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Positive solutions of m -point multi-term fractional integral BVP involving time-delay for fractional differential equations

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Abstract

In the paper, we establish sufficient conditions for the existence and multiplicity of positive solutions to a class of higher-order delayed nonlinear fractional differential equations with m -point multi-term fractional integral boundary conditions. The results are established by converting the problem into an equivalent integral equation and applying fixed point theorems of the cone expansion and compression of norm type. Our study improves the previous results in the literature. As an application, an example is also provided to illustrate our main results.

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Keywords: fractional differential equations; multiple positive solutions; integral boundary value conditions; time-delay

1 Introduction

The fractional differential equation has a significant role to play in many fields such as physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. The fractional differential equation also serves as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. There are a large number of papers dealing with the existence or multiplicity of solutions or positive solutions of initial or boundary value problems for some differential equations. For details and examples, see [1–23] and the references therein. In [8, 9, 16, 18–23], the authors have discussed the existence of multiple positive solutions for boundary value problems of integer or fractional differential equations.

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, and so on. For more details of nonlocal and integral boundary conditions, see [5, 10–12, 20] and the references therein. In addition, it is well known that delay arises naturally in practical systems due to the transmission of signal or the mechanical transmission. Though the theory of ordinary differential equations with delays is mature, not much has been done for fractional differential equations with delays; see [6, 10, 13, 14].

In [11], Tariboon *et al.* are concerned with the existence of at least one, two or three positive solutions for the boundary value problem with three-point multi-term fractional integral boundary conditions,

$$\begin{cases} D_{0+}^q u(t) + f(t, u(t)) = 0, & 1 < q \leq 2, t \in [0, 1], \\ u(0) = 0, & u(1) = \sum_{i=1}^m \alpha_i (I_{0+}^{p_i} u)(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

where $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$. D_{0+}^q is the standard Riemann-Liouville derivative of order q . $I_{0+}^{p_i}$ is the Riemann-Liouville fractional integral of order $p_i > 0$. $\alpha_i \geq 0$ ($i = 1, 2, \dots, m - 2$) are real constants.

To the best of our knowledge, no one has studied the existence of multiple positive solutions with delayed nonlinear fractional differential equations. In this article, motivated by the above-mentioned papers, we study the existence of multiple positive solutions for the following higher-order delayed nonlinear fractional differential equation with m -point multi-term fractional integral boundary conditions:

$$\begin{cases} D_{0+}^q u(t) + f(t, u(t - \tau_1), u(t + \tau_2)) = 0, & t \in [0, 1], \\ u^{(j)}(0) = 0, & u(1) = \sum_{i=1}^{m-2} \alpha_i (I_{0+}^{p_i} u)(\zeta_i), \quad 0 \leq j \leq n - 2, 0 < \zeta_i < 1, \end{cases} \tag{1.2}$$

subject to the following initial conditions:

$$\begin{cases} au(t) - bu'(t) = \eta(t), & t \in [-\tau_1, 0], \\ cu(t) + du'(t) = \xi(t), & t \in [1, 1 + \tau_2], \end{cases} \tag{1.3}$$

where $0 \leq \tau_1, \tau_2 < \theta \in (0, 1/2)$ are suitably small. D_{0+}^q is the standard Riemann-Liouville derivative of order $n - 1 < q \leq n, n \geq 3$. $a, b, c, d \geq 0, \eta(t) \in C([-\tau_1, 0], [0, +\infty)), \xi(t) \in C([1, 1 + \tau_2], [0, +\infty)), f \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$. $I_{0+}^{p_i}$ is the Riemann-Liouville fractional integral of order p_i . $\alpha_i \geq 0$ ($i = 1, 2, \dots, m - 2$) are real constants such that $\sum_{i=1}^{m-2} \frac{\alpha_i \zeta_i^{p_i+q-1} \Gamma(q)}{\Gamma(p_i+q)} < 1$.

Remark 1.1 By (1.2) and (1.3), we have $\eta(0) = 0$ and $\xi(1) = cu(1) + du'(1)$.

Remark 1.2 When $1 < q \leq 2, \tau_1 = \tau_2 = 0, \varsigma_i \equiv \eta, j = 0$, then (1.2) and (1.3) degenerate into (1.1). So our models extend (1.1).

2 Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the literature.

Definition 2.1 (see [24, 25]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 (see [24, 25]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^\alpha u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1} u(s) ds,$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 (see [17]) Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L(0,1)$. Then

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, where n is the smallest integer greater than or equal to α .

Lemma 2.2 (see [26]) Let E be a Banach space, $P \subseteq E$ is a cone, and Ω_1, Ω_2 are two bounded open balls of E centered at the origin with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ is a completely continuous operator such that either

- (i) $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, or
- (ii) $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$

holds. Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

For simplicity, we put

$$\Omega = 1 - \sum_{i=1}^{m-2} \frac{\alpha_i \zeta_i^{p_i+q-1} \Gamma(q)}{\Gamma(p_i + q)}. \tag{2.1}$$

Now we present the Green's function for BVP (1.2)-(1.3).

Lemma 2.3 Let $\sum_{i=1}^{m-2} \frac{\alpha_i \zeta_i^{p_i+q-1} \Gamma(q)}{\Gamma(p_i+q)} < 1$, $\alpha_i \geq 0, p_i > 0$ ($i = 1, 2, \dots, m-2$), and $h \in C([0,1], \mathbb{R})$, $1 \leq n - 1 < q \leq n$. The unique solution of

$$\begin{cases} D_{0+}^q u(t) + h(t) = 0, & t \in [0,1], \\ u^{(j)}(0) = 0, & u(1) = \sum_{i=1}^{m-2} \alpha_i (I_{0+}^{p_i} u)(\zeta_i), \quad j = 0, 1, \dots, n-2, 0 < \zeta_i < 1, \end{cases} \tag{2.2}$$

is

$$u(t) = \int_0^1 G(t,s)h(s) ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = g(t,s) + \sum_{i=1}^{m-2} \frac{\alpha_i t^{q-1}}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s), \tag{2.3}$$

where

$$g(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t^{q-1}(1-s)^{q-1} - (t-s)^{q-1}, & 0 \leq s \leq t \leq 1, \\ t^{q-1}(1-s)^{q-1}, & 0 \leq t \leq s \leq 1 \end{cases} \tag{2.4}$$

and

$$g_i(\zeta_i, s) = \begin{cases} \zeta_i^{p_i+q-1}(1-s)^{q-1} - (\zeta_i - s)^{p_i+q-1}, & 0 \leq s \leq \zeta_i < 1, \\ \zeta_i^{p_i+q-1}(1-s)^{q-1}, & 0 < \zeta_i \leq s \leq 1. \end{cases} \tag{2.5}$$

Proof Applying Lemma 2.1, BVP (2.2) can be expressed as an equivalent integral equation

$$u(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + C_1 t^{q-1} + C_2 t^{q-2} + \dots + C_n t^{q-n}, \tag{2.6}$$

for $C_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$) $\in \mathbb{R}$. $u^{(j)}(0) = 0$ ($j = 0, 1, 2, \dots, n-2$) implies that $C_2 = C_3 = \dots = C_n = 0$. Taking the Riemann-Liouville fractional integral of order $p_i > 0$ for (2.6), we get

$$\begin{aligned} (I_{0+}^{p_i} u)(t) &= \int_0^t \frac{(t-s)^{p_i-1}}{\Gamma(p_i)} \left(C_1 s^{q-1} - \int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} h(r) dr \right) ds \\ &= C_1 \int_0^t \frac{(t-s)^{p_i-1} s^{q-1}}{\Gamma(p_i)} ds - \int_0^t \frac{(t-s)^{p_i-1}}{\Gamma(p_i)} \int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} h(r) ds dr \\ &= C_1 \frac{t^{p_i+q-1} \Gamma(q)}{\Gamma(p_i+q)} - \frac{1}{\Gamma(p_i+q)} \int_0^t (t-s)^{p_i+q-1} h(s) ds. \end{aligned}$$

$u(1) = \sum_{i=1}^{m-2} \alpha_i (I_{0+}^{p_i} u)(\zeta_i)$ yields

$$\begin{aligned} C_1 - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s) ds \\ = C_1 \sum_{i=1}^{m-2} \frac{\alpha_i \zeta_i^{p_i+q-1} \Gamma(q)}{\Gamma(p_i+q)} - \sum_{i=1}^{m-2} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^{\zeta_i} (\zeta_i - s)^{p_i+q-1} h(s) ds. \end{aligned}$$

Then we have

$$C_1 = \frac{1}{\Omega} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s) ds - \sum_{i=1}^{m-2} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^{\zeta_i} (\zeta_i - s)^{p_i+q-1} h(s) ds \right].$$

Therefore, the unique solution of BVP (2.2) is written as

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + \frac{1}{\Omega \Gamma(q)} \int_0^1 (1-s)^{q-1} t^{q-1} h(s) ds \\ &\quad - \frac{1}{\Omega} \sum_{i=1}^{m-2} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^{\zeta_i} (\zeta_i - s)^{p_i+q-1} t^{q-1} h(s) ds. \end{aligned}$$

Hence, by taking into account (2.1), we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + \frac{1}{\Omega \Gamma(q)} \int_0^1 (1-s)^{q-1} t^{q-1} h(s) ds \\ &\quad - \frac{1}{\Omega} \sum_{i=1}^{m-2} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^{\zeta_i} (\zeta_i - s)^{p_i+q-1} t^{q-1} h(s) ds \\ &\quad + \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} t^{q-1} h(s) ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} t^{q-1} h(s) ds \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} t^{q-1} h(s) ds - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds \\
 &\quad + \sum_{i=1}^{m-2} \frac{\alpha_i t^{q-1}}{\Omega \Gamma(p_i + q)} \left(\int_0^1 \zeta_i^{p_i+q-1} (1-s)^{q-1} h(s) ds - \int_0^{\zeta_i} (\zeta_i-s)^{p_i+q-1} h(s) ds \right) \\
 &= \int_0^1 g(t,s) h(s) ds + \int_0^1 \sum_{i=1}^{m-2} \frac{\alpha_i t^{q-1}}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s) h(s) ds \\
 &= \int_0^1 G(t,s) h(s) ds.
 \end{aligned}$$

The proof is complete. □

Lemma 2.4 *The Green’s function $G(t, s)$ defined by (2.3)-(2.5) has the following properties:*

- (A1) $G(t, s) \in C([0, 1] \times [0, 1])$ and $G(t, s) \geq 0$, for all $t, s \in [0, 1]$.
- (A2) $G(t, s) \leq \max_{0 \leq t \leq 1} G(t, s) \leq G(s)$ and $G(t, s) \geq \min_{0 \leq t \leq 1} G(t, s) \geq \sigma(t)G(s)$, where

$$G(s) = \frac{s(1-s)^{q-1}}{\Gamma(q-1)} + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s), \quad \sigma(t) = \frac{t^{q-1}(1-t)}{q-1}.$$

- (A3) If $\theta \in (0, 1/2)$, then $\min_{t \in [\theta, 1-\theta]} G(t, s) \geq \sigma(\theta)G(s)$.

Proof From the expression of $G(t, s)$, it is obvious that $G(t, s) \in C([0, 1] \times [0, 1])$. To prove $G(t, s) \geq 0$, we will show that $g(t, s), g_i(\zeta_i, s) \geq 0, i = 1, 2, \dots, m-2$, for all $0 \leq s, t \leq 1$.

Let $k_1(t, s) = t^{q-1}(1-s)^{q-1} - (t-s)^{q-1}$ for $0 \leq s \leq t \leq 1$, then we have

$$k_1(t, s) = (t-ts)^{q-1} - (t-s)^{q-1} \geq (t-s)^{q-1} - (t-s)^{q-1} = 0.$$

For $0 \leq t \leq s \leq 1, k_2(t, s) = (t-ts)^{q-1} \geq 0$. Therefore, $g(t, s) \geq 0$ for all $0 \leq s, t \leq 1$. Now, let $k_3^i(\zeta_i, s) = \zeta_i^{p_i+q-1}(1-s)^{q-1} - (\zeta_i-s)^{p_i+q-1}$ for $0 \leq s \leq \zeta_i < 1$, then we have

$$\begin{aligned}
 k_3^i(\zeta_i, s) &= \zeta_i^{p_i+q-1}(1-s)^{q-1} - \zeta_i^{p_i+q-1} \left(1 - \frac{s}{\zeta_i} \right)^{p_i+q-1} \\
 &\geq \zeta_i^{p_i+q-1}(1-s)^{q-1} - \zeta_i^{p_i+q-1}(1-s)^{p_i+q-1} \\
 &= \zeta_i^{p_i+q-1} \left((1-s)^{q-1} - (1-s)^{p_i+q-1} \right) \geq 0.
 \end{aligned}$$

For $0 < \zeta_i \leq s \leq 1, k_4^i(\zeta_i, s) = \zeta_i^{p_i+q-1}(1-s)^{q-1} \geq 0$. Therefore $g_i(\zeta_i, s) \geq 0, i = 1, 2, \dots, m-2$, for all $0 \leq s \leq 1$.

In the following, we prove (A2). When $s \leq t$, we have $1-s \geq 1-t$, then

$$\begin{aligned}
 k_1(t, s) &= t^{q-1}(1-s)^{q-1} - (t-s)^{q-1} = (q-1) \int_{t-s}^{t-ts} x^{q-2} dx \\
 &\leq (q-1)(t-ts)^{q-2}((t-ts) - (t-s)) \\
 &= (q-1)t^{q-2}(1-s)^{q-2}(1-t)s \\
 &\leq (q-1)s(1-s)^{q-1}
 \end{aligned}$$

and

$$\begin{aligned}
 k_1(t, s) &= t^{q-1}(1-s)^{q-1} - (t-s)^{q-1} \\
 &= (t-ts)^{q-2}(t-ts) - (t-s)^{q-2}(t-s) \\
 &\geq (t-ts)^{q-2}(t-ts) - (t-ts)^{q-2}(t-s) \\
 &= t^{q-2}(1-s)^{q-2}(1-t)s \\
 &\geq t^{q-1}(1-t)s(1-s)^{q-1}.
 \end{aligned}$$

When $t \leq s$, we derive from $q > 2$ that

$$\begin{aligned}
 k_2(t, s) &= t^{q-1}(1-s)^{q-1} \leq (q-1)t^{q-2}t(1-s)^{q-1} \\
 &\leq (q-1)t^{q-2}s(1-s)^{q-1} \\
 &\leq (q-1)s(1-s)^{q-1}
 \end{aligned}$$

and

$$k_2(t, s) = t^{q-1}(1-s)^{q-1} \geq t^{q-1}(1-t)s(1-s)^{q-1}.$$

Thus,

$$\frac{t^{q-1}(1-t)s(1-s)^{q-1}}{\Gamma(q)} \leq g(t, s) \leq \frac{(q-1)s(1-s)^{q-1}}{\Gamma(q)} = \frac{s(1-s)^{q-1}}{\Gamma(q-1)}.$$

From the above analysis, we have for $0 \leq s \leq 1$,

$$\begin{aligned}
 G(t, s) &\leq \max_{0 \leq t \leq 1} G(t, s) = \max_{0 \leq t \leq 1} \left(g(t, s) + \sum_{i=1}^{m-2} \frac{\alpha_i t^{q-1}}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s) \right) \\
 &\leq \frac{s(1-s)^{q-1}}{\Gamma(q-1)} + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s) = G(s)
 \end{aligned}$$

and

$$\begin{aligned}
 G(t, s) &\geq \min_{0 \leq t \leq 1} G(t, s) = \min_{0 \leq t \leq 1} \left(g(t, s) + \sum_{i=1}^{m-2} \frac{\alpha_i t^{q-1}}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s) \right) \\
 &\geq \frac{t^{q-1}(1-t)s(1-s)^{q-1}}{\Gamma(q)} + \sum_{i=1}^{m-2} \frac{\alpha_i t^{q-1}}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s) \\
 &\geq \min_{0 \leq t \leq 1} \left\{ \frac{t^{q-1}(1-t)}{q-1}, t^{q-1} \right\} \left(\frac{(q-1)s(1-s)^{q-1}}{\Gamma(q)} + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} g_i(\zeta_i, s) \right) \\
 &= \frac{t^{q-1}(1-t)}{q-1} G(s) = \sigma(t)G(s).
 \end{aligned}$$

Now we prove (A3). By (A1) and (A2), we have

$$\min_{\theta \leq t \leq 1-\theta} G(t, s) \geq \min_{\theta \leq t \leq 1-\theta} \sigma(t)G(s)$$

and

$$\frac{\partial \sigma(t)}{\partial t} = \frac{(q-1)t^{q-2}(1-t) - t^{q-1}}{q-1} = \frac{(q-1)t^{q-2} - qt^{q-1}}{q-1} = \frac{t^{q-2}(q-1-qt)}{q-1}.$$

It is clear that $\sigma(t)$ is increasing in $t \in [0, \frac{q-1}{q}]$ and decreasing in $t \in [\frac{q-1}{q}, 1]$, respectively. By $q > 2$, we have

$$\frac{q-1}{q} = 1 - \frac{1}{q} > 1 - \frac{1}{2} = \frac{1}{2} > \theta,$$

thus, $\sigma(t)$ is increasing in $t \in [\theta, \frac{q-1}{q}]$, for $\frac{q-1}{q} \leq 1 - \theta$ or $\frac{q-1}{q} \geq 1 - \theta$, and we have $\min_{\theta \leq t \leq 1-\theta} \sigma(t) = \min\{\sigma(\theta), \sigma(1-\theta)\}$. For $0 < \theta < 1 - \theta < 1, q > 2$, we get

$$\sigma(\theta) - \sigma(1-\theta) = \frac{\theta^{q-1}(1-\theta)}{q-1} - \frac{(1-\theta)^{q-1}\theta}{q-1} = \frac{\theta^{q-1}(1-\theta)}{q-1} \left(1 - \left(\frac{1-\theta}{\theta} \right)^{q-2} \right) < 0.$$

Therefore

$$\min_{\theta \leq t \leq 1-\theta} G(t,s) \geq \min_{\theta \leq t \leq 1-\theta} \sigma(t)G(s) = \sigma(\theta)G(s).$$

This proof is complete. □

3 Existence of multiple positive solutions

In this section, we will consider the existence of multiple positive solutions for the BVP (1.2)-(1.3).

Let $E = \{u(t) : u(t) \in C[-\tau_1, 1 + \tau_2]\}$ denote a real Banach space with the norm $\|\cdot\|$ defined by $\|u\| = \max_{-\tau_1 \leq t \leq 1 + \tau_2} |u(t)|$. Define the cone $P \subset E$ by $P = \{u \in E : u(t) \geq 0\}$. Let

$$K = \{u \in P : u(t) \geq \sigma(t)\|u\|, \forall t \in [0, 1]\}, \tag{3.1}$$

$$K_r = \{u \in K : \|u\| < r\}, \quad \partial K_r = \{u \in K : \|u\| = r\}. \tag{3.2}$$

Suppose that $u(t)$ is a solution of (1.2)-(1.3); according to Lemma 2.3 and Remark 1.1, it can be written as

$$u(t) = \begin{cases} u(\tau_1, t), & -\tau_1 \leq t \leq 0, \\ \int_0^1 G(t,s)f(s, u(s-\tau_1), u(s+\tau_2)) ds, & 0 \leq t \leq 1, \\ u(\tau_2, t), & 1 \leq t \leq 1 + \tau_2, \end{cases}$$

where

$$u(\tau_1, t) = \begin{cases} e^{\frac{a}{b}t} \left(\frac{1}{b} \int_t^0 e^{-\frac{a}{b}s} \eta(s) ds + u(0) \right), & -\tau_1 \leq t \leq 0, b \neq 0, \\ \frac{\eta(t)}{a}, & -\tau_1 \leq t \leq 0, b = 0 \end{cases}$$

and

$$u(\tau_2, t) = \begin{cases} e^{-\frac{c}{d}t} \left(\frac{1}{d} \int_1^t e^{\frac{c}{d}s} \xi(s) ds + e^{\frac{c}{d}} u(1) \right), & 1 \leq t \leq 1 + \tau_2, d \neq 0, \\ \frac{\xi(t)}{c}, & 1 \leq t \leq 1 + \tau_2, d = 0. \end{cases}$$

Throughout this paper, we assume that $v_0(t)$ is the solution of (1.2)-(1.3) with $f \equiv 0$. Clearly, $v_0(t)$ can be expressed as follows:

$$v_0(t) = \begin{cases} v_0(\tau_1, t), & -\tau_1 \leq t \leq 0, \\ 0, & 0 \leq t \leq 1, \\ v_0(\tau_2, t), & 1 \leq t \leq 1 + \tau_2, \end{cases}$$

where

$$v_0(\tau_1, t) = \begin{cases} \frac{e^{\frac{a}{b}t}}{b} \int_t^0 e^{-\frac{a}{b}s} \eta(s) ds, & -\tau_1 \leq t \leq 0, b \neq 0, \\ \frac{\eta(t)}{a}, & -\tau_1 \leq t \leq 0, b = 0 \end{cases}$$

and

$$v_0(\tau_2, t) = \begin{cases} \frac{e^{-\frac{c}{d}t}}{d} \int_1^t e^{\frac{c}{d}s} \xi(s) ds, & 1 \leq t \leq 1 + \tau_2, d \neq 0, \\ \frac{\xi(t)}{c}, & 1 \leq t \leq 1 + \tau_2, d = 0. \end{cases}$$

Obviously, $v_0(t) \geq 0$ for each $t \in [-\tau_1, 1 + \tau_2]$.

Let $u(t)$ be a solution of (1.2)-(1.3) and $v(t) = u(t) - v_0(t)$. Noting that $v(t) \equiv u(t)$ for $0 \leq t \leq 1$, we have

$$v(t) = \begin{cases} v(\tau_1, t), & -\tau_1 \leq t \leq 0, \\ \int_0^1 G(t, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds, & 0 \leq t \leq 1, \\ v(\tau_2, t), & 1 \leq t \leq 1 + \tau_2, \end{cases}$$

where

$$\begin{aligned} (v + v_0)(s - \tau_1) &= v(s - \tau_1) + v_0(s - \tau_1), \\ (v + v_0)(s + \tau_2) &= v(s + \tau_2) + v_0(s + \tau_2), \\ v(\tau_1, t) &= \begin{cases} e^{\frac{a}{b}t} v(0), & -\tau_1 \leq t \leq 0, b \neq 0, \\ 0, & -\tau_1 \leq t \leq 0, b = 0 \end{cases} \end{aligned}$$

and

$$v(\tau_2, t) = \begin{cases} e^{-\frac{c}{d}(t-1)} v(1), & 1 \leq t \leq 1 + \tau_2, d \neq 0, \\ 0, & 1 \leq t \leq 1 + \tau_2, d = 0. \end{cases}$$

Define an operator $A : E \rightarrow E$ as follows:

$$(Av)(t) = \begin{cases} (B_1v)(t), & -\tau_1 \leq t \leq 0, \\ \int_0^1 G(t, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds, & 0 \leq t \leq 1, \\ (B_2v)(t), & 1 \leq t \leq 1 + \tau_2, \end{cases} \tag{3.3}$$

where

$$(B_1v)(t) = \begin{cases} e^{\frac{a}{b}t} \int_0^1 G(0, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds, & -\tau_1 \leq t \leq 0, b \neq 0, \\ 0, & -\tau_1 \leq t \leq 0, b = 0 \end{cases}$$

and

$$(B_2v)(t) = \begin{cases} e^{-\frac{c}{d}} \int_0^1 f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) \\ \quad \times G(1, s) ds, & 1 \leq t \leq 1 + \tau_2, d \neq 0, \\ 0, & 1 \leq t \leq 1 + \tau_2, d = 0. \end{cases}$$

It is easy to derive that u is a positive solution of BVP (1.2)-(1.3) if $v = u - v_0$ is a nontrivial fixed point of $A : K \rightarrow K$, where v_0 is defined as before.

Lemma 3.1 $A : K \rightarrow K$ defined by (3.3) is completely continuous.

Proof For $v \in K$, we find from Lemma 2.4 and the definition of A that $0 \leq (Av)(t) \leq (Av)(0)$, for $t \in [-\tau_1, 0]$, and $0 \leq (Av)(t) \leq (Av)(1)$ for $t \in [1, 1 + \tau_2]$. Thus, $\|Av\| = \|Av\|_{[0,1]} = \max_{0 \leq t \leq 1} |(Av)(t)|$. It follows from Lemma 2.4 that

$$\begin{aligned} (Av)(t) &\geq \sigma(t) \int_0^1 G(s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\geq \sigma(t) \|Av\|_{[0,1]} = \sigma(t) \|Av\|. \end{aligned}$$

Thus, $A(K) \subset K$. In addition, since f is continuous, it follows that A is continuous.

Let $Q_1 \in K$ be bounded, that is, there exists a positive constant $M_1 > 0$ such that $\|v\|_{[-\tau_1, 1 + \tau_2]} \leq M_1$ for all $v \in Q_1$. Then $\|v + v_0\|_{[-\tau_1, 1 + \tau_2]} \leq M_1 + M_0 \triangleq M_2$ for $v \in Q_1$, where v_0 is defined as before. Define a set $Q_2 \subset E$ as follows:

$$Q_2 = \{ \psi \in E : \|\psi\|_{[-\tau_1, 1 + \tau_2]} \leq M_2 \}.$$

Hence, $\max_{-\tau_1 \leq t \leq 1 + \tau_2} |(v + v_0)(t)| \leq \|v + v_0\|_{[-\tau_1, 1 + \tau_2]} \leq \|v\|_{[-\tau_1, 1 + \tau_2]} + \|v_0\|_{[-\tau_1, 1 + \tau_2]} \leq M_2$. Noting that f is continuous on $[0, 1] \times [0, M_2] \times [0, M_2]$, there exists a constant $M_3 > 0$ such that on $[0, 1] \times [0, M_2] \times [0, M_2]$,

$$|f(t, (v + v_0)(t - \tau_1), (v + v_0)(t + \tau_2))| \leq M_3.$$

Therefore,

$$\|Av\|_{[-\tau_1, 1 + \tau_2]} = \|Av\|_{[0,1]} \leq M_3 \int_0^1 G(s) ds.$$

Hence, $A(Q_1)$ is bounded.

Finally, we show the operator A is equicontinuous. For $v \in Q_1$, we have

$$\begin{aligned} (Av)'(t) &= \int_0^1 \frac{\partial}{\partial t} G(t, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds, \quad t \in [0, 1], \\ (Av)'(t) &= \frac{a}{b} e^{\frac{a}{b}t} \int_0^1 G(0, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds, \quad t \in [-\tau_1, 0] \end{aligned}$$

and

$$(Av)'(t) = -\frac{c}{d} e^{-\frac{c}{d}(t-1)} \int_0^1 G(1, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds, \quad t \in [1, 1 + \tau_2].$$

In the light of $f \leq M_3$ and

$$\begin{aligned} \frac{\partial}{\partial t} G(t, s) &\leq \frac{(q-1)t^{q-2}(1-s)^{q-1} + (q-1)(t-s)^{q-2}}{\Gamma(q)} + \sum_{i=1}^{m-2} \frac{\alpha_i(q-1)t^{q-2}}{\Omega\Gamma(p_i+q)} g_i(\zeta_i, s) \\ &\leq \frac{1}{\Gamma(q-1)} + \sum_{i=1}^{m-2} \frac{\alpha_i(q-1)}{\Omega\Gamma(p_i+q)} g_i(\zeta_i, s) \triangleq M', \end{aligned}$$

we have $\|(Av)'\| \leq M$ for some positive constant M . Thus, for $-\tau_1 \leq t_1 < t_2 \leq 1 + \tau_2$, we have

$$\|(Av)(t_2) - (Av)(t_1)\| \leq \int_{t_1}^{t_2} |(Av)'(s)| ds \leq M(t_2 - t_1) \rightarrow 0, \quad \text{as } t_1 \rightarrow t_2.$$

Therefore, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ which is independent of t_1, t_2 , and ν such that $\|(Av)(t_2) - (Av)(t_1)\| \leq \epsilon$, whenever $|t_2 - t_1| \leq \delta$. Thus, $A(Q_1)$ is equicontinuous. In view of the Ascoli-Arzelà theorem, we can easily see that $A : K \rightarrow K$ is a completely continuous operator. The proof is complete. \square

Further we make the following assumptions for $f(t, x, y)$:

- (H₁) There exists a constant $r_1 > 0$ such that $0 \leq x \leq r_1 + \|v_0\|_{[-\tau_1, 0]}$, $0 \leq y \leq r_1 + \|v_0\|_{[1, 1+\tau_2]}$, and $0 \leq t \leq 1$ implies $f(t, x, y) < \rho_1 r_1$, where $\rho_1 = \frac{1}{\int_0^1 G(s) ds}$.
- (H₂) There exists a constant $r_2 > 0$ such that $\sigma_1 r_2 \leq x \leq r_2$, $\sigma_2 r_2 \leq y \leq r_2$, and $\theta \leq t \leq 1 - \theta$ implies $f(t, x, y) > \rho_2 r_2$, where $\rho_2 = \frac{1}{\sigma(\theta) \int_\theta^{1-\theta} G(s) ds}$, $\sigma_1 = \min_{\theta-\tau_1 \leq t \leq 1-\theta-\tau_1} |\sigma(t)|$, $\sigma_2 = \min_{\theta+\tau_2 \leq t \leq 1-\theta+\tau_2} |\sigma(t)|$.
- (H₃) $f_\infty = \liminf_{x+y \rightarrow +\infty} \min_{t \in [0, 1]} \frac{f(t, x, y)}{x+y} = \infty$.
- (H₄) $f^\infty = \limsup_{x+y \rightarrow +\infty} \max_{t \in [0, 1]} \frac{f(t, x, y)}{x+y} = 0$.
- (H₅) $f_0 = \liminf_{x+y \rightarrow 0} \min_{t \in [0, 1]} \frac{f(t, x, y)}{x+y} = \infty$.
- (H₆) $f^0 = \limsup_{x+y \rightarrow 0} \max_{t \in [0, 1]} \frac{f(t, x, y)}{x+y} = 0$.

Theorem 3.1 *Assume that (H₁), (H₂), and (H₃) are satisfied. If $r_1 > r_2 > 0$, then BVP (1.2)-(1.3) has at least two positive solutions u_1 and u_2 such that*

$$0 < r_2 < \|u_1\|_{[0, 1]} < r_1 < \|u_2\|_{[0, 1]}.$$

Proof Let $A : K \rightarrow K$ be the cone preserving completely continuous that is defined by (3.3). Let $\Phi_{r_1} = \{v \in E : \|v\| < r_1\}$, then for any $v \in K \cap \partial\Phi_{r_1}$, we get

$$\begin{aligned} 0 &\leq (v + v_0)(t - \tau_1) \leq \|v\| + \|v_0\|_{[-\tau_1, 1]} = r_1 + \|v_0\|_{[-\tau_1, 0]}, \quad 0 \leq t \leq 1, \\ 0 &\leq (v + v_0)(t + \tau_2) \leq \|v\| + \|v_0\|_{[0, 1+\tau_2]} = r_1 + \|v_0\|_{[1, 1+\tau_2]}, \quad 0 \leq t \leq 1. \end{aligned}$$

Thus, from (A3) and (H₁), we have

$$\begin{aligned} \|Av\| &= \|Av\|_{[0, 1]} = \max_{t \in [0, 1]} (Av)(t) \\ &= \max_{t \in [0, 1]} \int_0^1 G(t, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 G(s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &< \rho_1 r_1 \int_0^1 G(s) ds = r_1 = \|v\|. \end{aligned}$$

Therefore,

$$\|Av\| < \|v\|, \quad v \in K \cap \partial\Phi_{r_1}. \tag{3.4}$$

Let $\Phi_{r_2} = \{v \in E : \|v\| < r_2\}$, then for any $v \in K \cap \partial\Phi_{r_2}$, we have

$$\begin{aligned} r_2 = \|v\| &\geq v(t - \tau_1) \geq \sigma(t - \tau_1)\|v\| \geq \sigma_1 r_2, \quad \theta \leq t \leq 1 - \theta, \\ r_2 = \|v\| &\geq v(t + \tau_2) \geq \sigma(t + \tau_2)\|v\| \geq \sigma_2 r_2, \quad \theta \leq t \leq 1 - \theta. \end{aligned}$$

Thus, from (H₂) and (A3) of Lemma 2.4, we get

$$\begin{aligned} \|Av\| &= \|Av\|_{[0,1]} = \max_{t \in [0,1]} (Av)(t) \\ &= \max_{t \in [0,1]} \int_0^1 G(t, s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\geq \min_{t \in [\theta, 1-\theta]} \int_0^1 G(t, s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\geq \sigma(\theta) \int_0^1 G(s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\geq \sigma(\theta) \int_\theta^{1-\theta} G(s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &= \sigma(\theta) \int_\theta^{1-\theta} G(s)f(s, v(s - \tau_1), v(s + \tau_2)) ds \\ &> \rho_2 r_2 \sigma(\theta) \int_\theta^{1-\theta} G(s) ds = r_2 = \|v\|. \end{aligned}$$

So

$$\|Av\| > \|v\|, \quad v \in K \cap \partial\Phi_{r_2}. \tag{3.5}$$

Choose $L > 0$ such that

$$(\sigma_1 + \sigma_2)L\sigma(\theta) \int_\theta^{1-\theta} G(s) ds > 1. \tag{3.6}$$

From (H₃), there exists $R_1 > 0$ such that

$$f(t, x, y) \geq L(x + y), \quad x, y \geq 0, x + y \geq R_1, 0 \leq t \leq 1. \tag{3.7}$$

Choose

$$R_0 > \max \left\{ r_1, \frac{R_1}{2 \min\{\sigma_1, \sigma_2\}} \right\}.$$

Let $\Phi_R = \{v \in E : \|v\| < R, R \geq R_0\}$, then for any $v \in K \cap \partial\Phi_R$, we have

$$v(t - \tau_1) \geq \sigma(t - \tau_1)\|v\| \geq \sigma_1 R \geq \min\{\sigma_1, \sigma_2\}R_0 > \frac{1}{2}R_1, \quad \theta \leq t \leq 1 - \theta,$$

$$v(t + \tau_2) \geq \sigma(t + \tau_2)\|v\| \geq \sigma_2 R \geq \min\{\sigma_1, \sigma_2\}R_0 > \frac{1}{2}R_1, \quad \theta \leq t \leq 1 - \theta.$$

Then, from (3.6) and (3.7) we have

$$\begin{aligned} \|Av\| &= \|Av\|_{[0,1]} = \max_{t \in [0,1]} (Av)(t) \\ &= \max_{t \in [0,1]} \int_0^1 G(t,s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\geq \min_{t \in [\theta, 1-\theta]} \int_0^1 G(t,s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\geq \sigma(\theta) \int_0^1 G(s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\geq \sigma(\theta) \int_\theta^{1-\theta} G(s)f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &= \sigma(\theta) \int_\theta^{1-\theta} G(s)f(s, v(s - \tau_1), v(s + \tau_2)) ds \\ &\geq L\sigma(\theta) \int_\theta^{1-\theta} G(s)[v(s - \tau_1) + v(s + \tau_2)] ds \\ &\geq (\sigma_1 + \sigma_2)RL\sigma(\theta) \int_\theta^{1-\theta} G(s) ds > R = \|v\|. \end{aligned}$$

Therefore,

$$\|Av\| > \|v\|, \quad v \in K \cap \partial\Phi_R. \tag{3.8}$$

Applying Lemma 2.2 to (3.4) and (3.5) yields the result that A has a fixed point $v_1 \in K \cap (\overline{\Phi}_{r_1} \setminus \Phi_{r_2})$ with $v_1(t) \geq \sigma_1\|u\| > 0, t \in [0, 1]$. Similarly, Lemma 2.2 associated with (3.4) and (3.8) shows that A has another fixed point $v_2 \in K \cap (\overline{\Phi}_R \setminus \Phi_{r_1})$ with $v_2(t) \geq \sigma_2\|u\| > 0, t \in [0, 1]$, which means that $u_1(t) = v_1(t) + v_0(t)$ and $u_2(t) = v_2(t) + v_0(t)$ are two positive solutions of BVP (1.2)-(1.3). Since

$$\|u_i\|_{[0,1]} = \|v_i + v_0\|_{[0,1]} = \|v_i\|_{[0,1]} = \|Av_i\|_{[0,1]} = \|Av_i\| = \|v_i\|, \quad i = 1, 2,$$

it follows that $u_1(t)$ and $u_2(t)$ satisfy

$$0 < r_2 < \|u_1\|_{[0,1]} = \|v_1\| < r_1 < \|v_2\| = \|u_2\|_{[0,1]}.$$

The proof is complete. □

Theorem 3.2 *Assume that $(H_1), (H_3),$ and (H_5) are satisfied. There exist constants $R > r_1 > r > 0,$ then BVP (1.2)-(1.3) has at least two positive solutions u_1 and u_2 such that*

$$0 < r < \|u_1\|_{[0,1]} < r_1 < \|u_2\|_{[0,1]}.$$

Proof Choose $0 < r < r_1 < R$, let $\Phi_r = \{v \in E : \|v\| < r, r < r_1\}$. For the same $L > 0$ satisfying (3.6), (H_5) implies that

$$f(t, x, y) \geq L(x + y), \quad x, y \geq 0, x + y \leq 2r, 0 \leq t \leq 1. \tag{3.9}$$

Then for any $v \in K \cap \partial\Phi_r$, we have

$$\begin{aligned} r = \|v\| &\geq v(t - \tau_1) \geq \sigma(t - \tau_1)\|v\| \geq \sigma_1 r, & \theta \leq t \leq 1 - \theta, \\ r = \|v\| &\geq v(t + \tau_2) \geq \sigma(t + \tau_2)\|v\| \geq \sigma_2 r, & \theta \leq t \leq 1 - \theta. \end{aligned}$$

Then, from (3.6) and (3.9) we have

$$\begin{aligned} \|Av\| &= \|Av\|_{[0,1]} = \max_{t \in [0,1]} (Av)(t) \\ &= \max_{t \in [0,1]} \int_0^1 G(t, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) \, ds \\ &\geq \sigma(\theta) \int_0^1 G(s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) \, ds \\ &\geq \sigma(\theta) \int_\theta^{1-\theta} G(s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) \, ds \\ &= \sigma(\theta) \int_\theta^{1-\theta} G(s) f(s, v(s - \tau_1), v(s + \tau_2)) \, ds \\ &\geq L\sigma(\theta) \int_\theta^{1-\theta} G(s) [v(s - \tau_1) + v(s + \tau_2)] \, ds \\ &\geq (\sigma_1 + \sigma_2)rL\sigma(\theta) \int_\theta^{1-\theta} G(s) \, ds > r = \|v\|. \end{aligned}$$

Therefore,

$$\|Av\| > \|v\|, \quad v \in K \cap \partial\Phi_r. \tag{3.10}$$

Applying Lemma 2.2 to (3.4) and (3.10) yields that A has a fixed point $v_1 \in K \cap (\overline{\Phi}_{r_1} \setminus \Phi_r)$ with $v_1(t) \geq \sigma_1 \|u\| > 0, t \in [0, 1]$. Similarly, Lemma 2.2 associated with (3.4) and (3.8) yields the result that A has another fixed point $v_2 \in K \cap (\overline{\Phi}_R \setminus \Phi_{r_1})$ with $v_2(t) \geq \sigma_2 \|u\| > 0, t \in [0, 1]$. This means that $u_1(t) = v_1(t) + v_0(t)$ and $u_2(t) = v_2(t) + v_0(t)$ are two positive solutions of BVP (1.2)-(1.3). Since

$$\|u_i\|_{[0,1]} = \|v_i + v_0\|_{[0,1]} = \|v_i\|_{[0,1]} = \|Av_i\|_{[0,1]} = \|Av_i\| = \|v_i\|, \quad i = 1, 2,$$

it follows that $u_1(t)$ and $u_2(t)$ satisfy

$$0 < r < \|u_1\|_{[0,1]} = \|v_1\| < r_1 < \|v_2\| = \|u_2\|_{[0,1]}.$$

The proof is complete. □

Theorem 3.3 *Assume that (H₂), (H₄), and (H₆) are satisfied. There exist constants $R > r_2 > r > 0$, then BVP (1.2)-(1.3) has at least two positive solutions u_1 and u_2 such that*

$$0 < r < \|u_1\|_{[0,1]} < r_2 < \|u_2\|_{[0,1]}.$$

Proof Choose $0 < r < r_2 < R$. By (H₄), for any $0 < \varepsilon < \frac{1}{2 \int_0^1 G(s) ds}$, there exists $R' > 0$ such that

$$f(t, x, y) \leq \varepsilon(x + y), \quad x, y \geq 0, x + y \geq R', 0 \leq t \leq 1.$$

Putting

$$C \triangleq \max_{0 \leq t \leq 1} \max_{0 \leq x, y, x+y \leq R'} |f(t, x, y)| + 1,$$

then

$$f(t, x, y) \leq \varepsilon(x + y) + C, \quad x, y \geq 0, 0 \leq t \leq 1.$$

Choose

$$R_0 > (C + 2\varepsilon \|v_0\|) \frac{\int_0^1 G(s) ds}{1 - 2\varepsilon \int_0^1 G(s) ds}.$$

Let $\Phi_R = \{v \in E : \|v\| < R, R \geq \max\{r_2, R_0\}\}$. Then, for any $v \in K \cap \partial\Phi_R$, we have

$$\begin{aligned} \|Av\| &= \|Av\|_{[0,1]} = \max_{t \in [0,1]} (Av)(t) \\ &= \max_{t \in [0,1]} \int_0^1 G(t, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\leq \int_0^1 G(s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\leq \int_0^1 G(s) [\varepsilon((v + v_0)(s - \tau_1) + (v + v_0)(s + \tau_2)) + C] ds \\ &\leq \int_0^1 G(s) [2\varepsilon(\|v\| + \|v_0\|) + C] ds \\ &= 2\varepsilon R \int_0^1 G(s) ds + (C + 2\varepsilon \|v_0\|) \int_0^1 G(s) ds < R = \|v\|. \end{aligned}$$

Therefore,

$$\|Av\| < \|v\|, \quad v \in K \cap \partial\Phi_R. \tag{3.11}$$

Let $\Phi_r = \{v \in E : \|v\| < r, r < r_2\}$. By (H₆), for any $0 < \varepsilon < \frac{1}{2 \int_0^1 G(s) ds}$, there exists $0 < r' < r$ such that

$$f(t, x, y) \leq \varepsilon(x + y), \quad x, y \geq 0, x + y < r', 0 \leq t \leq 1.$$

Putting

$$C' \triangleq \max_{0 \leq t \leq 1} \max_{0 \leq x, y, r' \leq x+y \leq 2r} |f(t, x, y)| + 1,$$

then

$$f(t, x, y) \leq \varepsilon(x + y) + C', \quad x, y \in \partial\Phi_r, 0 \leq t \leq 1.$$

Choose

$$r > (C' + 2\varepsilon\|v_0\|) \frac{\int_0^1 G(s) ds}{1 - 2\varepsilon \int_0^1 G(s) ds}.$$

Then, for any $v \in K \cap \partial\Phi_r$, we have

$$\begin{aligned} \|Av\| &= \|Av\|_{[0,1]} = \max_{t \in [0,1]} (Av)(t) \\ &= \max_{t \in [0,1]} \int_0^1 G(t, s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\leq \int_0^1 G(s) f(s, (v + v_0)(s - \tau_1), (v + v_0)(s + \tau_2)) ds \\ &\leq \int_0^1 G(s) [\varepsilon((v + v_0)(s - \tau_1) + (v + v_0)(s + \tau_2)) + C'] ds \\ &\leq \int_0^1 G(s) [2\varepsilon(\|v\| + \|v_0\|) + C'] ds \\ &= 2\varepsilon r \int_0^1 G(s) ds + (C' + 2\varepsilon\|v_0\|) \int_0^1 G(s) ds < r = \|v\|. \end{aligned}$$

So,

$$\|Av\| < \|v\|, \quad v \in K \cap \partial\Phi_r. \tag{3.12}$$

Applying Lemma 2.2 to (3.5) and (3.12) yields the result that A has a fixed point $v_1 \in K \cap (\overline{\Phi}_{r_1} \setminus \Phi_r)$ with $v_1(t) \geq \sigma_1\|u\| > 0, t \in [0, 1]$. Similarly, from Lemma 2.2 associated with (3.5) and (3.11) one derives that A has another fixed point $v_2 \in K \cap (\overline{\Phi}_R \setminus \Phi_{r_1})$ with $v_2(t) \geq \sigma_2\|u\| > 0, t \in [0, 1]$. This means that $u_1(t) = v_1(t) + v_0(t)$ and $u_2(t) = v_2(t) + v_0(t)$ are two positive solutions of BVP (1.2)-(1.3). Since

$$\|u_i\|_{[0,1]} = \|v_i + v_0\|_{[0,1]} = \|v_i\|_{[0,1]} = \|Av_i\|_{[0,1]} = \|Av_i\| = \|v_i\|, \quad i = 1, 2,$$

it follows that $u_1(t)$ and $u_2(t)$ satisfy

$$0 < r < \|u_1\|_{[0,1]} = \|v_1\| < r_2 < \|v_2\| = \|u_2\|_{[0,1]}.$$

This completes the proof. □

We account for the control functions

$$\begin{aligned} \varphi(r) &= \max\{f(t, x, y) : (t, x, y) \in [0, 1] \times [0, r + r_0] \times [0, r + r'_0]\}, \\ \psi(r) &= \min\{f(t, x, y) : (t, x, y) \in [\theta, 1 - \theta] \times [\sigma_1 r, r] \times [\sigma_2 r, r]\}, \end{aligned}$$

where $r_0 = \|v_0\|_{[-\tau_1, 0]}$, $r'_0 = \|v_0\|_{[1, 1+\tau_2]}$.

Theorem 3.4 *Suppose that there exist two positive numbers $\xi_2 < \xi_1$ such that one of the following conditions is satisfied:*

- (B₁) $\varphi(\xi_1) < \rho_1 \xi_1$, $\psi(\xi_2) > \rho_2 \xi_2$.
- (B₂) $\psi(\xi_1) > \rho_2 \xi_1$, $\varphi(\xi_2) < \rho_1 \xi_2$.

Then BVP (1.2)-(1.3) has at least one positive solution $u \in K$ such that

$$\xi_2 < \|u\|_{[0,1]} < \xi_1.$$

Proof Because of the similarity of the proof, we prove only this theorem under condition (B₁). By assumption (B₁), we have

$$f(t, x, y) \leq \varphi(\xi_1) < \rho_1 \xi_1, \quad f(t, x, y) \geq \psi(\xi_2) > \rho_2 \xi_2,$$

which are the assumptions (H₁) and (H₂). By Theorem 3.1, we find that A has a fixed point $v \in K \cap (\overline{\Phi}_{\xi_1} \setminus \Phi_{\xi_2})$, which means that (1.2)-(1.3) has at least one positive solution u and $\xi_2 < \|u\|_{[0,1]} < \xi_1$. This completes the proof. □

Similarly, we can obtain the existence of multiple positive solutions for BVP (1.2)-(1.3).

Theorem 3.5 *Suppose that there exist three positive numbers $\xi_3 < \xi_2 < \xi_1$ such that one of the following conditions is satisfied:*

- (B₃) $\varphi(\xi_1) < \rho_1 \xi_1$, $\psi(\xi_2) > \rho_2 \xi_2$, $\varphi(\xi_3) < \rho_1 \xi_3$.
- (B₄) $\psi(\xi_1) > \rho_2 \xi_1$, $\varphi(\xi_2) < \rho_1 \xi_2$, $\psi(\xi_3) > \rho_2 \xi_3$.

Then BVP (1.2)-(1.3) has at least two positive solutions $u_1, u_2 \in K$ such that

$$\xi_3 < \|u_1\|_{[0,1]} < \xi_2 < \|u_2\|_{[0,1]} < \xi_1.$$

Theorem 3.6 *Suppose that there exist four positive numbers $\xi_4 < \xi_3 < \xi_2 < \xi_1$ such that one of the following conditions is satisfied:*

- (B₅) $\varphi(\xi_1) < \rho_1 \xi_1$, $\psi(\xi_2) > \rho_2 \xi_2$, $\varphi(\xi_3) < \rho_1 \xi_3$, $\psi(\xi_4) > \rho_2 \xi_4$.
- (B₆) $\psi(\xi_1) > \rho_2 \xi_1$, $\varphi(\xi_2) < \rho_1 \xi_2$, $\psi(\xi_3) > \rho_2 \xi_3$, $\varphi(\xi_4) < \rho_1 \xi_4$.

Then BVP (1.2)-(1.3) has at least three positive solutions $u_1, u_2, u_3 \in K$ such that

$$\xi_4 < \|u_1\|_{[0,1]} < \xi_3 < \|u_2\|_{[0,1]} < \xi_2 < \|u_3\|_{[0,1]} < \xi_1.$$

Theorem 3.7 *Suppose that there exist $n + 1$ positive numbers $\xi_{n+1} < \xi_n < \dots < \xi_2 < \xi_1$ such that one of the following conditions is satisfied:*

- (B₇) $\varphi(\xi_{2k-1}) < \rho_1 \xi_{2k-1}, \psi(\xi_{2k}) > \rho_2 \xi_{2k}, k = 1, 2, \dots, [\frac{n+2}{2}].$
- (B₈) $\psi(\xi_{2k-1}) > \rho_2 \xi_{2k-1}, \varphi(\xi_{2k}) < \rho_1 \xi_{2k}, k = 1, 2, \dots, [\frac{n+2}{2}].$

Then BVP (1.2)-(1.3) has at least n positive solutions $u_i \in K (i = 1, 2, \dots, n)$ such that

$$\xi_{i+1} < \|u_i\|_{[0,1]} < \xi_i.$$

4 Example

Consider the following four-point BVP of delayed nonlinear fractional differential equations:

$$\begin{cases} D_{0+}^{\frac{5}{2}} u(t) + f(t, u(t - \frac{1}{6}), u(t + \frac{1}{9})), & t \in [0, 1], \\ u(0) = u'(0) = 0, & u(1) = 8(I_{0+}^{\frac{1}{2}} u)(\frac{1}{4}) + 5(I_{0+}^{\frac{3}{2}} u)(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $f(t, x, y) = \frac{4(t^2+t+1)}{3} + \frac{x^2+y^2}{8}, 0 \leq t \leq 1, x, y \geq 0, q = 5/2, \tau_1 = \frac{1}{6}, \tau_2 = \frac{1}{9}, \alpha_1 = 8, \alpha_2 = 5, p_1 = \frac{1}{2}, p_2 = \frac{3}{2}, \zeta_1 = \frac{1}{4}, \zeta_2 = \frac{1}{2}, m = 4.$ Choosing $\theta = \frac{1}{4}, r_1 = 2, r_2 = \frac{1}{50}, r_0 = r'_0 = \frac{1}{5}.$

By a simple calculation, we get

$$\begin{aligned} \sigma_1 &= \min_{\frac{1}{12} \leq t \leq \frac{7}{12}} \sigma(t) = \sigma\left(\frac{1}{12}\right) = \frac{11\sqrt{3}}{1,286}, \\ \sigma_2 &= \min_{\frac{13}{36} \leq t \leq \frac{31}{36}} \sigma(t) = \sigma\left(\frac{31}{36}\right) = \frac{155\sqrt{31}}{11,664}, \\ \Omega &= 1 - \sum_{i=1}^2 \frac{\alpha_i \zeta_i^{p_i+q-1} \Gamma(q)}{\Gamma(p_i + q)} \approx 0.5827572. \end{aligned}$$

By Lemmas 2.3 and 2.4 and the aid of a computer, we obtain

$$\begin{aligned} \rho_1 &= \left(\int_0^1 G(s) ds \right)^{-1} = \left(\int_0^1 \frac{s(1-s)^{q-1}}{\Gamma(q-1)} ds + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} \int_0^1 g_i(\zeta_i, s) ds \right)^{-1} \\ &= \left(\frac{q-1}{\Gamma(q+2)} + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} \left(\int_0^1 \zeta_i^{p_i+q-1} (1-s)^{q-1} ds - \int_0^{\zeta_i} (\zeta_i - s)^{p_i+q-1} ds \right) \right)^{-1} \\ &= \left(\frac{q-1}{\Gamma(q+2)} + \sum_{i=1}^{m-2} \frac{\alpha_i \zeta_i^{p_i+q-1} (p_i + q(1 - \zeta_i))}{\Omega \Gamma(p_i + q + 1) q} \right)^{-1} \\ &= \left(\frac{8}{35\sqrt{\pi}} + \frac{19}{240\Omega} + \frac{11}{384\Omega} \right)^{-1} \approx 3.025561 \end{aligned}$$

and

$$\begin{aligned} \rho_2 &= \left(\sigma(\theta) \int_{\theta}^{1-\theta} G(s) ds \right)^{-1} \\ &= \left(\sigma\left(\frac{1}{4}\right) \right)^{-1} \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s(1-s)^{q-1}}{\Gamma(q-1)} ds + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} \int_{\frac{1}{4}}^{\frac{3}{4}} g_i(\zeta_i, s) ds \right)^{-1} \\ &= 16 \left(\frac{117\sqrt{3} - 23}{1,120\sqrt{\pi}} + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} \right)^{-1} \end{aligned}$$

$$\begin{aligned} & \times \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \zeta_i^{p_i+q-1} (1-s)^{q-1} ds - \int_{\frac{1}{4}}^{\zeta_i} (\zeta_i - s)^{p_i+q-1} ds \right)^{-1} \\ & = 16 \left(\frac{117\sqrt{3} - 23}{1,120\sqrt{\pi}} + \sum_{i=1}^{m-2} \frac{\alpha_i}{\Omega \Gamma(p_i + q)} \left[\frac{\zeta_i^{p_i+q-1} (3^q - 1)}{q 4^q} - \frac{(\zeta_i - \frac{1}{4})^{p_i+q}}{p_i + q} \right] \right)^{-1} \\ & = 16 \left(\frac{117\sqrt{3} - 23}{1,120\sqrt{\pi}} + \frac{9\sqrt{3} - 1}{320\Omega} + \frac{72\sqrt{3} - 13}{6,144\Omega} \right)^{-1} \approx 80.029634. \end{aligned}$$

Therefore, for $(t, x, y) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{11\sqrt{3}}{64,800}, \frac{1}{50}] \times [\frac{155\sqrt{31}}{583,200}, \frac{1}{50}]$, we have

$$\begin{aligned} f(t, x, y) &= \frac{4(t^2 + t + 1)}{3} + \frac{x^2 + y^2}{8} \geq f\left(\frac{1}{4}, \frac{11\sqrt{3}}{64,800}, \frac{155\sqrt{31}}{583,200}\right) \\ &> f\left(\frac{1}{4}, 0, 0\right) = 1.75 > \rho_2 r_2 \approx 1.600593. \end{aligned}$$

For $(t, x, y) \in [0, 1] \times [0, \frac{11}{5}] \times [0, \frac{11}{5}]$, we obtain

$$f(t, x, y) = \frac{4(t^2 + t + 1)}{3} + \frac{x^2 + y^2}{8} \leq f\left(1, \frac{11}{5}, \frac{11}{5}\right) = 5.21 < \rho_1 r_1 \approx 6.051122$$

and

$$\begin{aligned} f_\infty &= \liminf_{x+y \rightarrow +\infty} \min_{t \in [0,1]} \frac{f(t, x, y)}{x + y} \\ &= \liminf_{x+y \rightarrow +\infty} \min_{t \in [0,1]} \frac{1}{x + y} \left(\frac{4(t^2 + t + 1)}{3} + \frac{x^2 + y^2}{8} \right) \\ &= \liminf_{x+y \rightarrow +\infty} \frac{1}{x + y} \left(\frac{4}{3} + \frac{x^2 + y^2}{8} \right) \\ &\geq \liminf_{x+y \rightarrow +\infty} \frac{1}{x + y} \left(\frac{4}{3} + \frac{(x + y)^2}{16} \right) = \infty. \end{aligned}$$

Thus (H₁)-(H₃) hold. With the use of Theorem 3.1, BVP (4.1) has at least two positive solutions u_1 and u_2 such that $0 < \frac{1}{50} < \|u_1\|_{[0,1]} < 2 < \|u_2\|_{[0,1]}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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