Boundary Value Problems
a SpringerOpen Journal

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Homoclinic solutions for n-dimensional prescribed mean curvature p-Laplacian equations

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Abstract

In this paper, a n-dimensional prescribed mean curvature Rayleigh p-Laplacian equation with a deviating argument, $(\varphi_p(\frac{u'(t)}{\sqrt{1+|u'(t)|^2}}))' + F(t,u'(t)) + G(t,u(t-\tau(t))) = e(t)$, is studied. By means of Mawhin's continuation theorem and some analysis methods, a new result on the existence of homoclinic solutions for the equation is obtained. Our research enriches the contents of prescribed mean curvature equations.

Keywords: homoclinic solution; Mawhin's continuation theorem; n-dimensional; prescribed mean curvature; p-Laplacian

1 Introduction

In recent years, the existence of homoclinic solutions has been studied widely for the Hamiltonian systems and the p-Laplacian systems (see [1–4] and the references therein). For example, in [1], Lzydorek and Janczewska studied the existence of homoclinic solutions for second-order Hamiltonian system in the following form:

$$\ddot{q} + V_q(t,q) = f(t),$$

where $q \in \mathbb{R}^n$ and $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $V(t,q) = -K(t,q) + W(t,q)$ is T-periodic with respect to t. Lu in [4] studied the existence of homoclinic solutions for a second-order p-Laplacian differential system with delay

$$\frac{d}{dt}\big[\varphi_p\big(u'(t)\big)\big] + \frac{d}{dt}\nabla F\big(u(t)\big) + \nabla G\big(u(t)\big) + \nabla H\big(u\big(t-\gamma(t)\big)\big) = e(t),$$

where $p \in (1,+\infty)$, $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi_p(u) = (|u_1|^{p-2}u_1, |u_2|^{p-2}u_2, \dots, |u_n|^{p-2}u_n)$ for $u \neq 0 = (0,0,\dots,0)$ and $\varphi_p(0) = (0,0,\dots,0)$, $F \in C^2(\mathbb{R}^n,\mathbb{R})$, $G,H \in C^1(\mathbb{R}^n,\mathbb{R})$, $e \in C(\mathbb{R},\mathbb{R}^n)$, and $\gamma(t)$ is a continuous T-periodic function with $\gamma(t) \geq 0$; T is a given constant.

In the recent past, the prescribed mean curvature equation

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' = f\big(u(t)\big),$$

and its modified forms, which arises from some problems associated to differential geometry and combustible gas dynamics, were studied extensively [5–10]. Also, we note that the existence of periodic solutions for the prescribed curvature mean equations has attracted much attention from researchers. For example, Feng in [11] studied the problem of the existence of periodic solution for a prescribed mean curvature Liénard equation

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f\big(u(t)\big)u'(t) + g\big(t, u\big(t-\tau(t)\big)\big) = e(t), \tag{1.1}$$

where $\tau, e \in C(\mathbb{R}, \mathbb{R})$ are T-periodic, and $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is T-periodic in the first argument, $T > 0$ is a constant. Aiming to apply Mawhin's continuation theorem, Feng made (1.1) equivalent to the following system through the transformation $v(t) = \frac{u'(t)}{\sqrt{1+(u'(t))^2}}$:

$$\begin{cases} u'(t) = \varphi(v(t)) = \frac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -f(t, \varphi(v(t))) - g(t, u(t-\tau(t))) + e(t). \end{cases}$$

Li in [12] further studied the existence of periodic solutions for a prescribed mean curvature Rayleigh equation of the form

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f\big(t, u'(t)\big) + g\big(t, u\big(t-\tau(t)\big)\big) = e(t),$$

and Wang in [13] discussed the following boundary valued problem:

$$\begin{cases} (\varphi_p(\frac{x'(t)}{\sqrt{1+(x'(t))^2}}))' + f(t, x'(t)) + g(t, x(t-\tau(t))) = e(t), \\ x_1(0) = x_1(\omega), \qquad x_2(0) = x_2(\omega), \end{cases} \tag{1.2}$$

where $p > 1$ and $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $e, \tau \in C(\mathbb{R}, \mathbb{R})$, $g(t+\omega, x) = g(t, x)$, $f(t+\omega, x) = f(t, x)$, $f(t, 0) = 0$, $e(t+\omega) = e(t)$ and $\tau(t+\omega) = \tau(t)$. By using a similar transformation in [11], (1.2) is converted to the following system:

$$\begin{cases} x_1'(t) = \phi(x_2(t)) = \frac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}}, \\ x_2'(t) = -f(t, \frac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}}) - g(t, x_1(t-\tau(t))) + e(t), \\ x_1(0) = x_1(\omega), \qquad x_2(0) = x_2(\omega). \end{cases} \tag{1.3}$$

Under the conditions imposed on f and g such as

$$f(t, x) \geq a|x|^r, \quad \forall (t, x) \in \mathbb{R}^2$$

and

$$g(t, x) - e(t) \geq -m_1|x| - m_2, \quad \forall t \in \mathbb{R}, x > d,$$

where $a, r \geq 1$; m_1 and m_2 are positive constants, the author found that (1.2) has at least one periodic solution. It is easy to see from the first equation of (1.3) that the function $\varphi_q(x_2(t))$ must satisfy $\max_{t\in[0,T]} |\varphi_q(x_2(t))| < 1$. This implies that the open and bounded set Ω associated to Mawhin's continuation theorem must satisfy $\overline{\Omega} \in \{(x_1; x_2)^\top \in X : \|x_1\|_\infty <$

$M; \|x_2\|_\infty < 1\}$. Thus, there must be a constant $\rho \in (0,1)$ such that $\Omega \in \{(x_1;x_2)^\top \in X : \|x_1\|_\infty < M; \|x_2\|_\infty < \rho\}$. But in [13], the author obtained $\Omega = \{(x_1;x_2)^\top \in X : \|x_1\|_\infty < M_1; \|x_2\|_\infty < M_2\}$ and there was no proof as regards $M_2 < 1$. A similar problem also occurred in [14].

Inspired by the above fact, the aim of this paper is to investigate the existence of homoclinic solution to the following n-dimensional prescribed mean curvature equation with a deviating argument:

$$\left(\varphi_p\left(\frac{u'(t)}{\sqrt{1+|u'(t)|^2}}\right)\right)' + F\bigl(t,u'(t)\bigr) + G\bigl(t,u\bigl(t-\tau(t)\bigr)\bigr) = e(t), \tag{1.4}$$

where $p \in (1,+\infty)$, $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi_p(u) = (|u_1|^{p-2}u_1, |u_2|^{p-2}u_2, \ldots, |u_n|^{p-2}u_n)$ for $u \neq 0 = (0,0,\ldots,0)$ and $\varphi_p(0) = (0,0,\ldots,0)$, $F \in C(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$, $G \in C(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$, $e \in C(\mathbb{R};\mathbb{R}^n)$, $\tau(t)$ is a continuous T-periodic function and $T > 0$ is given constant.

In order to study the homoclinic solution for (1.4), firstly, like in [1–4, 15] and [16], the existence of a homoclinic solution for (1.4) is obtained as a limit of a certain sequence of $2kT$-periodic solutions for the following equation:

$$\left(\varphi_p\left(\frac{u'(t)}{\sqrt{1+|u'(t)|^2}}\right)\right)' + F\bigl(t,u'(t)\bigr) + G\bigl(t,u\bigl(t-\tau(t)\bigr)\bigr) = e_k(t), \tag{1.5}$$

where $k \in \mathbb{N}$. $e_k : \mathbb{R} \to \mathbb{R}$ is a $2kT$-periodic function such that

$$e_k(t) = \begin{cases} e(t), & t \in [-kT, kT-\varepsilon_0), \\ e(kT-\varepsilon_0) + \frac{e(-kT)-e(kT-\varepsilon_0)}{\varepsilon_0}(t-kT+\varepsilon_0), & t \in [kT-\varepsilon_0, kT], \end{cases} \tag{1.6}$$

where $\varepsilon_0 \in (0,T)$ is a constant independent of k. Obviously, for each $k \in \mathbb{N}$, from (1.6) we observe that $e_k \in C(\mathbb{R},\mathbb{R}^n)$ with $e_k(t+2kT) \equiv e_k(t)$. In this paper, the approach for solving the $2kT$-periodic solutions to (1.5) is based on Mawhin's continuation theorem [17], which is different from the corresponding ones in [1–4] associated to critical point theory.

The rest of this paper organized as follows. In Section 2, we state some necessary definitions and lemmas. In Section 3, we prove the main result.

2 Preliminaries

First of all, we give the definition of the homoclinic solution. A solution $u(t)$ is named homoclinic (to 0) if $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to +\infty$. In addition, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

In the following, we recall some notations and lemmas, which are important for proving our main result.

Throughout this paper, $\|\cdot\|$ will denote the Euclidean norm on $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ denote the standard inner product.

For each $k \in \mathbb{N}$, define

$$C_{2kT} = \bigl\{u \mid u \in C\bigl(\mathbb{R},\mathbb{R}^n\bigr), u(t+2kT) \equiv u(t)\bigr\},$$
$$C^1_{2kT} = \bigl\{u \mid u \in C^1\bigl(\mathbb{R},\mathbb{R}^n\bigr), u(t+2kT) \equiv u(t)\bigr\}$$

and

$$C^2_{2kT} = \{u | u \in C^2(\mathbb{R}, \mathbb{R}^n), u(t + 2kT) \equiv u(t)\}.$$

If the norm of C_{2kT}, C^1_{2kT}, and C^2_{2kT} is defined by $\|\cdot\|_{C_{2kT}} = \|\cdot\|_0$, $\|x\|_{C^1_{2kT}} = \max\{\|x\|_0, \|x'\|_0\}$, and $\|x\|_{C^2_{2kT}} = \max\{\|x\|_0, \|x'\|_0, \|x''\|_0\}$, respectively, then C_{2kT}, C^1_{2kT}, and C^2_{2kT} are all Banach spaces.

Moreover, for any $\psi \in C_{2kT}$, define $\|\psi\|_r = (\int_{-kT}^{kT} |\psi(t)|^r\, dt)^{\frac{1}{r}}$, where $r \in (1, +\infty)$.

In order to use Mawhin's continuation theorem, we first recall it.

Let X and Y be two Banach spaces, a linear operator $L : D(L) \subset X \to Y$ is said to be a Fredholm operator of index zero provided that

(a) $\operatorname{Im} L$ is a closed subset of Y,

(b) $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$.

Let $\Omega \subset X$ be an open and bounded set, and let $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero. This means that there are continuous linear projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$, $X = \ker L \oplus \ker P$ and $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Obviously, $L : D(L) \cap \ker P \to \operatorname{Im} L$ has its right inverse. Let $K_P : \operatorname{Im} L \to D(L) \cap \ker P$ be the right inverse of $L : D(L) \cap \ker P \to \operatorname{Im} L$. A continuous operator $N : \Omega \subset X \to Y$ is said to be L-compact in $\overline{\Omega}$ provided that

(c) $K_p(I - Q)N(\overline{\Omega})$ is a relative compact set of X,

(d) $QN(\overline{\Omega})$ is a bounded set of Y.

Lemma 2.1 ([17]) *Let X and Y be two real Banach spaces, Ω be an open and bounded subset of X, $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero and the operator $N : \overline{\Omega} \subset X \to Y$ be L-compact in $\overline{\Omega}$. In addition, if the following conditions hold*:

(h_1) $Lx \neq \lambda Nx$, $\forall (x, \lambda) \in \partial\Omega \times (0, 1)$;

(h_2) $QNx \neq 0$, $\forall x \in \ker L \cap \partial\Omega$;

(h_3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, *where $J : \operatorname{Im} Q \to \ker L$ is a homeomorphism.*

Then $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Lemma 2.2 ([18]) *Let $0 < \alpha < T$ be a constant, $\tau \in C(\mathbb{R}, \mathbb{R})$ be a T-periodic function and $\max_{t\in[0,T]} |\tau(t)| = \alpha$, then for all $u \in C^1(\mathbb{R}, \mathbb{R})$ with $u(t + T) \equiv u(t)$, we have*

$$\int_0^T |u(t) - u(t - \tau(t))|^2\, dt \le 2\alpha^2 \int_0^T |u'(t)|^2\, dt.$$

Lemma 2.3 ([3]) *If $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}$, $a > 0$, $\mu > 1$, and $p > 1$ are constants, then for every $t \in \mathbb{R}$, the following inequality holds*:

$$|u(t)| \le (2a)^{-\frac{1}{\mu}} \left(\int_{t-a}^{t+a} |u(s)|^\mu\, ds\right)^{\frac{1}{\mu}} + a(2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} |u'(s)|^p\, ds\right)^{\frac{1}{p}}.$$

Lemma 2.4 ([4]) *Suppose $\tau \in C^1(\mathbb{R}, \mathbb{R})$ with $\tau(t + \omega) \equiv \tau(t)$ and $\tau'(t) < 1$, $\forall t \in [0, \omega]$. Then the function $t - \tau(t)$ has an inverse $\mu(t)$ satisfying $\mu \in C(\mathbb{R}, \mathbb{R})$ with $\mu(t + \omega) \equiv \mu(t) + \omega$, $\forall t \in [0, \omega]$.*

Throughout this paper, besides τ being a periodic function with period T, we suppose in addition that $\tau \in C^1(\mathbb{R},\mathbb{R})$ with $\tau'(t) < 1$, $\forall t \in [0,T]$.

Remark 2.1 From the above assumption, one can find from Lemma 2.4 that the function $(t-\tau(t))$ has an inverse denoted by $\mu(t)$. Define $\sigma_0 = -\min_{t\in[0,T]}\tau'(t)$, $\sigma_1 = \max_{t\in[0,T]}\tau'(t)$ and $\|\tau\|_0 = \max_{t\in[0,T]}|\tau(t)|$. Clearly, $\sigma_0 \geq 0$ and $0 \leq \sigma_1 < 1$.

Lemma 2.5 ([3]) *Let $u_k \in C^2_{2kT}$ be a $2kT$-periodic function for each $k \in \mathbb{N}$ with*

$$|u_k|_0 \leq A_0, \qquad \left|u_k'\right|_0 \leq A_1, \qquad \left|u_k''\right|_0 \leq A_2,$$

where A_0, A_1, and A_2 are constants independent of $k \in \mathbb{N}$. Then there exists a function $u \in C^1(\mathbb{R},\mathbb{R}^n)$ such that for each interval $[c,d] \subset \mathbb{R}$, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in\mathbb{N}}$ with $u'_{k_j}(t) \to u'_0(t)$ uniformly on $[c,d]$.

Equation (1.5) is equivalent to the following system:

$$\begin{cases} u'(t) = \phi(v(t)) = \frac{\varphi_q(v(t))}{\sqrt{1-|\varphi_q(v(t))|^2}}, \\ v'(t) = -F(t,\varphi(v(t))) - G(t,u(t-\tau(t))) + e_k(t), \end{cases} \tag{2.1}$$

where $\varphi_q(s) = |s|^{q-2}s$, $\frac{1}{p}+\frac{1}{q}=1$, $v(t) = \varphi_p(\frac{u'(t)}{\sqrt{1+|u'(t)|^2}}) = \phi^{-1}(u'(t))$.

Define

$$X_k = Y_k = \left\{\omega = \left(u(t),v(t)\right)^\top : u \in C_{2kT}, v \in C_{2kT}\right\},$$

and the norm $\|\omega\|_{X_k} = \|\omega\|_{Y_k} = \max\{\|u\|_{2kT}, \|v\|_{2kT}\}$. Obviously, X_k and Y_k are Banach spaces.

Now we define the operator

$$L : D(L) \subset X_k \to Y_k, \qquad L\omega = \omega' = \left(u'(t),v'(t)\right)^\top,$$

where $D(L) = \{\omega | \omega = (u(t),v(t))^\top : u \in C^1_{2kT}, u \in C^1_{2kT}\}$. Let

$$Z_k = \left\{\omega = \left(u(t),v(t)\right)^\top \in X_k : v \in C(R,B_k)\right\},$$

where $B_k = \{x \in \mathbb{R}^n : |x| < 1\}$. The nonlinear operator

$$N : \overline{\Omega} \subset Z_k \to Y_k$$

is defined as

$$N\omega = \left(\frac{\varphi_q(v(t))}{\sqrt{1-|\varphi_q(v(t))|^2}}, -F\left(t, \frac{\varphi_q(v(t))}{\sqrt{1-|\varphi_q(v(t))|^2}}\right) - G\left(t,u\left(t-\tau(t)\right)\right) + e_k(t)\right)^\top,$$

where Ω is an open bounded subset of Z_k. Clearly, the problem of the existence of a $2kT$-periodic solution to (2.1) is equivalent to the problem of the existence of a solution in $\overline{\Omega}$ for the equation $L\omega = N\omega$.

By simple calculating, we have $\ker L = \mathbb{R}^{2n}$ and $\operatorname{Im} L = \{z \in Y_k, \int_0^{2kT} z(s)\,ds = 0\}$. Therefore, L is a Fredholm operator of index zero.

Define

$$P : X_k \to \ker L, \qquad P\omega = \frac{1}{2kT}\int_0^{2kT} \omega(s)\,ds$$

and

$$Q : Y_k \to \operatorname{Im} Q, \qquad Qz = \frac{1}{2kT}\int_0^{2kT} z(s)\,ds.$$

If we define $K_p = L|^{-1}_{\operatorname{Ker} L\cap D(L)}$, then it is easy to see that

$$(K_p z)(t) = \int_0^{2kT} G_k(t,s)z(s)\,ds,$$

where

$$G_k(t) = \begin{cases} \frac{s-2kT}{2kT}, & 0 \le t \le s; \\ \frac{s}{2kT}, & s \le t \le 2kT. \end{cases}$$

For all $\overline{\Omega}$ such that $\overline{\Omega} \subset Z_k \subset X_k$, we can see that $K_p(I-Q)N(\overline{\Omega})$ is a relative compact set of X_k and $QN(\overline{\Omega})$ is a bounded set of Y_k, so the operator N is L-compact in $\overline{\Omega}$.

For the sake of convenience, we list the following assumptions:

(H_1) There are two constants $m_0 > 0$ and $m_1 > 0$ such that

$$\langle x, F(t,x)\rangle \le -m_0|x|^2 \quad \text{and} \quad \left|F(t,x)\right| \le m_1|x|, \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^n.$$

(H_2) There are two constants $\alpha > 0$ and $\beta > 0$ such that

$$\langle x, G(t,x)\rangle \le -\alpha|x|^2 \quad \text{and} \quad \left|G(t,x)\right| \le \beta|x|, \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^n.$$

(H_3) $e \in C(\mathbb{R},\mathbb{R}^n)$ is a bounded function with $e(t) \not\equiv 0 = (0,0,\ldots,0)^\top$ and

$$A := \left(\int_R \left|e(t)\right|^2 dt\right)^{\frac{1}{2}} + \sup_{t\in R}\left|e(t)\right| < +\infty.$$

Remark 2.2 From (1.6), we can see that $|e_k(t)| \le \sup_{t\in R}|e(t)|$. So if ($H_3$) holds, for each $k \in \mathbb{N}$, $(\int_{-kT}^{kT} |e(t)|^2\,dt)^{\frac{1}{2}} < A$.

3 Main results

In order to study the existence of $2kT$-periodic solutions to system (2.1), we firstly study some properties of all possible $2kT$-periodic solutions to the following system:

$$\begin{cases} u'(t) = \lambda\phi(v(t)) = \lambda\frac{\varphi_q(v(t))}{\sqrt{1-|\varphi_q(v(t))|^2}}, \\ v'(t) = -\lambda F(t,\varphi(v(t))) - \lambda G(t,u(t-\tau(t))) + \lambda e_k(t), \quad \lambda \in (0,1], \end{cases} \tag{3.1}$$

where $(u_k, v_k)^\top \in Z_k \subset X_k$. For each $k \in \mathbb{N}$ and all $\lambda \in (0,1]$. Let

$$\Delta = \{\omega = (u,v)^\top \in X_k : L\omega = \lambda N\omega, \lambda \in (0,1]\}.$$

This means that Δ represents the set of all the possible $2kT$-periodic solutions to (3.1).

Theorem 3.1 *Assume that assumptions* (H_1)-(H_3) *hold*, $\frac{\alpha}{1+\sigma_0} > \frac{m_1\beta\sqrt{1-\sigma_1}+\sqrt{2}\beta^2\|\tau\|_0}{m_0(1-\sigma_1)}$, *and*

$$\left[\frac{\beta\|\tau\|_0 d_0 d_1}{T\sqrt{2(1-\sigma_1)}} + \frac{m_1 d_0 d_1 + A d_0}{2T}\right]^{\frac{1}{q}} + \frac{\sqrt{T} d_0 \beta + \sqrt{T(1-\sigma_1)}(m_1 d_1 + A)}{\sqrt{2(1-\sigma_1)}} < 1,$$

where

$$d_0 := \frac{A(1-\sigma_1)(1+\sigma_0)(m_0+m_1) + \sqrt{2}A\beta\|\tau\|_0(1+\sigma_0)\sqrt{1-\sigma_1}}{\alpha m_0(1-\sigma_1) - m_1\beta(1+\sigma_0)\sqrt{1-\sigma_1} - \sqrt{2}\beta^2\|\tau\|_0(1+\sigma_0)}$$

and

$$d_1 := \frac{\beta d_0}{m_0\sqrt{1-\sigma_1}} + \frac{A}{m_0}.$$

Then, for each $k \in \mathbb{N}$, *if* $(u,v)^\top \in \Delta$, *there are positive constants* ρ_1, ρ_2, ρ_3, ρ_4, A_1, A_2, A_3, *and* A_4, *which are independent of k and* λ, *such that*

$$\|u\|_0 \le \rho_1, \qquad \|v\|_0 \le \rho_2 < 1, \qquad \|u'\|_0 \le \rho_3, \qquad \|v'\|_0 \le \rho_4,$$
$$\|u\|_2 \le A_1, \qquad \|u'\|_2 \le A_2, \qquad \|v\|_p \le A_3, \qquad \|v'\|_2 \le A_4.$$

Proof For each $k \in \mathbb{N}$, if $(u,v)^\top \in \Delta$, then $(u(t),v(t))^\top$ satisfies (3.1). Multiplying the second equation of (3.1) by $u(t)$ and integrating from $-kT$ to kT, we have

$$\begin{aligned}
&\int_{-kT}^{kT}\langle u'(t), v(t)\rangle\,dt \\
&\quad = -\int_{-kT}^{kT}\langle u(t), v'(t)\rangle\,dt \\
&\quad = \lambda\int_{-kT}^{kT}\left\langle u(t), F\left(t, \frac{u'(t)}{\lambda}\right)\right\rangle dt \\
&\qquad + \lambda\int_{-kT}^{kT}\langle u(t), G(t, u(t-\tau(t)))\rangle\,dt - \lambda\int_{-kT}^{kT}\langle u(t), e_k(t)\rangle\,dt \\
&\quad = \lambda\int_{-kT}^{kT}\left\langle u(t), F\left(t, \frac{u'(t)}{\lambda}\right)\right\rangle dt \\
&\qquad + \lambda\int_{-kT}^{kT}\langle u(t) - u(t-\tau(t)), G(t, u(t-\tau(t)))\rangle\,dt \\
&\qquad + \lambda\int_{-kT}^{kT}\langle u(t-\tau(t)), G(t, u(t-\tau(t)))\rangle\,dt - \lambda\int_{-kT}^{kT}\langle u(t), e_k(t)\rangle\,dt,
\end{aligned}$$

which combining with (H_1) and (H_2) gives

$$\begin{aligned}
&\int_{-kT}^{kT} \frac{|v(t)|^q}{\sqrt{1-|\varphi_q(v(t))|^2}}\,dt + \alpha \int_{-kT}^{kT} \bigl|u\bigl(t-\tau(t)\bigr)\bigr|^2\,dt \\
&\quad \le \frac{m_1}{\lambda} \int_{-kT}^{kT} \bigl|u(t)\bigr|\bigl|u'(t)\bigr|\,dt + \beta \int_{-kT}^{kT} \bigl|u(t)-u\bigl(t-\tau(t)\bigr)\bigr|\bigl|u\bigl(t-\tau(t)\bigr)\bigr|\,dt \\
&\qquad + \int_{-kT}^{kT} \bigl|u(t)\bigr|\bigl|e_k(t)\bigr|\,dt. \qquad (3.2)
\end{aligned}$$

Furthermore,

$$\int_{-kT}^{kT} \bigl|u\bigl(t-\tau(t)\bigr)\bigr|^2\,dt = \int_{-kT-\tau(-kT)}^{kT-\tau(kT)} \frac{1}{1-\tau'(\mu(s))} \bigl|u(s)\bigr|^2\,ds.$$

It follows from Lemma 2.4 that

$$\int_{-kT-\tau(-kT)}^{kT-\tau(kT)} \frac{1}{1-\tau'(\mu(s))} \bigl|u(s)\bigr|^2\,ds = \int_{-kT}^{kT} \frac{1}{1-\tau'(\mu(s))} \bigl|u(s)\bigr|^2\,ds.$$

By Remark 2.1, we have

$$\frac{1}{1+\sigma_0}\|u\|_2^2 \le \int_{-kT}^{kT} \frac{1}{1-\tau'(\mu(s))} \bigl|u(s)\bigr|^2\,ds \le \frac{1}{1-\sigma_1}\|u\|_2^2. \qquad (3.3)$$

Substituting (3.3) into (3.2) and combining with $\frac{|v(t)|^q}{\sqrt{1-|\varphi_q(v(t))|^2}} > |v(t)|^q$, we get

$$\begin{aligned}
&\int_{-kT}^{kT} \bigl|v(t)\bigr|^q\,dt + \frac{\alpha}{1+\sigma_0} \int_{-kT}^{kT} \bigl|u(t)\bigr|^2\,dt \\
&\quad \le \frac{m_1}{\lambda} \left(\int_{-kT}^{kT} \bigl|u(t)\bigr|^2\,dt\right)^{\frac12} \left(\int_{-kT}^{kT} \bigl|u'(t)\bigr|^2\,dt\right)^{\frac12} + \beta \left(\int_{-kT}^{kT} \bigl|u\bigl(t-\tau(t)\bigr)\bigr|^2\,dt\right)^{\frac12} \\
&\qquad \times \left(\int_{-kT}^{kT} \bigl|u(t)-u\bigl(t-\tau(t)\bigr)\bigr|^2\,dt\right)^{\frac12} + \left(\int_{-kT}^{kT} \bigl|e_k(t)\bigr|^2\,dt\right)^{\frac12} \left(\int_{-kT}^{kT} \bigl|u(t)\bigr|^2\,dt\right)^{\frac12}.
\end{aligned}$$

By applying Lemma 2.2 and (3.3), we see that

$$\begin{aligned}
&\int_{-kT}^{kT} \bigl|v(t)\bigr|^q\,dt + \frac{\alpha}{1+\sigma_0} \int_{-kT}^{kT} \bigl|u(t)\bigr|^2\,dt \\
&\quad \le \frac{m_1}{\lambda} \left(\int_{-kT}^{kT} \bigl|u(t)\bigr|^2\,dt\right)^{\frac12} \left(\int_{-kT}^{kT} \bigl|u'(t)\bigr|^2\,dt\right)^{\frac12} + \frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}} \left(\int_{-kT}^{kT} \bigl|u(t)\bigr|^2\,dt\right)^{\frac12} \\
&\qquad \times \left(\int_{-kT}^{kT} \bigl|u'(t)\bigr|^2\,dt\right)^{\frac12} + \left(\int_{-kT}^{kT} \bigl|e_k(t)\bigr|^2\,dt\right)^{\frac12} \left(\int_{-kT}^{kT} \bigl|u(t)\bigr|^2\,dt\right)^{\frac12},
\end{aligned}$$

i.e.,

$$\|v\|_q^q + \frac{\alpha}{1+\sigma_0}\|u\|_2^2 \le \frac{m_1}{\lambda}\|u\|_2\bigl\|u'\bigr\|_2 + \frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}}\|u\|_2\bigl\|u'\bigr\|_2 + A\|u\|_2.$$

This implies that

$$\|v\|_q^q \le \frac{m_1}{\lambda}\|u\|_2\|u'\|_2 + \frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}}\|u\|_2\|u'\|_2 + A\|u\|_2 \tag{3.4}$$

and

$$\frac{\alpha}{1+\sigma_0}\|u\|_2^2 \le \frac{m_1}{\lambda}\|u\|_2\|u'\|_2 + \frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}}\|u\|_2\|u'\|_2 + A\|u\|_2. \tag{3.5}$$

Multiplying the second equation of (3.1) by $u'(t)$ and integrating from $-kT$ to kT, we have

$$\begin{aligned}
0 &= \lambda\int_{-kT}^{kT}\left\langle \frac{\varphi_q(v(t))}{\sqrt{1-|\varphi_q(v(t))|^2}}, v'(t)\right\rangle dt = \int_{-kT}^{kT}\langle u'(t), v'(t)\rangle\, dt \\
&= -\lambda\int_{-kT}^{kT}\left\langle u'(t), F\left(t, \frac{u'(t)}{\lambda}\right)\right\rangle dt - \lambda\int_{-kT}^{kT}\big\langle u'(t), G\big(t, u\big(t-\tau(t)\big)\big)\big\rangle\, dt \\
&\quad + \lambda\int_{-kT}^{kT}\langle u'(t), e_k(t)\rangle\, dt.
\end{aligned}$$

Combining with (H_1), (H_2), and (3.3), we get

$$\begin{aligned}
&m_0\int_{-kT}^{kT}\big|u'(t)\big|^2\, dt \\
&\quad\le \left|\lambda^2\int_{-kT}^{kT}\left\langle \frac{u'(t)}{\lambda}, F\left(t, \frac{u'(t)}{\lambda}\right)\right\rangle dt\right| \\
&\quad\le \lambda\beta\int_{-kT}^{kT}\big|u'(t)\big|\big|u\big(t-\tau(t)\big)\big|\, dt + \lambda\int_{-kT}^{kT}\big|u'(t)\big|\big|e_k(t)\big|\, dt \\
&\quad\le \frac{\lambda\beta}{\sqrt{1-\sigma_1}}\|u'\|_2\|u\|_2 + \lambda A\|u'\|_2,
\end{aligned}$$

which results in

$$\|u'\|_2 \le \frac{\lambda\beta}{m_0\sqrt{1-\sigma_1}}\|u\|_2 + \frac{\lambda A}{m_0} \le \frac{\beta}{m_0\sqrt{1-\sigma_1}}\|u\|_2 + \frac{A}{m_0}. \tag{3.6}$$

Substituting (3.6) into (3.5), we obtain

$$\begin{aligned}
\frac{\alpha}{1+\sigma_0}\|u\|_2^2 &\le \frac{m_1}{\lambda}\|u\|_2\left(\frac{\lambda\beta}{m_0\sqrt{1-\sigma_1}}\|u\|_2 + \frac{\lambda A}{m_0}\right) \\
&\quad + \frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}}\|u\|_2\left(\frac{\beta}{m_0\sqrt{1-\sigma_1}}\|u\|_2 + \frac{A}{m_0}\right) + A\|u\|_2.
\end{aligned}$$

It follows from $\frac{\alpha}{1+\sigma_0} > \frac{m_1\beta\sqrt{1-\sigma_1}+\sqrt{2}\beta^2\|\tau\|_0}{m_0(1-\sigma_1)}$ that

$$\begin{aligned}
\|u\|_2 &\le \frac{A(1-\sigma_1)(1+\sigma_0)(m_0+m_1) + \sqrt{2}A\beta\|\tau\|_0(1+\sigma_0)\sqrt{1-\sigma_1}}{\alpha m_0(1-\sigma_1) - m_1\beta(1+\sigma_0)\sqrt{1-\sigma_1} - \sqrt{2}\beta^2\|\tau\|_0(1+\sigma_0)} \\
&:= d_0.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$\|u'\|_2 \leq \frac{\lambda d_0 \beta}{m_0\sqrt{1-\sigma_1}} + \frac{\lambda A}{m_0}, \tag{3.8}$$

i.e.,

$$\|u'\|_2 \leq \frac{d_0 \beta}{m_0\sqrt{1-\sigma_1}} + \frac{A}{m_0} := d_1. \tag{3.9}$$

Substituting (3.7), (3.8), and (3.9) into (3.4), we have

$$\begin{aligned} \|v\|_q^q &\leq \frac{m_1}{\lambda}\|u\|_2\|u'\|_2 + \frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}}\|u\|_2\|u'\|_2 + A\|u\|_2 \\ &\leq \frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}} d_0 d_1 + m_1 d_0 d_1 + A d_0. \end{aligned} \tag{3.10}$$

Moreover, it follows from Lemma 2.3 that

$$\begin{aligned} |u(t)| &\leq (2T)^{-\frac{1}{2}}\left(\int_{t-kT}^{t+kT}|u(s)|^2\,ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{t-kT}^{t+kT}|u'(s)|^2\,ds\right)^{\frac{1}{2}} \\ &= (2T)^{-\frac{1}{2}}\left(\int_{-kT}^{kT}|u(s)|^2\,ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{-kT}^{kT}|u'(s)|^2\,ds\right)^{\frac{1}{2}}, \end{aligned}$$

which combining with (3.7) and (3.9) yields

$$|u(t)| \leq (2T)^{-\frac{1}{2}} d_0 + T(2T)^{-\frac{1}{2}} d_1 := \rho_1, \quad \text{for all } t \in R,$$

and then

$$\|u\|_0 = \max_{t\in[-kT,kT]}|u(t)| \leq \rho_1. \tag{3.11}$$

Clearly, ρ_1 is independent of k and λ.

Multiplying the second equation of (3.1) by $v'(t)$ and integrating from $-kT$ to kT, in view of (H_1) and (H_2), we have

$$\begin{aligned} \int_{-kT}^{kT}|v'(t)|^2\,dt &= -\lambda\int_{-kT}^{kT}\left\langle v'(t), F\left(t, \frac{u'(t)}{\lambda}\right)\right\rangle dt - \lambda\int_{-kT}^{kT}\langle v'(t), G(t, u(t-\tau(t)))\rangle\,dt \\ &\quad + \lambda\int_{-kT}^{kT}\langle v'(t), e_k(t)\rangle\,dt \\ &\leq \frac{m_1}{\lambda}\int_{-kT}^{kT}|v'(t)||u'(t)|\,dt + \beta\int_{-kT}^{kT}|v'(t)||u(t-\tau(t))|\,dt \\ &\quad + \int_{-kT}^{kT}|v'(t)||e_k(t)|\,dt. \end{aligned}$$

By applying the Hölder inequality and (3.3), we have

$$\|v'\|_2 \leq \frac{m_1}{\lambda}\|u'\|_2 + \frac{\beta}{\sqrt{1-\sigma_1}}\|u\|_2 + A.$$

By (3.7), (3.8), and (3.9), we have

$$\begin{aligned}\|v'\|_2 &\le \frac{m_1}{\lambda}\|u'\|_2 + \frac{\beta}{\sqrt{1-\sigma_1}}\|u\|_2 + A \\ &\le \frac{m_1}{\lambda}\left(\frac{\lambda d_0\beta}{m_0\sqrt{1-\sigma_1}} + \frac{\lambda A}{m_0}\right) + \frac{\beta}{\sqrt{1-\sigma_1}}\|u\|_2 + A \\ &= m_1\left(\frac{d_0\beta}{m_0\sqrt{1-\sigma_1}} + \frac{A}{m_0}\right) + \frac{\beta}{\sqrt{1-\sigma_1}}\|u\|_2 + A \\ &\le \frac{d_0\beta}{\sqrt{1-\sigma_1}} + m_1 d_1 + A. \end{aligned} \tag{3.12}$$

By applying Lemma 2.3 again and combining with (3.10) and (3.12), we get

$$\begin{aligned}|v(t)| &\le (2T)^{-\frac{1}{q}}\left(\int_{t-kT}^{t+kT}|v(s)|^q\,ds\right)^{\frac{1}{q}} + T(2T)^{-\frac{1}{2}}\left(\int_{t-kT}^{t+kT}|v'(s)|^2\,ds\right)^{\frac{1}{2}} \\ &= (2T)^{-\frac{1}{q}}\left(\int_{-kT}^{kT}|v(s)|^q\,ds\right)^{\frac{1}{q}} + T(2T)^{-\frac{1}{2}}\left(\int_{-kT}^{kT}|v'(s)|^2\,ds\right)^{\frac{1}{2}} \\ &\le (2T)^{-\frac{1}{q}}\left(\frac{\sqrt{2}\beta\|\tau\|_0}{\sqrt{1-\sigma_1}}d_0d_1 + m_1d_0d_1 + Ad_0\right)^{\frac{1}{q}} + T(2T)^{-\frac{1}{2}}\left(\frac{d_0\beta}{\sqrt{1-\sigma_1}} + m_1d_1 + A\right) \\ &= \left[\frac{\beta\|\tau\|_0 d_0d_1}{T\sqrt{2(1-\sigma_1)}} + \frac{m_1d_0d_1 + Ad_0}{2T}\right]^{\frac{1}{q}} + \frac{\sqrt{T}d_0\beta + \sqrt{T}(m_1d_1+A)\sqrt{1-\sigma_1}}{\sqrt{2(1-\sigma_1)}} \\ &:= \rho_2. \end{aligned}$$

Since

$$\left[\frac{\beta\|\tau\|_0 d_0d_1}{T\sqrt{2(1-\sigma_1)}} + \frac{m_1d_0d_1 + Ad_0}{2T}\right]^{\frac{1}{q}} + \frac{\sqrt{T}d_0\beta + \sqrt{T}(m_1d_1+A)\sqrt{1-\sigma_1}}{\sqrt{2(1-\sigma_1)}} < 1,$$

we have

$$\|v\|_0 = \max_{t\in[-kT,kT]}|v(t)| \le \rho_2 < 1. \tag{3.13}$$

Clearly, ρ_2 is independent of k and λ.

Furthermore, it follows from (3.1) that

$$\begin{aligned}\|u'\|_0 &= \max_{t\in[-kT,kT]}|u'(t)| = \max_{t\in[-kT,kT]}\lambda\frac{\varphi_q(v(t))}{\sqrt{1-(\varphi_q(v(t)))^2}} \\ &\le \frac{\rho_2^{q-1}}{\sqrt{1-\rho_2^{2q-2}}} := \rho_3. \end{aligned} \tag{3.14}$$

Clearly, ρ_3 is independent of k and λ.

Define $F_{\rho_3} = \max_{|x|\le\rho_3, t\in[0,T]}|F(t,x)|$ and $G_{\rho_1} = \max_{|y|\le\rho_1, t\in[0,T]}|G(t,y)|$, then from the second equation of (3.1), we get

$$\|v'\|_0 = \max_{t\in[-kT,kT]}|v'(t)| \le F_{\rho_3} + G_{\rho_1} + A := \rho_4. \tag{3.15}$$

ρ_4 is also independent of k and λ. Therefore, from (3.7), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), and (3.15), we know that all the conclusions of Theorem 3.1 hold. □

Theorem 3.2 *Assume that the conditions of Theorem* 3.1 *are satisfied. Then, for each* $k \in \mathbb{N}$, *system* (3.1) *has at least one* $2kT$*-periodic solution* $(u_k(t), v_k(t))^\top$ *in* $\Delta \subset X_k$ *such that*

$$\|u_k\|_0 \le \rho_1, \qquad \|v_k\|_0 \le \rho_2 < 1, \qquad \|u_k'\|_0 \le \rho_3, \qquad \|v_k'\|_0 \le \rho_4,$$
$$\|u_k\|_2 \le A_1, \qquad \|u_k'\|_2 \le A_2, \qquad \|v_k\|_p \le A_3, \qquad \|v_k'\|_2 \le A_4,$$

where ρ_1, ρ_2, ρ_3, ρ_4, A_1, A_2, A_3, *and* A_4 *are constants defined by Theorem* 3.1.

Proof In order to use Lemma 2.1, for each $k \in \mathbb{N}$, we consider the following system:

$$\begin{cases} u'(t) = \lambda\varphi(v(t)) = \lambda\frac{\varphi_q(v(t))}{\sqrt{1-|\varphi_q(v(t))|^2}}, \\ v'(t) = -\lambda F(t, \varphi(v(t))) - \lambda G(t, u(t-\tau(t))) + \lambda e_k(t), \quad \lambda \in (0,1), \end{cases} \tag{3.16}$$

where $v(t) = \varphi_p(\frac{\frac{u'(t)}{\lambda}}{\sqrt{1+|\frac{u'(t)}{\lambda}|^2}})$. Let $\Omega_1 \subset X_k$ represents the set of all possible $2kT$-periodic solutions of (3.16). Since $(0,1) \subset (0,1]$, then $\Omega_1 \subset \Delta$, where Δ is defined by Theorem 3.1. If $(u,v)^\top \in \Omega_1$, by using Theorem 3.1, we have

$$\|u\|_0 \le \rho_1, \qquad \|u'\|_0 \le \rho_3, \qquad \|v\|_0 \le \rho_2 < 1, \qquad \|v'\|_0 \le \rho_4.$$

Define $\Omega_2 = \{\omega = (u,v)^\top \in \ker L, QN\omega = 0\}$. If $(u,v)^\top \in \Omega_2$, then $(u,v)^\top = (a_1,a_2)^\top \in \mathbb{R}^2$ (constant vector) such that

$$\begin{cases} \int_{-kT}^{kT} \frac{\varphi_q(a_2)}{\sqrt{1-|\varphi_q(a_2)|^2}}\,dt = 0, \\ \int_{-kT}^{kT} [-F(t, \frac{\varphi_q(a_2)}{\sqrt{1-|\varphi_q(a_2)|^2}}) - G(t,a_1) + e_k(t)]\,dt = 0, \end{cases}$$

i.e.,

$$\begin{cases} a_2 = 0, \\ \int_{-kT}^{kT} [-F(t,0) - G(t,a_1) + e_k(t)]\,dt = 0. \end{cases} \tag{3.17}$$

Multiplying the second equation of (3.17) by a_1 and combining with (H_1) and (H_2), we have

$$\begin{aligned} 2kT\alpha|a_1|^2 &\le \int_{-kT}^{kT} |F(t,0)||a_1|\,dt + \int_{-kT}^{kT} |a_1||e_k(t)|\,dt \\ &\le 2kT|a_1|A. \end{aligned}$$

Thus,

$$|a_1| \le \frac{A}{\alpha} := \varpi.$$

Now, if we define $\Omega = \{\omega = (u,v)^\top \in X_k, \|u\|_0 < \rho_1 + \varpi, \|v\|_0 < \frac{1+\rho_2}{2} < 1\}$, it is easy to see that $\Omega_1 \cup \Omega_2 \subset \Omega$. So, condition ($h_1$) and condition ($h_2$) of Lemma 2.1 are satisfied. In order to verify the condition (h_3) of Lemma 2.1, define

$$H(\omega,\mu) : (\Omega \cap \mathbb{R}^2) \times [0,1] \to \mathbb{R}: \quad H(\omega,\mu) = \mu\omega + (1-\mu)JQN(\omega),$$

where $J : \operatorname{Im} Q \to \ker L$ is a linear isomorphism, $J(u,v) = (v,u)^\top$. From assumption (H_1), we have

$$\omega^\top H(\omega,\mu) \neq 0, \quad \forall(\omega,\mu) \in \partial\Omega \cap \mathbb{R}^2 \times [0,1].$$

Hence,

$$\begin{aligned}\deg\{JQN, \Omega \cap \mathbb{R}^2, 0\} &= \deg\{H(\omega,0), \Omega \cap \mathbb{R}^2, 0\} \\ &= \deg\{H(\omega,1), \Omega \cap \mathbb{R}^2, 0\} \\ &\neq 0.\end{aligned}$$

Thus, the condition (h_3) of Lemma 2.1 is also satisfied. Therefore, by using Lemma 2.1, we can see that (2.1) has a $2kT$-periodic solution $(u_k,v_k)^\top \in \overline{\Omega}$. Clearly, u_k is a $2kT$-periodic solution to (1.5), and $(u_k,v_k)^\top$ must be in Δ for the case of $\lambda = 1$. Thus, by using Theorem 3.1, we have

$$\begin{aligned}&\|u_k\|_0 \le \rho_1, \qquad \|v_k\|_0 \le \rho_2 < 1, \qquad \|u_k'\|_0 \le \rho_3, \qquad \|v_k'\|_0 \le \rho_4, \\ &\|u_k\|_2 \le A_1, \qquad \|u_k'\|_2 \le A_2, \qquad \|v_k\|_p \le A_3, \qquad \|v_k'\|_2 \le A_4.\end{aligned}$$

Hence, all the conclusions of Theorem 3.2 hold. □

Theorem 3.3 *Suppose that the conditions in Theorem* 3.1 *hold, then* (1.4) *has a nontrivial homoclinic solution.*

Proof From Theorem 3.2, we see that for each $k \in \mathbb{N}$, there exists a $2kT$-periodic solution $(u_k,v_k)^\top$ to (2.1) with $(u_k,v_k)^\top \in X_k$ and

$$\|u_k\|_0 \le \rho_1, \qquad \|v_k\|_0 \le \rho_2 < 1, \qquad \|u_k'\|_0 \le \rho_3, \qquad \|v_k'\|_0 \le \rho_4, \tag{3.18}$$

where ρ_1, ρ_2, ρ_3, ρ_4 are constants independent of $k \in \mathbb{N}$. Equation (3.18) together with Lemma 2.5 shows that there are a function $w_0 := (u_0,u_0)^\top \in C(\mathbb{R},\mathbb{R}^{2n})$ and a subsequence $\{(u_{k_j},v_{k_j})^\top\}$ of $\{(u_k,v_k)^\top\}_{k\in\mathbb{N}}$ such that for each interval $[a,b] \subset \mathbb{R}$, $u_{k_j}(t) \to u_0(t)$, and $v_{k_j}(t) \to v_0(t)$ uniformly on $[a,b]$. Below, we will show that $(u_0(t),v_0(t))^\top$ is just a homoclinic solution to (1.4).

Since $(u_k(t),v_k(t))^\top$ is a $2kT$-periodic solution of (2.1), it follows that

$$\begin{cases} u_k'(t) = \phi(v_k(t)) = \frac{\varphi_q(v_k(t))}{\sqrt{1-|\varphi_q(v_k(t))|^2}}, \\ v_k'(t) = -F(t,\varphi(v_k(t))) - G(t,u_k(t-\tau(t))) + e_k(t). \end{cases} \tag{3.19}$$

For all $a, b \in R$ with $a < b$, there must be a positive integer j_0 such that for $j > j_0$, $[-k_jT, k_jT - \varepsilon_0] \supset [a - \|\tau\|_0, b + \|\tau\|_0]$. So for $j > j_0$, from (1.6) and (3.19) we see that

$$\begin{cases} u'_{k_j}(t) = \varphi(y_{k_j}(t)) = \frac{\varphi_q(v_{k_j}(t))}{\sqrt{1-|\varphi_q(v_{k_j}(t))|^2}}, \\ v'_{k_j}(t) = -F(t, \varphi(y_{k_j}(t))) - G(t, u_{k_j}(t - \tau(t))) + e(t), \quad t \in (a, b), \end{cases}$$

which results in

$$u'_{k_j}(t) = \frac{\varphi_q(v_{k_j}(t))}{\sqrt{1 - |\varphi_q(v_{k_j}(t))|^2}} \to \frac{\varphi_q(v_0(t))}{\sqrt{1 - |\varphi_q(v_0(t))|^2}} \tag{3.20}$$

and

$$\begin{aligned} v'_{k_j}(t) &= -F\bigl(t, \varphi\bigl(v_{k_j}(t)\bigr)\bigr) - G\bigl(t, u_{k_j}\bigl(t - \tau(t)\bigr)\bigr) + e(t) \\ &\to -F\bigl(t, \varphi\bigl(v_0(t)\bigr)\bigr) - G\bigl(t, u_0\bigl(t - \tau(t)\bigr)\bigr) + e(t) \end{aligned} \tag{3.21}$$

uniformly for $t \in [a, b]$ as $j \to +\infty$. Since $u_{k_j}(t) \to u_0(t)$ and $u_{k_j}(t)$ is continuously differentiable for $t \in (a, b)$, it follows that

$$u'_{k_j}(t) \to u'_0(t) \quad \text{uniformly for } t \in [a, b] \text{ as } j \to +\infty,$$

which together with (3.20) yields

$$u'_0(t) = \frac{\varphi_q(v_0(t))}{\sqrt{1 - |\varphi_q(v_0(t))|^2}}, \quad t \in (a, b).$$

Similarly, by (3.21) we have

$$v'_0(t) = -F\bigl(t, \varphi\bigl(v_0(t)\bigr)\bigr) - G\bigl(t, u_0\bigl(t - \tau(t)\bigr)\bigr) + e(t), \quad t \in (a, b).$$

Considering a, b to be two arbitrary constants with $a < b$, it is easy to see that $(u_0(t), v_0(t))^\top$, $t \in R$, is a solution to the following equation:

$$\begin{cases} u'(t) = \phi(v(t)) = \frac{\varphi_q(v(t))}{\sqrt{1-|\varphi_q(v(t))|^2}}, \\ v'(t) = -F(t, \varphi(v(t))) - G(t, u(t - \tau(t))) + e(t), \end{cases}$$

i.e.,

$$\begin{cases} u'_0(t) = \phi(v_0(t)) = \frac{\varphi_q(v_0(t))}{\sqrt{1-|\varphi_q(v_0(t))|^2}}, \\ v'_0(t) = -F(t, \varphi(v_0(t))) - G(t, u_0(t - \tau(t))) + e(t). \end{cases} \tag{3.22}$$

Now, we will prove $u_0(t) \to 0$ and $u'_0(t) \to 0$ as $|t| \to +\infty$.

Since

$$\begin{aligned} \int_{-\infty}^{+\infty} \bigl(\bigl|u_0(t)\bigr|^2 + \bigl|u'_0(t)\bigr|^2\bigr)\,dt &= \lim_{i \to +\infty} \int_{-iT}^{iT} \bigl(\bigl|u_0(t)\bigr|^2 + \bigl|u'_0(t)\bigr|^2\bigr)\,dt \\ &= \lim_{i \to +\infty} \lim_{j \to +\infty} \int_{-iT}^{iT} \bigl(\bigl|u_0(t)\bigr|^2 + \bigl|u'_0(t)\bigr|^2\bigr)\,dt. \end{aligned}$$

By using the conclusion of Theorem 3.2, we have

$$\int_{-iT}^{iT}\left(\left|u_{k_j}(t)\right|^2+\left|u'_{k_j}(t)\right|^2\right)dt\le\int_{-k_jT}^{k_jT}\left(\left|u_{k_j}(t)\right|^2+\left|u'_{k_j}(t)\right|^2\right)dt\le A_0^2+A_1^2.$$

Let $i\to+\infty$ and $j\to+\infty$, we have

$$\int_{-\infty}^{+\infty}\left(\left|u_0(t)\right|^2+\left|u'_0(t)\right|^2\right)dt\le A_0^2+A_1^2,$$

and then

$$\int_{|t|\ge r}\left(\left|u_0(t)\right|^2+\left|u'_0(t)\right|^2\right)dt\to 0$$

as $r\to+\infty$. So by using Lemma 2.3, we obtain

$$\begin{aligned}\left|u_0(t)\right|&\le(2T)^{-\frac{1}{2}}\left(\int_{t-T}^{t+T}\left|u_0(s)\right|^{l_1+1}ds\right)^{\frac{1}{2}}+T(2T)^{-\frac{1}{2}}\left(\int_{t-T}^{t+T}\left|u'_0(s)\right|^2ds\right)^{\frac{1}{2}}\\&\le\left[(2T)^{-\frac{1}{2}}+T(2T)^{-\frac{1}{2}}\right]\left[\left(\int_{t-T}^{t+T}\left|x(s)\right|^2ds\right)^{1/(2)}+\left(\int_{t-T}^{t+T}\left|u'_0(s)\right|^2ds\right)^{\frac{1}{2}}\right]\\&\to 0\quad\text{as }|t|\to+\infty,\end{aligned}$$

which implies that

$$u_0(t)\to 0\quad\text{as }|t|\to+\infty.\tag{3.23}$$

Similarly, we can prove that

$$v_0(t)\to 0\quad\text{as }|t|\to+\infty,$$

which together with the first equation of (3.22) gives

$$u'_0(t)\to 0\quad\text{as }|t|\to+\infty.\tag{3.24}$$

It is easy to see from (3.22) that $u_0(t)$ is a solution for (1.4). Thus, by (3.23) and (3.24), $u_0(t)$ is just a homoclinic solution to (1.4). Clearly, $u_0(t)\not\equiv 0$, otherwise, $e(t)\equiv 0$, which contradicts assumption (H_3). Hence, the conclusion of Theorem 3.3 holds. □

Remark 3.1 Obviously, the prescribed mean curvature equations studied in [11, 12, 15, 16] are special cases of (1.4). This implies that the main result in this paper is essentially new.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equally contributed to obtaining new results in this article and also read and approved the final manuscript.

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Acknowledgements
The authors express their thanks to the referee for his (or her) valuable suggestions. The research was supported by the National Natural Science Foundation of China (Grant No. 11271197).

Received: 25 February 2015 Accepted: 28 May 2015 Published online: 19 June 2015

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