,v)(t_1) - T_2(u,v)(\sigma_{u,v})\bigr| + \bigl|T_2(u,v)(\sigma_{u,v}) - T_2(u,v)(t_2)\bigr|,
\end{aligned}$$

and the proof follows as in previous cases.

Moreover, the map T is compact since the components T_i are compact maps. □

3 Existence results

For our index calculations, given $\rho_1, \rho_2 > 0$ we use the following (relative) open bounded sets in K:

$$K_{\rho_1,\rho_2} = \bigl\{(u,v) \in K : \|u\|_\infty < \rho_1 \text{ and } \|v\|_\infty < \rho_2\bigr\}$$

and

$$V_{\rho_1,\rho_2} = \Bigl\{(u,v) \in K : \min_{t\in[0,b_1]} u(t) < c_1\rho_1 \text{ and } \min_{t\in[a_2,b_2]} v(t) < c_2\rho_2\Bigr\},$$

and if $\rho_1 = \rho_2 = \rho$, we write simply K_ρ and V_ρ. The set V_ρ was introduced in [43] as an extension to the case of systems of a set given by Lan [44]. The use of different radii, in the spirit of the paper [40], allows more freedom in the growth of the nonlinearities.

The following lemma is similar to Lemma 5 of [43] and therefore its proof is omitted.

Lemma 3.1 *The sets defined above have the following properties*:

- $K_{c_1\rho_1,c_2\rho_2} \subset V_{\rho_1,\rho_2} \subset K_{\rho_1,\rho_2}$.
- $(w_1,w_2) \in \partial K_{\rho_1,\rho_2}$ *iff* $(w_1,w_2) \in K$ *and* $\|w_i\|_\infty = \rho_i$ *for some* $i \in \{1,2\}$ *and* $\|w_j\|_\infty \le \rho_j$ *for* $j \ne i$.
- $(w_1,w_2) \in \partial V_{\rho_1,\rho_2}$ *iff* $(w_1,w_2) \in K$ *and* $\min_{t\in[a_i,b_i]} w_i(t) = c_i\rho_i$ *for some* $i \in \{1,2\}$ *and* $\min_{t\in[a_j,b_j]} w_j(t) \le c_j\rho_j$ *for* $j \ne i$.
- *If* $(w_1,w_2) \in \partial V_{\rho_1,\rho_2}$, *then for some* $i \in \{1,2\}$, $c_i\rho_i \le w_i(t) \le \rho_i$ *for each* $t \in [a_i,b_i]$ *and* $\|w_i\|_\infty \le \rho_i$; *moreover, for* $j \ne i$, *we have* $\|w_j\|_\infty \le \rho_j$.

We firstly prove that the fixed point index is 1 on the set K_{ρ_1,ρ_2}.

Lemma 3.2 *Assume that*

$(\mathrm{I}^1_{\rho_1,\rho_2})$ *there exist* $\rho_1, \rho_2 > 0$ *such that for every* $i = 1, 2$

$$f_i^{\rho_1,\rho_2} < \varphi_{p_i}(m_i), \tag{3.1}$$

where

$$f_i^{\rho_1,\rho_2} = \sup\biggl\{\frac{f_i(t,u,v)}{\rho_i^{p_i-1}} : (t,u,v) \in [0,1]\times[0,\rho_1]\times[0,\rho_2]\biggr\},$$

$$\frac{1}{m_1} = \int_0^1 \varphi_{p_1}^{-1}\biggl(\int_0^s g_1(\tau)\,d\tau\biggr)\,ds + h_{12}\varphi_{p_1}^{-1}\biggl(\int_0^1 g_1(\tau)\,d\tau\biggr)$$

and

$$\frac{1}{m_2} = \max\left[\int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)\,d\tau\right)ds + h_{22}\varphi_{p_2}^{-1}\left(\int_0^{\frac{1}{2}} g_2(\tau)\,d\tau\right),\right.$$
$$\left.\int_{\frac{1}{2}}^1 \varphi_{p_2}^{-1}\left(\int_{\frac{1}{2}}^s g_2(\tau)\,d\tau\right)ds\right].$$

Then $i_K(T, K_{\rho_1,\rho_2}) = 1$.

Proof We show that $\lambda(u,v) \neq T(u,v)$ for every $(u,v) \in \partial K_{\rho_1,\rho_2}$ and for every $\lambda \geq 1$; this ensures that the index is 1 on K_{ρ_1,ρ_2}. In fact, if this does not happen, there exist $\lambda \geq 1$ and $(u,v) \in \partial K_{\rho_1,\rho_2}$ such that $\lambda(u,v) = T(u,v)$.

Firstly we assume that $\|u\|_\infty = \rho_1$ and $\|v\|_\infty \leq \rho_2$.

Then we have

$$\begin{aligned}
\lambda u(t) &= \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + B_1\left(\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)\right) \\
&\leq \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right) \\
&= \rho_1 \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\frac{f_1(\tau,u(\tau),v(\tau))}{\rho_1^{p_1-1}}\,d\tau\right)ds \\
&\quad + \rho_1 h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\frac{f_1(\tau,u(\tau),v(\tau))}{\rho_1^{p_1-1}}\,d\tau\right).
\end{aligned}$$

Taking $t = 0$ gives

$$\begin{aligned}
\lambda u(0) &= \lambda\rho_1 \\
&\leq \rho_1 \int_0^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1^{\rho_1,\rho_2}\,d\tau\right)ds + \rho_1 h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1^{\rho_1,\rho_2}\,d\tau\right) \\
&= \rho_1\varphi_{p_1}^{-1}\big(f_1^{\rho_1,\rho_2}\big)\left(\int_0^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\,d\tau\right)ds + h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\,d\tau\right)\right) \\
&= \rho_1\frac{1}{m_1}\varphi_{p_1}^{-1}\big(f_1^{\rho_1,\rho_2}\big).
\end{aligned}$$

Using hypothesis (3.1) and the strict monotonicity of $\varphi_{p_1}^{-1}$, we obtain $\lambda\rho_1 < \rho_1$. This contradicts the fact that $\lambda \geq 1$ and proves the result.

Now we assume $\|v\|_\infty = \rho_2$ and $\|u\|_\infty \leq \rho_1$.

Then we have

$$\lambda\rho_2 = \big\|T_2(u,v)\big\|_\infty = T_2(u,v)(\sigma_{u,v}).$$

If $\sigma_{u,v} \le \frac{1}{2}$, we have

$$\begin{aligned}
\lambda\rho_2 &= \|T_2(u,v)\|_\infty = T_2(u,v)(\sigma_{u,v}) \\
&= \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + B_2\left(\varphi_{p_2}^{-1}\left(\int_0^{\sigma_{u,v}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)\right) \\
&\le \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + h_{22}\varphi_{p_2}^{-1}\left(\int_0^{\sigma_{u,v}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right) \\
&\le \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + h_{22}\varphi_{p_2}^{-1}\left(\int_0^{\frac{1}{2}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right) \\
&= \rho_2\int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_2-1}}\,d\tau\right)ds \\
&\quad + \rho_2 h_{22}\varphi_{p_2}^{-1}\left(\int_0^{\frac{1}{2}} g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_2-1}}\,d\tau\right);
\end{aligned}$$

thus we obtain

$$\lambda\rho_2 \le \rho_2\varphi_{p_2}^{-1}\bigl(f_2^{\rho_1,\rho_2}\bigr)\left(\int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)\,d\tau\right)ds + h_{22}\varphi_{p_2}^{-1}\left(\int_0^{\frac{1}{2}} g_2(\tau)\,d\tau\right)\right).$$

If $\sigma_{u,v} > \frac{1}{2}$, we have

$$\begin{aligned}
\lambda\rho_2 &= \|T_2(u,v)\|_\infty = T_2(u,v)(\sigma_{u,v}) \\
&= \int_{\sigma_{u,v}}^1 \varphi_{p_2}^{-1}\left(\int_{\sigma_{u,v}}^s g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\le \int_{\frac{1}{2}}^1 \varphi_{p_2}^{-1}\left(\int_{\frac{1}{2}}^s g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&= \rho_2\int_{\frac{1}{2}}^1 \varphi_{p_2}^{-1}\left(\int_{\frac{1}{2}}^s g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_2-1}}\,d\tau\right)ds \\
&\le \rho_2\varphi_{p_2}^{-1}\bigl(f_2^{\rho_1,\rho_2}\bigr)\int_{\frac{1}{2}}^1 \varphi_{p_2}^{-1}\left(\int_{\frac{1}{2}}^s g_2(\tau)\,d\tau\right)ds.
\end{aligned}$$

Then, in both cases, we have

$$\begin{aligned}
\lambda\rho_2 &= \|T_2(u,v)\|_\infty = T_2(u,v)(\sigma_{u,v}) \\
&\le \rho_2\varphi_{p_2}^{-1}\bigl(f_2^{\rho_1,\rho_2}\bigr)\max\left[\int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)\,d\tau\right)ds + h_{22}\varphi_{p_2}^{-1}\left(\int_0^{\frac{1}{2}} g_2(\tau)\,d\tau\right),\right.
\end{aligned}$$

$$\left.\int_{\frac{1}{2}}^{1}\varphi_{p_2}^{-1}\left(\int_{\frac{1}{2}}^{s}g_2(\tau)\,d\tau\right)ds\right]$$
$$=\rho_2\varphi_{p_2}^{-1}\left(f_2^{\rho_1,\rho_2}\right)\frac{1}{m_2}.$$

Using hypothesis (3.1) and the strict monotonicity of $\varphi_{p_2}^{-1}$, we obtain $\lambda\rho_2<\rho_2$. This contradicts the fact that $\lambda\geq 1$ and proves the result. □

We give a first lemma that shows that the index is 0 on a set V_{ρ_1,ρ_2}.

Lemma 3.3 *Assume that*:

$(\mathrm{I}^0_{\rho_1,\rho_2})$ *there exist* $\rho_1,\rho_2>0$ *such that for every* $i=1,2$

$$f_{i,(\rho_1,\rho_2)}>\varphi_{p_i}(M_i),\tag{3.2}$$

where

$$f_{1,(\rho_1,\rho_2)}=\inf\left\{\frac{f_1(t,u,v)}{\rho_1^{p_1-1}}:(t,u,v)\in[0,b_1]\times[c_1\rho_1,\rho_1]\times[0,\rho_2]\right\},$$
$$f_{2,(\rho_1,\rho_2)}=\inf\left\{\frac{f_2(t,u,v)}{\rho_2^{p_2-1}}:(t,u,v)\in[a_2,b_2]\times[0,\rho_1]\times[c_2\rho_2,\rho_2]\right\},$$
$$\frac{1}{M_1}=\int_0^{b_1}\varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\,d\tau\right)ds+h_{11}\varphi_{p_1}^{-1}\left(\int_0^{b_1}g_1(\tau)\,d\tau\right),$$

and

$$\frac{1}{M_2}=\frac{1}{2}\min_{a_2\leq\nu\leq b_2}\left[\int_{a_2}^{\nu}\varphi_{p_2}^{-1}\left(\int_s^{\nu}g_2(\tau)\,d\tau\right)ds\right.$$
$$\left.+\int_{\nu}^{b_2}\varphi_{p_2}^{-1}\left(\int_{\nu}^{s}g_2(\tau)\,d\tau\right)ds+h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{\nu}g_2(\tau)\,d\tau\right)\right].$$

Then $i_K(T,V_{\rho_1,\rho_2})=0$.

Proof Let $e(t)\equiv 1$ for $t\in[0,1]$. Then $(e,e)\in K$. We prove that $(u,v)\neq T(u,v)+\lambda(e,e)$ for $(u,v)\in\partial V_{\rho_1,\rho_2}$ and $\lambda\geq 0$. In fact, if this does not happen, there exist $(u,v)\in\partial V_{\rho_1,\rho_2}$ and $\lambda\geq 0$ such that $(u,v)=T(u,v)+\lambda(e,e)$. We examine two cases.

Case (1): $c_1\rho_1\leq u(t)\leq\rho_1$ for $t\in[0,b_1]$ and $0\leq v(t)\leq\rho_2$ for $t\in[0,1]$.

Thus we have, for $t\in[0,b_1]$,

$$\begin{aligned}\rho_1&\geq u(t)\\&=\int_t^1\varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\&\quad+B_1\left(\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)\right)+\lambda\\&\geq\int_t^{b_1}\varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\&\quad+h_{11}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)+\lambda\end{aligned}$$

$$\begin{aligned}
&\geq \int_t^{b_1} \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau) f_1\big(\tau, u(\tau), v(\tau)\big)\, d\tau\right) ds \\
&\quad + h_{11}\varphi_{p_1}^{-1}\left(\int_0^{b_1} g_1(\tau) f_1\big(\tau, u(\tau), v(\tau)\big)\, d\tau\right) + \lambda \\
&= \rho_1 \int_t^{b_1} \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau) \frac{f_1(\tau, u(\tau), v(\tau))}{\rho_1^{p_1-1}}\, d\tau\right) ds \\
&\quad + \rho_1 h_{11}\varphi_{p_1}^{-1}\left(\int_0^{b_1} g_1(\tau) \frac{f_1(\tau, u(\tau), v(\tau))}{\rho_1^{p_1-1}}\, d\tau\right) + \lambda.
\end{aligned}$$

For $t = 0$, we obtain

$$\begin{aligned}
\rho_1 &\geq \rho_1\varphi_{p_1}^{-1}(f_{1,(\rho_1,\rho_2)}) \int_0^{b_1} \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\, d\tau\right) ds \\
&\quad + \rho_1\varphi_{p_1}^{-1}(f_{1,(\rho_1,\rho_2)}) h_{11}\varphi_{p_1}^{-1}\left(\int_0^{b_1} g_1(\tau)\, d\tau\right) + \lambda \\
&> \rho_1\varphi_{p_1}^{-1}(f_{1,(\rho_1,\rho_2)}) \frac{1}{M_1} + \lambda.
\end{aligned}$$

Using hypothesis (3.2) we obtain $\rho_1 > \rho_1 + \lambda$, a contradiction.

Case (2): $0 \leq u(t) \leq \rho_1$ for $t \in [0,1]$ and $c_2\rho_2 \leq v(t) \leq \rho_2$.

We distinguish three cases as follows.

Case (2_1): $0 < \sigma_{u,v} \leq a_2$.

Therefore we get

$$\begin{aligned}
\rho_2 &\geq v(\sigma_{u,v}) = T_2(u,v)(\sigma_{u,v}) + \lambda \\
&= \int_{\sigma_{u,v}}^1 \varphi_{p_2}^{-1}\left(\int_{\sigma_{u,v}}^s g_2(\tau) f_2\big(\tau, u(\tau), v(\tau)\big)\, d\tau\right) ds + \lambda \\
&\geq \int_{a_2}^{b_2} \varphi_{p_2}^{-1}\left(\int_{a_2}^s g_2(\tau) f_2\big(\tau, u(\tau), v(\tau)\big)\, d\tau\right) ds + \lambda \\
&= \rho_2 \int_{a_2}^{b_2} \varphi_{p_2}^{-1}\left(\int_{a_2}^s g_2(\tau) \frac{f_2(\tau, u(\tau), v(\tau))}{\rho_2^{p_2-1}}\, d\tau\right) ds + \lambda \\
&\geq \rho_2\varphi_{p_2}^{-1}(f_{2,(\rho_1,\rho_2)}) \int_{a_2}^{b_2} \varphi_{p_2}^{-1}\left(\int_{a_2}^s g_2(\tau)\, d\tau\right) ds + \lambda \\
&\geq \rho_2\varphi_{p_2}^{-1}(f_{2,(\rho_1,\rho_2)}) \frac{1}{M_2} + \lambda.
\end{aligned}$$

Using hypothesis (3.2) we obtain $\rho_2 > \rho_2 + \lambda$, a contradiction.

Case (2_2): $\sigma_{u,v} \geq b_2$.

$$\begin{aligned}
\rho_2 &\geq v(\sigma_{u,v}) = T_2(u,v)(\sigma_{u,v}) + \lambda \\
&= \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau) f_2\big(\tau, u(\tau), v(\tau)\big)\, d\tau\right) ds \\
&\quad + B_2\left(\varphi_{p_2}^{-1}\left(\int_0^{\sigma_{u,v}} g_2(\tau) f_2\big(\tau, u(\tau), v(\tau)\big)\, d\tau\right)\right) + \lambda \\
&\geq \int_{a_2}^{b_2} \varphi_{p_2}^{-1}\left(\int_s^{b_2} g_2(\tau) f_2\big(\tau, u(\tau), v(\tau)\big)\, d\tau\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+ h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{b_2} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right) + \lambda \\
&= \rho_2\int_{a_2}^{b_2}\varphi_{p_2}^{-1}\left(\int_s^{b_2} g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_1-1}}\,d\tau\right)ds \\
&\quad + \rho_2 h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{b_2} g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_2-1}}\,d\tau\right) + \lambda \\
&\geq \rho_2\varphi_{p_2}^{-1}(f_{2,(\rho_1,\rho_2)})\int_{a_2}^{b_2}\varphi_{p_2}^{-1}\left(\int_s^{b_2} g_2(\tau)\,d\tau\right)ds \\
&\quad + \rho_2\varphi_{p_2}^{-1}(f_{2,(\rho_1,\rho_2)})h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{b_2} g_2(\tau)\,d\tau\right) + \lambda \\
&\geq \rho_2\varphi_{p_2}^{-1}(f_{2,(\rho_1,\rho_2)})\frac{1}{M_2} + \lambda.
\end{aligned}
$$

Using hypothesis (3.2) we obtain $\rho_2 > \rho_2 + \lambda$, a contradiction.

Case (2_3): $a_2 < \sigma_{u,v} < b_2$.

$$
\begin{aligned}
2\rho_2 &\geq 2v(\sigma_{u,v}) = 2\lambda + 2T_2(u,v)(\sigma_{u,v}) \\
&= 2\lambda + \int_0^{\sigma_{u,v}}\varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + B_2\left(\varphi_{p_2}^{-1}\left(\int_0^{\sigma_{u,v}} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)\right) \\
&\quad + \int_{\sigma_{u,v}}^1\varphi_{p_2}^{-1}\left(\int_{\sigma_{u,v}}^s g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\geq 2\lambda + \int_{a_2}^{\sigma_{u,v}}\varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{\sigma_{u,v}} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right) \\
&\quad + \int_{\sigma_{u,v}}^{b_2}\varphi_{p_2}^{-1}\left(\int_{\sigma_{u,v}}^s g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&= 2\lambda + \rho_2\int_{a_2}^{\sigma_{u,v}}\varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_2-1}}\,d\tau\right)ds \\
&\quad + \rho_2 h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{\sigma_{u,v}} g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_2-1}}\,d\tau\right) \\
&\quad + \rho_2\int_{\sigma_{u,v}}^{b_2}\varphi_{p_2}^{-1}\left(\int_{\sigma_{u,v}}^s g_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{\rho_2^{p_2-1}}\,d\tau\right)ds \\
&\geq 2\lambda + \rho_2\varphi_{p_2}^{-1}(f_{2,(\rho_1,\rho_2)})\left[\int_{a_2}^{\sigma_{u,v}}\varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)\,d\tau\right)ds\right. \\
&\quad \left. + h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{\sigma_{u,v}} g_2(\tau)\,d\tau\right) + \int_{\sigma_{u,v}}^{b_2}\varphi_{p_2}^{-1}\left(\int_{\sigma_{u,v}}^s g_2(\tau)\,d\tau\right)ds\right] \\
&\geq 2\lambda + 2\rho_2\varphi_{p_2}^{-1}(f_{2,(\rho_1,\rho_2)})\frac{1}{M_2}.
\end{aligned}
$$

Using hypothesis (3.2) we obtain $\rho_2 > \lambda + \rho_2$, a contradiction. □

Remark 3.4 We point out that a stronger, but easier to check, hypothesis than (3.2) is

$$f_{i,(\rho_1,\rho_2)} > \varphi_{p_i}(\tilde{M}_i),$$

where

$$\frac{1}{\tilde{M}_1} = \int_0^{b_1} \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\,d\tau\right) ds$$

and

$$\frac{1}{\tilde{M}_2} = \frac{1}{2}\min_{a_2\le \nu\le b_2}\left\{\int_{a_2}^{\nu} \varphi_{p_2}^{-1}\left(\int_s^{\nu} g_2(\tau)\,d\tau\right) ds + \int_{\nu}^{b_2} \varphi_{p_2}^{-1}\left(\int_{\nu}^{s} g_2(\tau)\,d\tau\right) ds\right\}.$$

In the following lemma we exploit an idea that was used in [37, 38, 40], and we provide a result of index 0 controlling the growth of just one nonlinearity f_i, at the cost of having a larger domain. Nonlinearities with different growths were considered, for example, in [45–47].

Lemma 3.5 *Assume that*

$(\mathrm{I}^0_{\rho_1,\rho_2})^\star$ *there exist* $\rho_1, \rho_2 > 0$ *such that for some* $i \in \{1,2\}$ *we have*

$$f^*_{i,(\rho_1,\rho_2)} > \varphi_{p_i}(M_i), \tag{3.3}$$

where

$$f^*_{i,(\rho_1,\rho_2)} = \inf\left\{\frac{f_i(t,u,v)}{\rho_i^{p_i-1}} : (t,u,v) \in [a_i,b_i]\times[0,\rho_1]\times[0,\rho_2]\right\}.$$

Then $i_K(T, V_{\rho_1,\rho_2}) = 0$.

Proof Suppose that condition (3.3) holds for $i = 1$. Let $(u,v) \in \partial V_{\rho_1,\rho_2}$ and $\lambda \ge 0$ such that $(u,v) = T(u,v) + \lambda(e,e)$. Thus we proceed as in the proof of Lemma 3.3. □

The proof of the next result regarding the existence of at least one, two or three positive solutions follows by the properties of fixed point index and is omitted. It is possible to state results for four or more positive solutions, in a similar way as in [48], by expanding the lists in conditions (S_5), (S_6).

Theorem 3.6 *System* (2.1) *has at least one positive solution in* K *if one of the following conditions holds.*

- (S_1) *For* $i = 1,2$, *there exist* $\rho_i, r_i \in (0,\infty)$ *with* $\rho_i < r_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ [*or* $(\mathrm{I}^0_{\rho_1,\rho_2})^\star$], $(\mathrm{I}^1_{r_1,r_2})$ *hold.*
- (S_2) *For* $i = 1,2$, *there exist* $\rho_i, r_i \in (0,\infty)$ *with* $\rho_i < c_i r_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ *hold.*

System (2.1) *has at least two positive solutions in* K *if one of the following conditions holds.*

- (S_3) *For* $i = 1,2$, *there exist* $\rho_i, r_i, s_i \in (0,\infty)$ *with* $\rho_i < r_i < c_i s_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ [*or* $(\mathrm{I}^0_{\rho_1,\rho_2})^\star$], $(\mathrm{I}^1_{r_1,r_2})$ *and* $(\mathrm{I}^0_{s_1,s_2})$ *hold.*

(S_4) *For $i = 1,2$, there exist $\rho_i, r_i, s_i \in (0,\infty)$ with $\rho_i < c_i r_i$ and $r_i < s_i$ such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ *and* $(\mathrm{I}^1_{s_1,s_2})$ *hold.*

System (2.1) *has at least three positive solutions in K if one of the following conditions holds.*

(S_5) *For $i = 1,2$, there exist $\rho_i, r_i, s_i, \delta_i \in (0,\infty)$ with $\rho_i < r_i < c_i s_i$ and $s_i < \delta_i$ such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ *[or* $(\mathrm{I}^0_{\rho_1,\rho_2})^\star$*],* $(\mathrm{I}^1_{r_1,r_2})$, $(\mathrm{I}^0_{s_1,s_2})$ *and* $(\mathrm{I}^1_{\delta_1,\delta_2})$ *hold.*

(S_6) *For $i = 1,2$, there exist $\rho_i, r_i, s_i, \delta_i \in (0,\infty)$ with $\rho_i < c_i r_i$ and $r_i < s_i < c_i\delta_i$ such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$, $(\mathrm{I}^1_{s_1,s_2})$ *and* $(\mathrm{I}^0_{\delta_1,\delta_2})$ *hold.*

4 Non-existence results

We now provide some non-existence results for system (2.1). We use an argument similar to the ones of [40, 49–53].

Theorem 4.1 *Assume that one of the following conditions holds.*

1. *For $i = 1,2$,*

$$f_i(t,u_1,u_2) < \varphi_{p_i}(m_i u_i) \quad \text{for every } t \in [0,1] \text{ and } u_i > 0, \tag{4.1}$$

where m_i is defined in Lemma 3.2.

2. *For $i = 1,2$,*

$$f_i(t,u_1,u_2) > \varphi_{p_i}\left(\frac{M_i}{c_i}u_i\right) \quad \text{for every } t \in [a_i,b_i] \text{ and } u_i > 0, \tag{4.2}$$

where M_i is defined in Lemma 3.3.

3. *There exists $k \in \{1,2\}$ such that* (4.1) *is verified for f_k and for $j \neq k$ condition* (4.2) *is verified for f_j.*

Then there is no positive solution of system (2.1) *in K.*

Proof (1) Assume, on the contrary, that there exists $(u,v) \in K$ such that $(u,v) = T(u,v)$ and $(u,v) \neq (0,0)$. We distinguish two cases.

- Let $\|u\|_\infty \neq 0$. Then we have

$$\begin{aligned} u(t) &= \int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\quad + B_1\left(\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)\right) \\ &< m_1\int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\varphi_{p_1}\big(u(\tau)\big)\,d\tau\right)ds + m_1h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\varphi_{p_1}\big(u(\tau)\big)\,d\tau\right) \\ &\leq m_1\|u\|_\infty\left(\int_t^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\,d\tau\right)ds + h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\,d\tau\right)\right). \end{aligned}$$

Taking $t = 0$ gives

$$\begin{aligned} \|u\|_\infty = u(0) &< m_1\|u\|_\infty\int_0^1 \varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\,d\tau\right)ds + m_1\|u\|_\infty h_{12}\varphi_{p_1}^{-1}\left(\int_0^1 g_1(\tau)\,d\tau\right) \\ &= m_1\|u\|_\infty\frac{1}{m_1}, \end{aligned}$$

a contradiction.

- Let $\|v\|_\infty \neq 0$.

  Reasoning as in Lemma 3.2 we distinguish the cases $\sigma_{u,v} \leq 1/2$ and $\sigma_{u,v} > 1/2$.

  In the first case we have

$$\begin{aligned}
\|v\|_\infty &= \bigl\| T_2(u,v)\bigr\|_\infty \\
&= T_2(u,v)(\sigma_{u,v}) \\
&= \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1}\biggl(\int_s^{\sigma_{u,v}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad + B_2\biggl(\varphi_{p_2}^{-1}\biggl(\int_0^{\sigma_{u,v}} g_2(\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\biggr) \\
&< m_2\|v\|_\infty \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1}\biggl(\int_s^{\sigma_{u,v}} g_2(\tau)\,d\tau\biggr)\,ds \\
&\quad + h_{22}m_2\|v\|_\infty\varphi_{p_2}^{-1}\biggl(\int_0^{\sigma_{u,v}} g_2(\tau)\,d\tau\biggr) \\
&\leq m_2\|v\|_\infty\biggl(\int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\biggl(\int_s^{\frac{1}{2}} g_2(\tau)\,d\tau\biggr)\,ds + h_{22}\varphi_{p_2}^{-1}\biggl(\int_0^{\frac{1}{2}} g_2(\tau)\,d\tau\biggr)\biggr) \\
&\leq m_2\|v\|_\infty\frac{1}{m_2},
\end{aligned}$$

a contradiction.

In a similar manner we proceed in the case $\sigma_{u,v} > 1/2$.

(2) Assume, on the contrary, that there exists $(u,v) \in K$ such that $(u,v) = T(u,v)$ and $(u,v) \neq (0,0)$. We distinguish two cases.

- Let $\|u\|_\infty \neq 0$. Then, for $t \in [a_1,b_1] = [0,b_1]$, we have

$$\begin{aligned}
u(t) &= \int_t^1 \varphi_{p_1}^{-1}\biggl(\int_0^s g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad + B_1\biggl(\varphi_{p_1}^{-1}\biggl(\int_0^1 g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\biggr) \\
&\geq \int_t^{b_1} \varphi_{p_1}^{-1}\biggl(\int_0^s g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad + h_{11}\varphi_{p_1}^{-1}\biggl(\int_0^1 g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr) \\
&\geq \int_t^{b_1} \varphi_{p_1}^{-1}\biggl(\int_0^s g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad + h_{11}\varphi_{p_1}^{-1}\biggl(\int_0^{b_1} g_1(\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr) \\
&> \frac{M_1}{c_1}\int_t^{b_1} \varphi_{p_1}^{-1}\biggl(\int_0^s g_1(\tau)\varphi_{p_1}\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad + \frac{M_1}{c_1}h_{11}\varphi_{p_1}^{-1}\biggl(\int_0^{b_1} g_1(\tau)\varphi_{p_1}\bigl(u(\tau)\bigr)\,d\tau\biggr) \\
&> \frac{M_1}{c_1}\int_t^{b_1} \varphi_{p_1}^{-1}\biggl(\int_0^s g_1(\tau)\varphi_{p_1}\bigl(c_1\|u\|_\infty\bigr)\,d\tau\biggr)\,ds
\end{aligned}$$

$$\begin{aligned}
&+\frac{M_1}{c_1}h_{11}\varphi_{p_1}^{-1}\left(\int_0^{b_1} g_1(\tau)\varphi_{p_1}(c_1\|u\|_\infty)\,d\tau\right)\\
&= M_1\|u\|_\infty\left(\int_t^{b_1}\varphi_{p_1}^{-1}\left(\int_0^s g_1(\tau)\,d\tau\right)ds + h_{11}\varphi_{p_1}^{-1}\left(\int_0^{b_1} g_1(\tau)\,d\tau\right)\right).
\end{aligned}$$

For $t = 0$ we obtain

$$u(0) = \|u\|_\infty > M_1\|u\|_\infty\frac{1}{M_1},$$

a contradiction.

- Let $\|v\|_\infty \neq 0$. We examine the case $\sigma_{u,v} \geq b_2$. We have

$$\begin{aligned}
\|v\|_\infty &= v(\sigma_{u,v}) = T_2(u,v)(\sigma_{u,v})\\
&= \int_0^{\sigma_{u,v}}\varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\\
&\quad + B_2\left(\varphi_{p_2}^{-1}\left(\int_0^{\sigma_{u,v}} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)\right)\\
&\geq \int_{a_2}^{b_2}\varphi_{p_2}^{-1}\left(\int_s^{b_2} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\\
&\quad + h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{b_2} g_2(\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)\\
&> M_2\|v\|_\infty\left(\int_{a_2}^{b_2}\varphi_{p_2}^{-1}\left(\int_s^{b_2} g_2(\tau)\,d\tau\right)ds + h_{21}\varphi_{p_2}^{-1}\left(\int_{a_2}^{b_2} g_2(\tau)\,d\tau\right)\right)\\
&\geq M_2\|v\|_\infty\frac{1}{M_2},
\end{aligned}$$

a contradiction. By similar proofs, the cases $0 < \sigma_{u,v} \leq a_2$ and $a_2 < \sigma_{u,v} < b_2$ can be examined.

(3) Assume, on the contrary, that there exists $(u,v) \in K$ such that $(u,v) = T(u,v)$ and $(u,v) \neq (0,0)$. If $\|u\|_\infty \neq 0$, then the function f_1 satisfies either (4.1) or (4.2), and the proof follows as in the previous cases. If $\|v\|_\infty \neq 0$, then the function f_2 satisfies either (4.1) or (4.2), and the proof follows as in the previous cases. □

5 An example

We illustrate in the following example that all the constants that occur in Theorem 3.6 can be computed.

Consider the system

$$\begin{aligned}
&\big(\varphi_{p_1}(u')\big)'(t) + g_1(t)f_1\big(t,u(t),v(t)\big) = 0, \quad t \in (0,1),\\
&\big(\varphi_{p_2}(v')\big)'(t) + g_2(t)f_2\big(t,u(t),v(t)\big) = 0, \quad t \in (0,1),
\end{aligned} \tag{5.1}$$

subject to the boundary conditions

$$u'(0) = 0, \qquad u(1) + B_1\big(u'(1)\big) = 0, \qquad v(0) = B_2\big(v'(0)\big), \qquad v(1) = 0, \tag{5.2}$$

where B_1 and B_2 are defined by

$$B_1(w) = \begin{cases} w, & w \le 0, \\ \frac{w}{2}, & 0 \le w \le 1, \\ \frac{w}{6} + \frac{1}{3}, & w \ge 1, \end{cases}$$

and

$$B_2(w) = \begin{cases} \frac{w}{3}, & 0 \le w \le 1, \\ \frac{w}{9} + \frac{2}{9}, & w \ge 1. \end{cases}$$

Now we assume $g_1 = g_2 \equiv 1$. Thus we have

$$\begin{aligned} \frac{1}{m_1} &= \frac{p_1 - 1}{p_1} + h_{12}, \\ \frac{1}{m_2} &= \frac{p_2 - 1}{p_2}\left(\frac{1}{2}\right)^{\frac{p_2}{p_2-1}} + h_{22}\left(\frac{1}{2}\right)^{\frac{1}{p_2-1}}, \\ \frac{1}{M_1} &= \frac{1}{M_1[0,b_1]} = \frac{p_1-1}{p_1} b_1^{\frac{p_1}{p_1-1}} + h_{11} b_1^{\frac{1}{p_1-1}} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{M_2} &= \frac{1}{M_2[a_2,b_2]} \\ &= \frac{1}{2} \min_{a_2 \le v \le b_2} \left(\frac{p_2-1}{p_2} \left((v-a_2)^{\frac{p_2}{p_2-1}} + (b_2 - v)^{\frac{p_2}{p_2-1}} \right) + h_{21}(v-a_2)^{\frac{1}{p_2-1}} \right). \end{aligned}$$

The choice $p_1 = \frac{3}{2}$, $p_2 = 3$, $b_1 = \frac{2}{3}$, $a_2 = \frac{1}{4}$, $b_2 = \frac{3}{4}$, $h_{11} = \frac{1}{6}$, $h_{12} = \frac{1}{2}$, $h_{21} = \frac{1}{9}$ and $h_{22} = \frac{1}{3}$ gives, by direct computation,

$$c_1 = \frac{1}{3}; \qquad c_2 = \frac{1}{4}; \qquad m_1 = 1.2; \qquad M_1 = 5.785; \qquad m_2 = 2.121; \qquad M_2 = 9.145.$$

Let us now consider

$$f_1(t,u,v) = \frac{1}{16}\left(u^4 + t^3 v^3\right) + \frac{27}{50}, \qquad f_2(t,u,v) = \sqrt{tu} + 10v^9.$$

Then, with the choice of $\rho_1 = \rho_2 = \frac{1}{20}$, $r_1 = 1$, $r_2 = \frac{2}{3}$, $s_1 = s_2 = 9$, we obtain

$$\begin{aligned} &\inf\left\{f_1(t,u,v) : (t,u,v) \in \left[0, \frac{2}{3}\right] \times [0,\rho_1] \times [0,\rho_2]\right\} \\ &\quad = f_1(0,0,0) = 0.54 > \sqrt{M_1\rho_1} = 0.538, \\ &\sup\{f_1(t,u,v) : (t,u,v) \in [0,1] \times [0,r_1] \times [0,r_2]\} \\ &\quad = f_1(1,r_1,r_2) = 0.62 < \sqrt{m_1 r_1} = 1.095, \\ &\sup\{f_2(t,u,v) : (t,u,v) \in [0,1] \times [0,r_1] \times [0,r_2]\} \\ &\quad = f_2(1,r_1,r_2) = 1.260 < (m_2 r_2)^2 = 2, \end{aligned}$$

$$\begin{aligned}
&\inf\left\{f_1(t,u,v):(t,u,v)\in\left[0,\frac{2}{3}\right]\times[c_1s_1,s_1]\times[0,s_2]\right\}\\
&\quad=f_1(0,c_1s_1,0)=5.602>\sqrt{M_1s_1}=1.247,\\
&\inf\left\{f_2(t,u,v):(t,u,v)\in\left[\frac{1}{4},\frac{3}{4}\right]\times[0,s_1]\times[c_2s_2,s_2]\right\}\\
&\quad=f_2(t,0,c_2s_2)=14778.9>M_2^2s_2^2=6774.07.
\end{aligned}$$

Thus the conditions $(\mathrm{I}^0_{\frac{1}{20},\frac{1}{20}})^\star$, $(\mathrm{I}^1_{1,\frac{2}{3}})$ and $(\mathrm{I}^0_{9,9})$ are satisfied; therefore system (5.1)-(5.2) has at least two positive solutions (u_1,v_1) and (u_2,v_2) such that $\frac{1}{20}<\|(u_1,v_1)\|\leq 1$ and $1<\|(u_2,v_2)\|\leq 9$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the anonymous reviewers for the careful reading of the manuscript and for the constructive comments.

Received: 5 June 2015 Accepted: 11 August 2015 Published online: 17 September 2015

References

1. Wang, J: The existence of positive solutions for the one-dimensional *p*-Laplacian. Proc. Am. Math. Soc. **125**(8), 2275-2283 (1997)
2. Karakostas, GL: Positive solutions for the Φ-Laplacian when Φ is a sup-multiplicative-like function. Electron. J. Differ. Equ. **2004**, 68 (2004)
3. Agarwal, RP, Lü, H, O'Regan, D: Eigenvalues and the one-dimensional *p*-Laplacian. J. Math. Anal. Appl. **266**(2), 383-400 (2002)
4. Bai, D, Chen, Y: Three positive solutions for a generalized Laplacian boundary value problem with a parameter. Appl. Math. Comput. **219**(9), 4782-4788 (2013)
5. Infante, G, Maciejewski, M, Precup, R: A topological approach to the existence and multiplicity of positive solutions of (p,q)-Laplacian systems. Dyn. Partial Differ. Equ. **12**, 193-215 (2015)
6. Kim, CG: The three-solutions theorem for *p*-Laplacian boundary value problems. Nonlinear Anal. **75**(2), 924-931 (2012)
7. Lü, H, O'Regan, D, Zhong, C: Multiple positive solutions for the one-dimensional singular *p*-Laplacian. Appl. Math. Comput. **133**(2-3), 407-422 (2002)
8. Sim, I, Lee, YH: A new solution operator of one-dimensional *p*-Laplacian with a sign-changing weight and its application. Abstr. Appl. Anal. **2012**, Article ID 243740 (2012)
9. Wang, Z, Zhang, Z: Positive solutions for one-dimensional *p*-Laplacian boundary value problems with dependence on the first order derivative. J. Math. Anal. Appl. **314**(2), 618-630 (2006)
10. Yang, Z, O'Regan, D: Positive solutions of a focal problem for one-dimensional *p*-Laplacian equations. Math. Comput. Model. **55**(7-8), 1942-1950 (2012)
11. Avery, R, Henderson, J: Existence of three positive pseudo-symmetric solutions for a one-dimensional *p*-Laplacian. J. Math. Anal. Appl. **277**(2), 395-404 (2003)
12. Bachouche, K, Djebali, S, Moussaoui, T: ϕ-Laplacian BVPS with linear bounded operator conditions. Arch. Math. **48**(2), 121-137 (2012)
13. Bai, C, Fang, J: Existence of multiple positive solutions for nonlinear *m*-point boundary value problems. J. Math. Anal. Appl. **281**(1), 76-85 (2003)
14. Calvert, BD: One-dimensional nonlinear Laplacians under a 3-point boundary condition. Acta Math. Sin. Engl. Ser. **26**(9), 1641-1652 (2010)
15. Feng, H, Ge, W, Jiang, M: Multiple positive solutions for m-point boundary-value problems with a one-dimensional *p*-Laplacian. Nonlinear Anal. **68**(8), 2269-2279 (2008)
16. He, X, Ge, W: A remark on some three-point boundary value problems for the one-dimensional *p*-Laplacian. Z. Angew. Math. Mech. **82**(10), 728-731 (2002)
17. Karakostas, GL: Solvability of the Φ-Laplacian with nonlocal boundary conditions. Appl. Math. Comput. **215**(2), 514-523 (2009)
18. Wang, Y, Ge, W: Existence of multiple positive solutions for multipoint boundary value problems with a one-dimensional *p*-Laplacian. Nonlinear Anal. **67**(2), 476-485 (2007)
19. Zhang, Y: Existence and multiplicity results for a class of generalized one-dimensional *p*-Laplacian problem. Nonlinear Anal. **72**(2), 748-756 (2010)
20. Li, J, Shen, J: Existence of three positive solutions for boundary value problems with *p*-Laplacian. J. Math. Anal. Appl. **311**(2), 457-465 (2005)

21. Ma, DX, Du, ZJ, Ge, W: Existence and iteration of monotone positive solutions for multipoint boundary value problem with p-Laplacian operator. Comput. Math. Appl. **50**(5-6), 729-739 (2005)
22. Wang, Y, Ge, W: Positive solutions for multipoint boundary value problems with a one-dimensional p-Laplacian. Nonlinear Anal. **66**(6), 1246-1256 (2007)
23. Wang, Y, Hou, C: Existence of multiple positive solutions for one-dimensional p-Laplacian. J. Math. Anal. Appl. **315**(1), 144-153 (2006)
24. Cheng, X, Lü, H: Multiplicity of positive solutions for a (p_1, p_2)-Laplacian system and its applications. Nonlinear Anal., Real World Appl. **13**(5), 2375-2390 (2012)
25. Lan, KQ, Zhang, Z: Nonzero positive weak solutions of systems of p-Laplace equations. J. Math. Anal. Appl. **394**(2), 581-591 (2012)
26. Lee, EK, Lee, YH: A multiplicity result for generalized Laplacian systems with multiparameters. Nonlinear Anal. **71**(12), e366-e376 (2009)
27. Xu, X, Lee, YH: Some existence results of positive solutions for ϕ-Laplacian systems. Abstr. Appl. Anal. **2014**, Article ID 814312 (2014)
28. Jebelean, P, Precup, R: Solvability of (p, q)-Laplacian systems with potential boundary conditions. Appl. Anal. **89**(2), 221-228 (2010)
29. Prasad, K, Kumar, K, Murali, P: Solvability of higher order (p, q)-Laplacian two-point boundary value problems. J. Appl. Computat. Math. **3**(6), 182 (2014)
30. Su, H, Wei, Z, Xu, F: The existence of countably many positive solutions for a system of nonlinear singular boundary value problems with the p-Laplacian operator. J. Math. Anal. Appl. **325**(1), 319-332 (2007)
31. Yang, Z: Positive solutions for a system of p-Laplacian boundary value problems. Comput. Math. Appl. **62**(12), 4429-4438 (2011)
32. Goodrich, CS: Positive solutions to boundary value problems with nonlinear boundary conditions. Nonlinear Anal. **75**, 417-432 (2012)
33. Goodrich, CS: On nonlinear boundary conditions satisfying certain asymptotic behavior. Nonlinear Anal. **76**, 58-67 (2013)
34. Infante, G: Nonlocal boundary value problems with two nonlinear boundary conditions. Commun. Appl. Anal. **12**, 279-288 (2008)
35. Infante, G, Minhós, FM, Pietramala, P: Non-negative solutions of systems of ODEs with coupled boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **17**, 4952-4960 (2012)
36. Infante, G, Pietramala, P: A cantilever equation with nonlinear boundary conditions. Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I **2009**, 15 (2009)
37. Infante, G, Pietramala, P: Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations. Nonlinear Anal. **71**, 1301-1310 (2009)
38. Infante, G, Pietramala, P: Multiple nonnegative solutions of systems with coupled nonlinear boundary conditions. Math. Methods Appl. Sci. **37**(14), 2080-2090 (2014)
39. Pietramala, P: A note on a beam equation with nonlinear boundary conditions. Bound. Value Probl. **2011**, Article ID 376782 (2011)
40. Infante, G, Pietramala, P: Nonzero radial solutions for a class of elliptic systems with nonlocal BCs on annular domains. Nonlinear Differ. Equ. Appl. **22**, 979-1003 (2015). doi:10.1007/s00030-015-0311-8
41. Amann, H: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. **18**, 620-709 (1976)
42. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, Boston (1988)
43. Franco, D, Infante, G, O'Regan, D: Nontrivial solutions in abstract cones for Hammerstein integral systems. Dyn. Contin. Discrete Impuls. Syst. **14**, 837-850 (2007)
44. Lan, KQ: Multiple positive solutions of semilinear differential equations with singularities. J. Lond. Math. Soc. **63**, 690-704 (2001)
45. Precup, R: Componentwise compression-expansion conditions for systems of nonlinear operator equations and applications. In: Mathematical Models in Engineering, Biology and Medicine. AIP Conf. Proc., vol. 1124, pp. 284-293. Amer. Inst. Phys, Melville, NY (2009)
46. Precup, R: Existence, localization and multiplicity results for positive radial solutions of semilinear elliptic systems. J. Math. Anal. Appl. **352**, 48-56 (2009)
47. Yang, Z: Positive solutions to a system of second-order nonlocal boundary value problems. Nonlinear Anal. **62**, 1251-1265 (2005)
48. Lan, KQ: Multiple positive solutions of Hammerstein integral equations with singularities. Differ. Equ. Dyn. Syst. **8**, 175-195 (2000)
49. Dunninger, DR, Wang, H: Multiplicity of positive solutions for a nonlinear differential equation with nonlinear boundary conditions. Ann. Pol. Math. **69**(2), 155-165 (1998)
50. Graef, JR, Qian, C, Yang, B: A three point boundary value problem for nonlinear fourth order differential equations. J. Math. Anal. Appl. **287**(1), 217-233 (2003)
51. Infante, G, Pietramala, P, Tojo, FAF: Nontrivial solutions of local and nonlocal Neumann boundary value problems. Proc. Roy. Soc. Edinburgh Sect. A. to appear
52. Karakostas, GL: Nonexistence of solutions to some boundary-value problems for second-order ordinary differential equations. Electron. J. Differ. Equ. **2012**, 20 (2012)
53. Karakostas, GL: Non-existence of solutions for two-point fractional and third-order boundary-value problems. Electron. J. Differ. Equ. **2013**, 152 (2013)