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Existence and multiplicity of nontrivial solutions for a nonlocal problem

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Abstract

Purpose: In this paper, we study the existence and multiplicity of nontrivial solutions for a new nonlocal problem.

Methods: Variational method, mountain pass lemma.

Results: Some existence and multiplicity results of nontrivial solutions are obtained.
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Keywords: nonlocal problem; nontrivial solution; existence; multiplicity

1 Introduction and main results

In this paper, we study the existence and multiplicity of nontrivial solutions for a new nonlocal Dirichlet boundary value problem

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,\mathrm{d}x)\triangle u = |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

by using the mountain pass lemma, where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $N \geq 1$, $a, b > 0$ are constants and

$$2 < p < 2^* = \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ \infty, & N = 1, 2. \end{cases} \tag{2}$$

Recently, the Kirchhoff type problem on a bounded domain

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x)\triangle u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

has been studied by many authors, for example [1–5]. Many solvability conditions of problem (3) have been considered. Moreover, some scholars have studied the existence of nontrivial solutions for the more general Kirchhoff type problems

$$\begin{cases} -M(\int_\Omega |\nabla u|^2\,\mathrm{d}x)\triangle u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

where M is a certain continuous function, for example [6–8]. However, such problems cannot contain problem (1) because the function M is assumed to be bounded from below. For more results, please refer to [9, 10] and the references therein. Using a standard method, we can prove that the energy functional J (see Section 2 below) of problem (1) possesses a mountain pass energy c_0. To deal with the difficulty caused by the noncompactness due to the nonlocal term, we should estimate precisely the value of c_0 and give a threshold value (see Lemma 2.1 below) under which the $(PS)_{c_0}$ condition for J is satisfied. Therefore, the study of the existence and multiplicity of a nontrivial solution for problem (1) presents different difficulties from those in problem (4). Our main results are as follows.

Theorem 1.1 *Problem* (1) *possesses at least a nontrivial weak solution.*

Theorem 1.2 *Problem* (1) *possesses at least a nontrivial non-negative solution and a nontrivial non-positive solution.*

The novelty of our results lies in two aspects. Firstly, differently from [1–5], where the nonlocal term is $a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x$, we put forward a new nonlocal term $a - b\int_\Omega |\nabla u|^2\,\mathrm{d}x$ which presents interesting difficulties. Secondly, we obtain the precise threshold value under which the (PS) condition for J is satisfied.

2 The proof of main results

Let X be the usual Sobolev space $H_0^1(\Omega)$ equipped with the inner product $(u,v) = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x$ and the norm $\|u\| = \sqrt{(u,u)}$. We denote by $|u|_r$, $1 \le r \le 2^*$, the norm of the space $L^r(\Omega)$. It is well known that $X \hookrightarrow L^r(\Omega)$ continuously for $r \in [1,2^*]$, compactly for $r \in [1,2^*)$. Hence, there exists $\gamma_r > 0$ such that

$$|u|_r \le \gamma_r \|u\|, \quad \forall u \in X, r \in \left[1, 2^*\right]. \tag{5}$$

A function $u \in X$ is called a weak solution of problem (1) if

$$a\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x - b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x = \int_\Omega |u|^{p-2}uv\,\mathrm{d}x, \quad \forall v \in X.$$

Define a functional by

$$J(u) := \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x, \quad \forall u \in X.$$

From (2) we know that $J \in C^1(X, \mathbb{R}^1)$ and

$$\langle J'(u), v\rangle = a\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x - b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x - \int_\Omega |u|^{p-2}uv\,\mathrm{d}x, \quad \forall u, v \in X.$$

Thus u is a weak solution of problem (1) if and only if u is a critical point of the functional J on X.

Firstly, we give two preliminary results.

Lemma 2.1 *There exists a sequence* $\{u_n\} \subset X$ *satisfying* $J(u_n) \to c_0$, $J'(u_n) \to 0$, *where* $0 < c_0 < \frac{a^2}{4b}$.

Proof From inequality (5) we have

$$\begin{aligned} J(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\gamma_p^p\|u\|^p. \end{aligned}$$

Noting that $2 < p < 2^*$, we can choose small $0 < \rho \leq \min\{[\frac{a}{2b}]^{\frac{1}{2}}, [\frac{ap}{8\gamma_p^p}]^{\frac{1}{p-2}}\}$. Then for all $u \in X$, $\|u\| = \rho$, it holds that $J(u) \geq \frac{a}{4}\rho^2 = \gamma > 0$. On the other hand, for $\tau \in \mathbb{R}^1$, and fixed $u \neq 0$, with $\int_\Omega |u|^p \,\mathrm{d}x > 0$,

$$J(\tau u) = \frac{a}{2}\|u\|^2\tau^2 - \frac{b}{4}\|u\|^4\tau^4 - |\tau|^p\frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x,$$

then $J(\tau u) \to -\infty$ ($|\tau| \to \infty$). So there exists $\tau_1 > 0$ such that $u_1 = \tau_1 u \in X$, $\|u_1\| > \rho$, $J(u_1) < 0$. Hence, by the mountain pass lemma without (PS) condition (see [11]), we obtain a sequence $\{u_n\} \subset X$ such that $J(u_n) \to c_0$, $J'(u_n) \to 0$ for

$$c_0 = \inf_{g\in\Gamma}\max_{u\in g([0,1])} J(u) \geq \gamma > 0,$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1], X\bigr) : g(0) = 0, g(1) = u_1\bigr\}.$$

Due to

$$\begin{aligned} \max_{t\in[0,1]} J(tu_1) &= \max_{t\in[0,1]}\left\{\frac{a}{2}\|u_1\|^2t^2 - \frac{b}{4}\|u_1\|^4t^4 - |t|^p\frac{1}{p}\int_\Omega |u_1|^p \,\mathrm{d}x\right\} \\ &< \max_{t\in[0,1]}\left\{\frac{a}{2}\|u_1\|^2t^2 - \frac{b}{4}\|u_1\|^4t^4\right\} \\ &\leq \frac{a^2}{4b}, \end{aligned}$$

and from the definition of c_0 we have $0 < c_0 < \frac{a^2}{4b}$. □

Lemma 2.2 *Under the condition* $c < \frac{a^2}{4b}$, J *satisfies the* $(PS)_c$ *condition, i.e., any* $(PS)_c$ *sequence of* J *has a convergent subsequence.*

Proof Let $\{u_n\} \subset X$ be such that $J(u_n) \to c$, $J'(u_n) \to 0$. Since

$$\begin{aligned} 2J(u_n) - \bigl\langle J'(u_n), u_n\bigr\rangle &= \frac{b}{2}\|u_n\|^4 + \left(1 - \frac{2}{p}\right)\int_\Omega |u_n|^p \,\mathrm{d}x \\ &\geq \frac{b}{2}\|u_n\|^4, \end{aligned}$$

$J(u_n) \to c$, $J'(u_n) \to 0$, $\{u_n\}$ is bounded in X. By passing to a subsequence if necessary, we may assume that there exists $u \in X$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X, \\ &u_n \to u \quad \text{in } L^r(\Omega), \text{for } r \in [1, 2^*), \\ &u_n(x) \to u(x) \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Noting that

$$\begin{aligned} \left| \int_\Omega |u_n|^{p-2} u_n (u - u_n) \,\mathrm{d}x \right| &\le \left(\int_\Omega \left(|u_n|^{p-1}\right)^{\frac{p}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_\Omega |u - u_n|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\ &= |u_n|_p^{p-1} |u - u_n|_p \\ &\le \gamma_p^{p-1} \|u_n\|^{p-1} |u - u_n|_p \to 0 \quad (n \to \infty), \end{aligned}$$

using the previous conditions and the fact that

$$\left(a - b\|u_n\|^2\right) \int_\Omega \nabla u_n \cdot \nabla (u - u_n) \,\mathrm{d}x - \int_\Omega |u_n|^{p-2} u_n (u - u_n) \,\mathrm{d}x = \left\langle J'(u_n), (u - u_n) \right\rangle \to 0,$$

it follows

$$\left(a - b\|u_n\|^2\right) \int_\Omega \nabla u_n \cdot \nabla (u - u_n) \,\mathrm{d}x \to 0 \quad (n \to \infty). \tag{6}$$

If there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) satisfying $\|u_n\|^2 \to \frac{a}{b}$, define a functional by

$$\varphi(u) := \frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x, \quad u \in X,$$

then

$$\left\langle \varphi'(u), v \right\rangle = \int_\Omega |u|^{p-2} u v \,\mathrm{d}x, \quad u, v \in X.$$

Since $u_n \to u$ in $L^p(\Omega)$, then $|u_n|^{p-2} u_n \to |u|^{p-2} u$ in $L^{\frac{p}{p-1}}(\Omega)$, and yet

$$\left\langle \varphi'(u_n) - \varphi'(u), v \right\rangle = \int_\Omega \left[|u_n|^{p-2} u_n - |u|^{p-2} u \right] v \,\mathrm{d}x.$$

Due to Hölder's inequality, we have

$$\begin{aligned} \left| \left\langle \varphi'(u_n) - \varphi'(u), v \right\rangle \right| &\le \int_\Omega \left| |u_n|^{p-2} u_n - |u|^{p-2} u \right| |v| \,\mathrm{d}x \\ &\le \left| |u_n|^{p-2} u_n - |u|^{p-2} u \right|_{\frac{p}{p-1}} |v|_p \\ &\le \left| |u_n|^{p-2} u_n - |u|^{p-2} u \right|_{\frac{p}{p-1}} \gamma_p \|v\|. \end{aligned}$$

Thus

$$\left\|\varphi'(u_n)-\varphi'(u)\right\|_{X'} \le \left||u_n|^{p-2}u_n-|u|^{p-2}u\right|_{\frac{p}{p-1}}\gamma_p \to 0,$$

and $\varphi'(u_n)\to\varphi'(u)$. While $\langle J'(u_n),v\rangle=(a-b\|u_n\|^2)\langle u_n,v\rangle-\langle\varphi'(u_n),v\rangle$, $\langle J'(u_n),v\rangle\to 0$, and $(a-b\|u_n\|^2)\to 0$, hence $\varphi'(u_n)\to 0$ $(n\to\infty)$, *i.e.*,

$$\left\langle\varphi'(u),v\right\rangle=\int_\Omega |u|^{p-2}uv\,\mathrm{d}x=0,\quad \forall v\in X.$$

Then we have

$$\left|u(x)\right|^{p-2}u(x)=0,\quad \text{a.e. } x\in\Omega$$

by the variational method fundamental lemma (see [12]). It follows from that $u=0$. So

$$\varphi(u_n)=\frac{1}{p}\int_\Omega |u_n|^p\,\mathrm{d}x\to\frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x=0.$$

Hence we see that $J(u_n)=\frac{a}{2}\|u_n\|^2-\frac{b}{4}\|u_n\|^4-\frac{1}{p}\int_\Omega|u_n|^p\,\mathrm{d}x\to\frac{a^2}{4b}$ from $\|u_n\|^2\to\frac{a}{b}$. This is a contradiction with $J(u_n)\to c<\frac{a^2}{4b}$. Then $(a-b\|u_n\|^2)\to 0$ $(n\to\infty)$ is not true and any subsequence of $\{a-b\|u_n\|^2\}$ does not converge to zero. Therefore there exists $\delta>0$ such that $|a-b\|u_n\|^2|>\delta$ when n is large enough. It is clear that $\{a-b\|u_n\|^2\}$ is bounded. It follows from (6) that $\int_\Omega\nabla u_n\cdot\nabla(u-u_n)\,\mathrm{d}x\to 0$ $(n\to\infty)$. So $\|u_n\|\to\|u\|$. Hence $u_n\to u$ $(n\to\infty)$ in X due to the uniform convexity of X. □

Remark 2.1 The $(PS)_c$ condition is not satisfied for $c\ge\frac{a^2}{4b}$.

(1) The case $c>\frac{a^2}{4b}$. It follows from

$$J(u)\le\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4\le\frac{a^2}{4b} \tag{7}$$

that if $\{u_n\}$ is a $(PS)_c$ sequence of J, then we have $c\le\frac{a^2}{4b}$. This is a contradiction and the claim is proved.

(2) The case $c=\frac{a^2}{4b}$. Now we suppose that J satisfies the $(PS)_{\frac{a^2}{4b}}$ condition on the contrary, that is to say, if $\{u_n\}\subset X$ is such that $J(u_n)\to\frac{a^2}{4b}$, $J'(u_n)\to 0$, then $\{u_n\}$ possesses a convergent subsequence (still denoted by $\{u_n\}$) and converges to u. Hence $u_n\to u$ in $L^r(\Omega)$, $r\in[1,2^*)$. It follows from $J(u_n)\to\frac{a^2}{4b}$ and (7) that $(\frac{a}{2}\|u_n\|^2-\frac{b}{4}\|u_n\|^4)\to\frac{a^2}{4b}$. And then $\frac{1}{p}\int_\Omega|u_n|^p\,\mathrm{d}x\to 0$ by the definition of energy functional J. Noting that $u_n\to u$ in $L^p(\Omega)$, we obtain $\frac{1}{p}\int_\Omega|u|^p\,\mathrm{d}x=0$. Hence $u=0$ a.e. and $J(u)=0$. However, $J(u_n)\to J(u)=\frac{a^2}{4b}$. This is a contradiction and the claim is proved.

Now, we prove our main result Theorem 1.1 by using Lemma 2.1 and Lemma 2.2.

Proof of Theorem 1.1 According to Lemma 2.1, there exists $\{u_n\}\subset X$ satisfying $J(u_n)\to c_0>0$, $J'(u_n)\to 0$ $(n\to\infty)$. By Lemma 2.2, $\{u_n\}$, which is the sequence obtained by Lemma 2.1, possesses a convergent subsequence (still denoted by $\{u_n\}$) and converges to u.

So it follows from the continuity of J and J' that $J(u) = c_0 > 0$, $J'(u) = 0$. But $J(0) = 0$, therefore $u \neq 0$, namely u is a nontrivial solution of problem (1). □

Proof of Theorem 1.2 We only establish the existence of a nontrivial non-negative solution for problem (1), and the existence of a nontrivial non-positive solution for problem (1) can obtained similarly.

Define a functional by

$$\bar{J}(u) := \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega |u^+|^p \,\mathrm{d}x, \quad \forall u \in X.$$

Then $\bar{J} \in C^1(X, \mathbb{R}^1)$ and

$$\langle \bar{J}'(u), v\rangle = a\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega (u^+)^{p-1} v \,\mathrm{d}x, \quad \forall u, v \in X.$$

From inequality (5) we have

$$\bar{J}(u) \geq J(u) \geq \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\gamma_p^p \|u\|^p.$$

So we can choose small $0 < \rho \leq \min\{[\frac{a}{2b}]^{\frac{1}{2}}, [\frac{ap}{8\gamma_p^p}]^{\frac{1}{p-2}}\}$ such that for all $u \in X$, $\|u\| = \rho$, $J(u) \geq \frac{a}{4}\rho^2 = \gamma > 0$ holds. On the other hand, for $\tau > 0$ and fixed $u \geq 0$, with $\int_\Omega |u^+|^p \,\mathrm{d}x > 0$,

$$\bar{J}(\tau u) = \frac{a}{2}\|u\|^2\tau^2 - \frac{b}{4}\|u\|^4\tau^4 - |\tau|^p \frac{1}{p}\int_\Omega |u^+|^p \,\mathrm{d}x.$$

Then $\bar{J}(\tau u) \to -\infty$ $(\tau \to \infty)$. So there exists $\tau_2 > 0$ such that $u_2 = \tau_2 u \in X$, $\|u_2\| > \rho$, $\bar{J}(u_2) < 0$. Hence, by the mountain pass lemma without (PS) condition (see [11]), we obtain a sequence $\{u_n\} \subset X$ such that $\bar{J}(u_n) \to c_1$, $\bar{J}'(u_n) \to 0$ for

$$c_1 = \inf_{g\in\Gamma} \max_{u\in g([0,1])} \bar{J}(u) \geq \gamma > 0,$$

where

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = u_2\}.$$

Due to

$$\max_{t\in[0,1]} \bar{J}(tu_2) = \max_{t\in[0,1]} J(tu_2) < \frac{a^2}{4b},$$

and from the definition of c_1, we have $0 < c_1 < \frac{a^2}{4b}$. Similarly to the arguments of Lemma 2.2, we can show that under the condition $c_1 < \frac{a^2}{4b}$, $\bar{J}$ satisfies the $(PS)_{c_1}$ condition, *i.e.*, $\{u_n\}$ possesses a convergent subsequence (still denoted by $\{u_n\}$) and converges to u. So it follows from the continuity of $\bar{J}$ and $\bar{J}'$ that $\bar{J}(u) = c_1 > 0$, $\bar{J}'(u) = 0$. But $\bar{J}(0) = 0$, therefore $u \neq 0$.

By the mountain pass theorem, $\bar{J}$ has a positive critical value and the problem

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2 \,\mathrm{d}x)\triangle u = |u^+|^{p-2}u^+, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{8}$$

has a nontrivial solution u. Multiplying the equation by u^- and integrating over Ω, we find

$$(a - b\|u\|^2)\|u^-\|^2 = 0.$$

Noting $u_n \to u$ and $\|u_n\|^2 \nrightarrow \frac{a}{b}$, we obtain $\|u^-\|^2 = 0$. Hence $u^- = 0$ and $u(x) \geq 0$, $x \in \overline{\Omega}$. Therefore, u is a nontrivial non-negative solution of (1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
GY participated in the design of the study and drafted the manuscript. JL carried out the theoretical studies and helped to draft the manuscript. All authors read and approved the final manuscript.

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