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Blow-up criteria of smooth solutions to the three-dimensional magneto-micropolar fluid equations

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Abstract
In this short article, the initial value problem for the 3D magneto-micropolar fluid equations is investigated. Some new blow-up criteria of smooth solutions in terms of the vorticity and the velocity in a homogenous Besov space are established, respectively.

MSC: 76A15; 76B03

Keywords: magneto-micropolar fluid equations; smooth solutions; blow-up criteria

1 Introduction

In the short article, we consider the initial value problem for three-dimensional magneto-micropolar fluid equations

$$\begin{cases} \partial_t u - (\mu+\chi)\Delta u + u\cdot\nabla u - b\cdot\nabla b + \nabla(p+\frac{1}{2}|b|^2) - \chi\nabla\times v = 0, \\ \partial_t v - \gamma\Delta v - \kappa\nabla\nabla\cdot v + 2\chi v + u\cdot\nabla v - \chi\nabla\times u = 0, \\ \partial_t b - \nu\Delta b + u\cdot\nabla b - b\cdot\nabla u = 0, \\ \nabla\cdot u = 0, \qquad \nabla\cdot b = 0, \end{cases} \tag{1.1}$$

with the initial value

$$t=0: \quad u=u_0(x), \qquad v=v_0(x), \qquad b=b_0(x), \tag{1.2}$$

where $u(t,x)$, $v(t,x)$, $b(t,x)$ and $p(t,x)$ denote the velocity of the fluid, the micro-rotational velocity, magnetic field and hydrostatic pressure, respectively. μ is the kinematic viscosity, χ is the vortex viscosity, γ and κ are spin viscosities, and $\frac{1}{\nu}$ is the magnetic Reynold.

Lots of physicists and mathematicians have studied the incompressible magneto-micropolar fluid equations because the equations have rich phenomena, important physical background and mathematical complexity and challenges. On the one hand, for well-posedness of solutions to problem (1.1), (1.2), we refer to [1–4] and [5] and the references cited therein. On the other hand, for the blow-up criteria of smooth solutions and regularity criteria of weak solutions, we refer to [6–8] and [5, 9, 10].

If $b=0$, (1.1) reduces to micropolar fluid equations. The micropolar fluid equations were first proposed by Eringen [11] (see also [12]). The study of the micropolar fluid equations attracts lots of physicists and mathematicians' attention, and many interesting results have been established. For instance, we refer to [13–18] and [19]. If both $\nu = 0$ and $\chi = 0$, then equations (1.1) reduce to being the magneto-hydrodynamic (MHD) equations. The MHD equations govern the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas, liquid metals, salt water, *etc.* (see [20]). The field of MHD was initiated by Hannes Alfvén, for which he received the Nobel Prize in physics in 1970. For global well-posedness of solutions to the MHD equations, there are a few results, we refer to [21, 22]. When the magnetic fields are purely swirling and prependicular to the velocity fields, Lei proved global existence of solutions. Wang and Wang proved global existence of solutions in the critical space χ^{-1}, which was introduced in [23] and used in studying the global well-posedness of the incompressible Navier-Stokes equations by Lei and Lin [24] provided that the norm of initial norm of the initial value are bounded exactly by the minimal value of the viscosity coefficients. We also emphasize the various regularity criteria and blow-up criteria in [25–33] and [34]. Regularity criterion of weak solutions to the MHD equations in terms of the vorticity was established in [34]. Lei and Zhou [31] derived a criterion for the breakdown of classical solutions to the incompressible magneto-hydrodynamic equations with zero viscosity and positive resistivity.

In the absence of global well-posedness, the development of blow-up/non blow-up theory is of major importance for both theoretical and practical purposes. The purpose of this paper is to establish the blow-up criteria of smooth solutions to (1.1), (1.2). The results obtained in this paper extend the MHD results in [34] to complex fluid equations (1.1). We state our main results as follows.

Theorem 1.1 *Assume that* $u_0, v_0, b_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$ *with* $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. *Let* (u, v, b) *be a smooth solution to problem* (1.1), (1.2) *for* $0 \leq t < T$. *If* u *satisfies*

$$\int_0^T \|\nabla \times u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}} \, dt < \infty, \tag{1.3}$$

then the solution (u, v, b) *can be extended beyond* $t = T$.

We have the following corollary immediately.

Corollary 1.1 *Assume that* $u_0, v_0, b_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$ *with* $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. *Let* (u, v, b) *be a smooth solution to problem* (1.1), (1.2) *for* $0 \leq t < T$. *Suppose that* T *is the maximal existence time, then*

$$\int_0^T \|\nabla \times u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}} \, dt = \infty. \tag{1.4}$$

Noticing the equivalence of the norm $\|\nabla \times u\|_{\dot{B}^{-1}_{\infty,\infty}}$ *and* $\|u(t)\|_{\dot{B}^{0}_{\infty,\infty}}$, *from Theorem* 1.1, *we immediately obtain the following.*

Corollary 1.2 *Assume that* $u_0, v_0, b_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$ *with* $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. *Let* (u, v, b) *be a smooth solution to problem* (1.1), (1.2) *for* $0 \leq t < T$. *If* u *satisfies*

$$\int_0^T \|u(t)\|^2_{\dot{B}^{0}_{\infty,\infty}} \, dt < \infty, \tag{1.5}$$

then the solution (u, v, b) *can be extended beyond* $t = T$.

Corollary 1.2 implies the following result.

Corollary 1.3 *Assume that $u_0, v_0, b_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$ with $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. Let (u, v, b) be a smooth solution to problem* (1.1), (1.2) *for $0 \leq t < T$. Suppose that T is the maximal existence time, then*

$$\int_0^T \|u(t)\|^2_{\dot{B}^0_{\infty,\infty}}\,dt = \infty. \tag{1.6}$$

The paper is organized as follows. We first state some function spaces and important inequalities in Section 2. Then we prove our main results in Section 3.

2 Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx,$$

and for any given $g \in \mathcal{S}(\mathbb{R}^n)$, its inverse Fourier transform $\mathcal{F}^{-1}g = \check{g}$ is defined by

$$\check{g}(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

Firstly, we recall the Littlewood-Paley decomposition. Choose a nonnegative radial function $\phi \in \mathcal{S}(\mathbb{R}^n)$, supported in $\mathcal{C} = \{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, such that

$$\sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}.$$

The frequency localization operator is defined by

$$\triangle_k f = \int_{\mathbb{R}^n} \check{\phi}(y) f(x - 2^{-k}y)\,dy.$$

Next we recall the definition of homogeneous function spaces (see [35]). For $(p, q) \in [1, \infty]^2$ and $s \in \mathbb{R}$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined as the set of f up to polynomials such that

$$\|f\|_{\dot{B}^s_{p,q}} \triangleq \big\| 2^{ks} \|\triangle_k f\|_{L^p} \big\|_{l^q(\mathbb{Z})} < \infty.$$

BMO denotes the homogenous space of bounded mean oscillations associated with the norm

$$\|f\|_{BMO} \triangleq \sup_{x\in\mathbb{R}^n, R>0} \frac{1}{|B_R(x)|} \int_{B_R(x)} \left| f(y) - \frac{1}{|B_R(y)|} \int_{B_R(y)} f(z)\,dz \right| dy.$$

The following inequality is the well-known Gagliardo-Nirenberg inequality.

Lemma 2.1 *Let j, m be any integers satisfying $0 \le j < m$, and let $1 \le q, r \le \infty$, and $p \in \mathbb{R}$, $\frac{j}{m} \le \theta \le 1$ such that*

$$\frac{1}{p} - \frac{j}{n} = \theta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q}.$$

Then, for all $f \in L^q(\mathbb{R}^n) \cap W^{m,r}(\mathbb{R}^n)$, there is a positive constant C depending only on n, m, j, q, r, θ such that the following inequality holds:

$$\|\nabla^j f\|_{L^p} \le C\|f\|_{L^q}^{1-\theta}\|\nabla^m f\|_{L^r}^{\theta} \tag{2.1}$$

with the following exception: *if $1 < r < \infty$ and $m - j - \frac{n}{r}$ is a nonnegative integer, then* (2.1) *holds only for satisfying $\frac{j}{m} \le \theta < 1$.*

In order to prove our main result, we need the following lemma, which may be found in [36].

Lemma 2.2 *There exists a positive constant C such that*

$$\|f\|_{BMO} \le C\big(1 + \|f\|_{\dot{B}^0_{\infty,\infty}}\sqrt{\ln\big(e + \|\nabla^3 f\|_{L^2}\big)}\big). \tag{2.2}$$

We also need the following lemma, which may be found in [37].

Lemma 2.3 *Assume that f, g satisfy $\nabla \cdot f = 0$ and $\nabla \times g = 0$. Then*

$$\|fg\|_{\mathcal{H}^1} \le C\|f\|_{L^2}\|g\|_{L^2}. \tag{2.3}$$

3 Proof of main results

Proof of Theorem 1.1 It follows from (1.1) and energy estimate that

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \big(\mu\|\nabla u(\tau)\|_{L^2}^2 + \gamma\|\nabla v(\tau)\|_{L^2}^2\big)\,d\tau \\
&\quad + 2\int_0^t \Big(\kappa\|\nabla\cdot v(\tau)\|_{L^2}^2 + \frac{\chi}{2}\|v(\tau)\|_{L^2}^2 + \nu\|\nabla b(\tau)\|_{L^2}^2\Big)\,d\tau \\
&\quad \le \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.
\end{aligned} \tag{3.1}$$

Applying ∇ to the first equation in (1.1) and multiplying the resulting equation by ∇u and integrating with respect to x on $\mathbb{R}^3$, using integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + (\mu+\chi)\|\nabla^2 u(t)\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \nabla(u\cdot\nabla u)\cdot\nabla u\,dx \\
&\qquad + \int_{\mathbb{R}^3} \nabla(b\cdot\nabla b)\cdot\nabla u\,dx + \chi\int_{\mathbb{R}^3} \nabla(\nabla\times v)\cdot\nabla u\,dx.
\end{aligned} \tag{3.2}$$

Similarly, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla v(t)\right\|_{L^2}^2+\gamma\left\|\nabla^2 v(t)\right\|_{L^2}^2+\kappa\|\nabla\cdot\nabla v\|_{L^2}^2+2\chi\|\nabla v\|_{L^2}^2 = \chi\int_{\mathbb{R}^3}\nabla(\nabla\times u)\cdot\nabla v\,dx-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla v)\cdot\nabla v\,dx \tag{3.3}$$

and

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla b(t)\right\|_{L^2}^2+\nu\left\|\nabla^2 b(t)\right\|_{L^2}^2=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b\,dx+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla u)\cdot\nabla b\,dx. \tag{3.4}$$

Summing up (3.2)-(3.4), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla u(t)\right\|_{L^2}^2+\left\|\nabla v(t)\right\|_{L^2}^2+\left\|\nabla b(t)\right\|_{L^2}^2\right)+(\mu+\chi)\left\|\nabla^2 u(t)\right\|_{L^2}^2\\
&\quad+\gamma\left\|\nabla^2 v(t)\right\|_{L^2}^2+\kappa\|\nabla\cdot\nabla v\|_{L^2}^2+2\chi\|\nabla v\|_{L^2}^2+\nu\left\|\nabla^2 b(t)\right\|_{L^2}^2\\
&\quad=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\,dx+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla b)\cdot\nabla u\,dx+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times v)\cdot\nabla u\,dx\\
&\quad+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times u)\cdot\nabla v\,dx-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla v)\cdot\nabla v\,dx-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b\,dx\\
&\quad+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla u)\cdot\nabla b\,dx.
\end{aligned} \tag{3.5}$$

By integration by parts and the Cauchy inequality, we obtain

$$\chi\int_{\mathbb{R}^3}\nabla(\nabla\times v)\cdot\nabla u\,dx+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times u)\cdot\nabla v\,dx\le\chi\left\|\nabla^2 u\right\|_{L^2}^2+\frac{3\chi}{2}\|\nabla v\|_{L^2}^2. \tag{3.6}$$

Using integration by parts, (2.3) and the Cauchy inequality, we arrive at

$$\begin{aligned}
-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\,dx &= \int_{\mathbb{R}^3}u_i\partial_i u_j\partial_k^2 u_j\,dx\\
&= -\int_{\mathbb{R}^3}\partial_k(u_i\partial_i u_j)\partial_k u_j\,dx\\
&= -\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j\,dx\\
&= \int_{\mathbb{R}^3}\partial_k(\partial_k u_i\partial_i u_j)u_j\,dx\\
&= \int_{\mathbb{R}^3}\partial_k u_i\partial_k\partial_i u_j u_j\,dx\\
&= \int_{\mathbb{R}^3}\partial_k u\cdot\nabla\partial_k u\cdot u\,dx\\
&\le C\|u\|_{BMO}\|\partial_k u\cdot\nabla\partial_k u\|_{\mathcal{H}^1}\\
&\le C\|u\|_{BMO}\|\nabla u\|_{L^2}\left\|\nabla^2 u\right\|_{L^2}\\
&\le \frac{\mu}{2}\left\|\nabla^2 u\right\|_{L^2}^2+C\|u\|_{BMO}^2\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.7}$$

where we have used $\nabla\cdot\partial_k u=0$ and $\nabla\times\nabla\partial_k u=0$.

Integration by parts, $\nabla\cdot\partial_i b = 0$, $\nabla\times\nabla\partial_i b = 0$ and $\nabla\cdot\partial_i^2 b = 0$, $\nabla\times\nabla b = 0$, (2.3) and the Cauchy inequality give

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla(b\cdot\nabla b)\cdot\nabla u\,dx+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla u)\cdot\nabla b\,dx\\
&\quad=\int_{\mathbb{R}^3}\big[\partial_i(b_k\partial_k b_j)\partial_i u_j+\partial_i(b_k\partial_k u_j)\partial_i b_j\big]\,dx\\
&\quad=\int_{\mathbb{R}^3}[\partial_i b_k\partial_k b_j\partial_i u_j+\partial_i b_k\partial_k u_j\partial_i b_j]\,dx\\
&\quad=-\int_{\mathbb{R}^3}u_j\big[\partial_i b_k\partial_k\partial_i b_j+\partial_i^2 b_k\partial_k b_j+\partial_i b_k\partial_i\partial_k b_j\big]\,dx\\
&\quad=-\int_{\mathbb{R}^3}u\cdot\big[(\partial_i b\cdot\nabla)\cdot\partial_i b+\big(\partial_i^2 b\cdot\nabla\big)b+(\partial_i b\cdot\nabla)\partial_i b\big]\,dx\\
&\quad=C\|u\|_{BMO}\|\partial_i b\cdot\nabla\cdot\partial_i b\|_{\mathcal{H}^1}+C\|u\|_{BMO}\big\|\partial_i^2 b\cdot\nabla b\big\|_{\mathcal{H}^1}+C\|u\|_{BMO}\|\partial_i b\cdot\nabla\partial_i b\|_{\mathcal{H}^1}\\
&\quad\le\frac{\nu}{4}\big\|\nabla^2 b\big\|_{L^2}^2+C\|u\|_{BMO}^2\|\nabla b\|_{L^2}^2.
\end{aligned}\tag{3.8}$$

By the method to obtain (3.16) in [38], we have

$$-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla v)\cdot\nabla v\,dx\le\frac{\gamma}{2}\big\|\nabla^2 v\big\|_{L^2}^2+C\|u\|_{BMO}^2\|\nabla v\|_{L^2}^2.\tag{3.9}$$

Similar to the proof of (3.9), we arrive at

$$-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b\,dx\le\frac{\nu}{4}\big\|\nabla^2 b\big\|_{L^2}^2+C\|u\|_{BMO}^2\|\nabla b\|_{L^2}^2.\tag{3.10}$$

Inserting (3.6)-(3.10) into (3.5) yields

$$\begin{aligned}
&\frac{d}{dt}\big(\big\|\nabla u(t)\big\|_{L^2}^2+\big\|\nabla v(t)\big\|_{L^2}^2+\big\|\nabla b(t)\big\|_{L^2}^2\big)+\big\|\nabla^2 u(t)\big\|_{L^2}^2\\
&\qquad+\big\|\nabla^2 v(t)\big\|_{L^2}^2+\|\nabla\cdot\nabla v\|_{L^2}^2+\big\|\nabla v(t)\big\|_{L^2}^2+\big\|\nabla^2 b(t)\big\|_{L^2}^2\\
&\quad\le C\|u\|_{BMO}^2\big(\big\|\nabla u(t)\big\|_{L^2}^2+\big\|\nabla v(t)\big\|_{L^2}^2+\big\|\nabla b(t)\big\|_{L^2}^2\big).
\end{aligned}\tag{3.11}$$

Owing to (1.3), we know that for any small constant $\varepsilon > 0$, there exists $T_\star < T$ such that

$$\int_{T_\star}^{T}\big\|\nabla\times u(t)\big\|_{\dot{B}_{\infty,\infty}^{-1}}^2\,dt\le\varepsilon.\tag{3.12}$$

Let

$$X(t)=\sup_{T_\star\le\tau\le t}\big(\big\|\nabla^3 u(\tau)\big\|_{L^2}^2+\big\|\nabla^3 v(\tau)\big\|_{L^2}^2+\big\|\nabla^3 b(\tau)\big\|_{L^2}^2\big),\quad T_\star\le t<T.\tag{3.13}$$

Integrating (3.11) with respect to t, we have

$$\begin{aligned}
&\big\|\nabla u(t)\big\|_{L^2}^2+\big\|\nabla v(t)\big\|_{L^2}^2+\big\|\nabla b(t)\big\|_{L^2}^2+\int_{T_*}^{t}\big(\big\|\nabla^2 u(\tau)\big\|_{L^2}^2+\big\|\nabla^2 v(\tau)\big\|_{L^2}^2\big)\,d\tau\\
&\qquad+\int_{T_*}^{t}\big(\big\|\nabla\cdot\nabla v(\tau)\big\|_{L^2}^2+\big\|\nabla v(\tau)\big\|_{L^2}^2+\big\|\nabla^2 b(\tau)\big\|_{L^2}^2\big)\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\|u(T_*)\|_{L^2}^2 + \|v(T_*)\|_{L^2}^2 + \|b(T_*)\|_{L^2}^2\right) \exp\left\{C \int_{T_*}^t \|u(\tau)\|_{BMO}^2 \, d\tau\right\} \\
&\le \left(\|u(T_*)\|_{L^2}^2 + \|v(T_*)\|_{L^2}^2 + \|b(T_*)\|_{L^2}^2\right) \\
&\quad \times \exp\left\{C \int_{T_*}^t \left[1 + \|u(\tau)\|_{\dot{B}^0_{\infty,\infty}}^2 \ln\left(e + \|\nabla^3 u(\tau)\|_{L^2}\right)\right] d\tau\right\} \\
&\le \left(\|u(T_*)\|_{L^2}^2 + \|v(T_*)\|_{L^2}^2 + \|b(T_*)\|_{L^2}^2\right) \exp\left\{C \ln\left(e + X(t)\right) \int_{T_*}^t \|u(\tau)\|_{\dot{B}^0_{\infty,\infty}}^2 \, d\tau\right\} \\
&\le \left(\|u(T_*)\|_{L^2}^2 + \|v(T_*)\|_{L^2}^2 + \|b(T_*)\|_{L^2}^2\right) \exp\left\{C \ln\left(e + X(t)\right) \int_{T_*}^t \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2 \, d\tau\right\} \\
&\le \left(\|u(T_*)\|_{L^2}^2 + \|v(T_*)\|_{L^2}^2 + \|b(T_*)\|_{L^2}^2\right) \\
&\quad \times \exp\left\{C \ln\left(e + X(t)\right) \int_{T_*}^t \|\nabla \times u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2 \, d\tau\right\} \\
&\le C\left(e + X(t)\right)^{C_1 \varepsilon}. \qquad (3.14)
\end{aligned}
$$

We apply ∇^m to the first equation in (1.1) and multiply the resulting equation by $\nabla^m u$ and integrate with respect to x on $\mathbb{R}^3$, use integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^m u(t)\|_{L^2}^2 + (\mu + \chi)\|\nabla^{m+1} u(t)\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \nabla^m (u \cdot \nabla u) \cdot \nabla^m u \, dx \\
&\qquad + \int_{\mathbb{R}^3} \nabla^m (b \cdot \nabla b) \cdot \nabla^m u \, dx + \chi \int_{\mathbb{R}^3} \nabla^m (\nabla \times v) \cdot \nabla^m u \, dx. \qquad (3.15)
\end{aligned}
$$

Similarly, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^m v(t)\|_{L^2}^2 + \gamma\|\nabla^{m+1} v(t)\|_{L^2}^2 + \kappa\|\nabla \cdot \nabla^m v\|_{L^2}^2 + 2\chi\|\nabla^m v\|_{L^2}^2 \\
&\quad = \chi \int_{\mathbb{R}^3} \nabla^m (\nabla \times u) \cdot \nabla^m v \, dx - \int_{\mathbb{R}^3} \nabla^m (u \cdot \nabla v) \cdot \nabla^m v \, dx \qquad (3.16)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^m b(t)\|_{L^2}^2 + \nu\|\nabla^{m+1} b(t)\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \nabla^m (u \cdot \nabla b) \cdot \nabla^m b \, dx + \int_{\mathbb{R}^3} \nabla^m (b \cdot \nabla u) \cdot \nabla^m b \, dx. \qquad (3.17)
\end{aligned}
$$

In what follows, for simplicity, we shall set $m = 3$.

Summing up (3.15)-(3.17) and noting $\nabla \cdot u = 0$, $\nabla \cdot b = 0$, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3 v(t)\|_{L^2}^2 + \|\nabla^3 b(t)\|_{L^2}^2\right) + (\mu + \chi)\|\nabla^4 u(t)\|_{L^2}^2 \\
&\qquad + \gamma\|\nabla^4 v(t)\|_{L^2}^2 + \kappa\|\nabla \cdot \nabla^3 v\|_{L^2}^2 + 2\chi\|\nabla^3 v\|_{L^2}^2 + \nu\|\nabla^4 b(t)\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \left[\nabla^3 (u \cdot \nabla u) - u \cdot \nabla\nabla^3 u\right] \cdot \nabla^3 u \, dx + \int_{\mathbb{R}^3} \left[\nabla^3 (b \cdot \nabla b) - b \cdot \nabla\nabla^3 b\right] \cdot \nabla^3 u \, dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \chi \int_{\mathbb{R}^3} \nabla^3(\nabla \times v) \cdot \nabla^3 u \, dx + \chi \int_{\mathbb{R}^3} \nabla^3(\nabla \times u) \cdot \nabla^3 v \, dx \\
&- \int_{\mathbb{R}^3} \bigl[\nabla^3(u \cdot \nabla v) - u \cdot \nabla\nabla^3 v\bigr] \cdot \nabla^3 v \, dx - \int_{\mathbb{R}^3} \bigl[\nabla^3(u \cdot \nabla b) - u \cdot \nabla\nabla^3 b\bigr] \cdot \nabla^3 b \, dx \\
&+ \int_{\mathbb{R}^3} \bigl[\nabla^3(b \cdot \nabla u) - b \cdot \nabla\nabla^3 u\bigr] \cdot \nabla^3 b \, dx \\
=:\ & J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \qquad (3.18)
\end{aligned}
$$

It follows from integration by parts, the Hölder inequality, Gagliardo-Nirenberg inequality (2.1), the Cauchy inequality and (3.14) that

$$
\begin{aligned}
J_1 &\le 4\|\nabla u\|_{L^\infty} \bigl\|\nabla^3 u\bigr\|_{L^2}^2 + 3\bigl\|\nabla^2 u\bigr\|_{L^4} \bigl\|\nabla^3 u\bigr\|_{L^2}^2 \\
&\le C\|\nabla u\|_{L^2}^{\frac{7}{6}} \bigl\|\nabla^4 u\bigr\|_{L^2}^{\frac{11}{6}} \\
&\le \frac{\mu}{8} \bigl\|\nabla^4 u\bigr\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{14} \\
&\le \frac{\mu}{8} \bigl\|\nabla^4 u\bigr\|_{L^2}^2 + C\bigl(e + X(t)\bigr)^{7C_1\epsilon}. \qquad (3.19)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J_2 + J_6 &\le \|\nabla b\|_{L^\infty} \bigl\|\nabla^3 u\bigr\|_{L^2} \bigl\|\nabla^3 b\bigr\|_{L^2} + 3\bigl\|\nabla^2 u\bigr\|_{L^4} \bigl\|\nabla^2 b\bigr\|_{L^4} \bigl\|\nabla^3 b\bigr\|_{L^2} + 3\|\nabla u\|_{L^\infty} \bigl\|\nabla^3 b\bigr\|_{L^2}^2 \\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{3}} \bigl\|\nabla^4 u\bigr\|_{L^2}^{\frac{2}{3}} \|\nabla b\|_{L^2}^{\frac{5}{6}} \bigl\|\nabla^4 b\bigr\|_{L^2}^{\frac{7}{6}} + \|\nabla u\|_{L^2}^{\frac{5}{12}} \bigl\|\nabla^4 u\bigr\|_{L^2}^{\frac{7}{12}} \|\nabla b\|_{L^2}^{\frac{3}{4}} \bigl\|\nabla^4 b\bigr\|_{L^2}^{\frac{5}{4}} \\
&\quad + C\|\nabla u\|_{L^2}^{\frac{1}{2}} \bigl\|\nabla^4 u\bigr\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{2}{3}} \bigl\|\nabla^4 b\bigr\|_{L^2}^{\frac{4}{3}} \\
&\le \frac{\mu}{8} \bigl\|\nabla^4 u\bigr\|_{L^2}^2 + \frac{\nu}{4} \bigl\|\nabla^4 b\bigr\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\nabla b\|_{L^2}^{10} \\
&\quad + C\|\nabla u\|_{L^2}^5 \|\nabla b\|_{L^2}^9 + C\|\nabla u\|_{L^2}^6 \|\nabla b\|_{L^2}^8 \\
&\le \frac{\mu}{8} \bigl\|\nabla^4 u\bigr\|_{L^2}^2 + \frac{\nu}{4} \bigl\|\nabla^4 b\bigr\|_{L^2}^2 + C\bigl(e + X(t)\bigr)^{7C_1\epsilon}. \qquad (3.20)
\end{aligned}
$$

The Cauchy inequality gives

$$
J_3 \le \chi \bigl\|\nabla^4 u\bigr\|_{L^2}^2 + \frac{3\chi}{2} \bigl\|\nabla^3 v\bigr\|_{L^2}^2. \qquad (3.21)
$$

We may obtain the following estimate by making use of integration by parts, the Hölder inequality, Gagliardo-Nirenberg inequality (2.1), the Cauchy inequality and (3.14):

$$
\begin{aligned}
J_4 &\le \|\nabla v\|_{L^\infty} \bigl\|\nabla^3 u\bigr\|_{L^2} \bigl\|\nabla^3 v\bigr\|_{L^2} + 3\bigl\|\nabla^2 u\bigr\|_{L^4} \bigl\|\nabla^2 v\bigr\|_{L^4} \bigl\|\nabla^3 v\bigr\|_{L^2} + 3\|\nabla u\|_{L^\infty} \bigl\|\nabla^3 v\bigr\|_{L^2}^2 \\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{3}} \bigl\|\nabla^4 u\bigr\|_{L^2}^{\frac{2}{3}} \|\nabla v\|_{L^2}^{\frac{5}{6}} \bigl\|\nabla^4 v\bigr\|_{L^2}^{\frac{7}{6}} + \|\nabla u\|_{L^2}^{\frac{5}{12}} \bigl\|\nabla^4 u\bigr\|_{L^2}^{\frac{7}{12}} \|\nabla v\|_{L^2}^{\frac{3}{4}} \bigl\|\nabla^4 v\bigr\|_{L^2}^{\frac{5}{4}} \\
&\quad + C\|\nabla u\|_{L^2}^{\frac{1}{2}} \bigl\|\nabla^4 u\bigr\|_{L^2}^{\frac{1}{2}} \|\nabla v\|_{L^2}^{\frac{2}{3}} \bigl\|\nabla^4 v\bigr\|_{L^2}^{\frac{4}{3}} \\
&\le \frac{\mu}{8} \bigl\|\nabla^4 u\bigr\|_{L^2}^2 + \frac{\gamma}{2} \bigl\|\nabla^4 v\bigr\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\nabla v\|_{L^2}^{10} \\
&\quad + C\|\nabla u\|_{L^2}^5 \|\nabla v\|_{L^2}^9 + C\|\nabla u\|_{L^2}^6 \|\nabla v\|_{L^2}^8 \\
&\le \frac{\mu}{8} \bigl\|\nabla^4 u\bigr\|_{L^2}^2 + \frac{\gamma}{2} \bigl\|\nabla^4 v\bigr\|_{L^2}^2 + C\bigl(e + X(t)\bigr)^{7C_1\epsilon}. \qquad (3.22)
\end{aligned}
$$

Similarly, we deduce that

$$\begin{aligned}
J_5 &\le \|\nabla b\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^3 b\|_{L^2} + 3\|\nabla^2 u\|_{L^4}\|\nabla^2 b\|_{L^4}\|\nabla^3 b\|_{L^2} + 3\|\nabla u\|_{L^\infty}\|\nabla^3 b\|_{L^2}^2 \\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{3}}\|\nabla^4 u\|_{L^2}^{\frac{2}{3}}\|\nabla b\|_{L^2}^{\frac{5}{6}}\|\nabla^4 b\|_{L^2}^{\frac{7}{6}} + \|\nabla u\|_{L^2}^{\frac{5}{12}}\|\nabla^4 u\|_{L^2}^{\frac{7}{12}}\|\nabla b\|_{L^2}^{\frac{3}{4}}\|\nabla^4 b\|_{L^2}^{\frac{5}{4}} \\
&\quad + C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^4 u\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{2}{3}}\|\nabla^4 b\|_{L^2}^{\frac{4}{3}} \\
&\le \frac{\mu}{8}\|\nabla^4 u\|_{L^2}^2 + \frac{\nu}{4}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla b\|_{L^2}^{10} \\
&\quad + C\|\nabla u\|_{L^2}^5\|\nabla b\|_{L^2}^9 + C\|\nabla u\|_{L^2}^6\|\nabla b\|_{L^2}^8 \\
&\le \frac{\mu}{8}\|\nabla^4 u\|_{L^2}^2 + \frac{\nu}{4}\|\nabla^4 b\|_{L^2}^2 + C\big(e + X(t)\big)^{7C_1\epsilon}.
\end{aligned} \tag{3.23}$$

Inserting estimates (3.19)-(3.23) into (3.18) yields

$$\frac{d}{dt}\big(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3 v(t)\|_{L^2}^2 + \|\nabla^3 b(t)\|_{L^2}^2\big) \le C\big(e + X(t)\big)^{7C_1\varepsilon}, \quad t \in [T^*, T). \tag{3.24}$$

Integrating (3.24) with respect to time from T^* to $t \in [T^*, T)$, we have

$$e + \|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3 v(t)\|_{L^2}^2 + \|\nabla^3 b(t)\|_{L^2}^2 \le C + C\big(e + X(t)\big)^{7C_1\varepsilon}. \tag{3.25}$$

By choosing $\varepsilon < \frac{1}{7C_1}$ and noting (3.1), we know that $(u, v, b) \in L^\infty(0, T; H^3(\mathbb{R}^3))$. Thus, (u, v, b) can be extended smoothly beyond $t = T$. We have completed the proof of Theorem 1.1. □

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author completed the paper herself. The author read and approved the final manuscript.

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