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Nontrivial solution for asymmetric $(p,2)$-Laplacian Dirichlet problem

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Abstract

We consider a class of particular $(p,2)$-Laplacian Dirichlet problems with a right-hand side nonlinearity which exhibits an asymmetric growth at $+\infty$ and $-\infty$. Namely, it is linear at $-\infty$ and superlinear at $+\infty$. However, it need not satisfy the Ambrosetti-Rabinowitz condition on the positive semi-axis. Some existence results for a nontrivial solution are established by the mountain pass theorem and a variant version of the mountain pass theorem in the general case $2 < p < N$. Similar results are also established by combining the mountain pass theorem and a variant version of the mountain pass theorem with the Moser-Trudinger inequality in the case of $p = N$.

Keywords: asymmetric Dirichlet problem; subcritical exponential growth; one side resonance

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N > 2)$ with smooth boundary $\partial\Omega$. We consider the following quasilinear elliptic boundary problem:

$$\begin{cases} -\triangle_p u(x) - \mu\triangle u = a(x)|u|^{s-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $2 < p < \infty$ and $1 < s < p$, $\triangle_p$ denotes the p-Laplacian operator defined by $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\mu > 0$ is a real parameter, $a(x) \in L^\infty(\Omega)$ and $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$.

It is known that the nontrivial solutions of problem (1.1) are equivalent to the corresponding nonzero critical points of the C^1-energy functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{\mu}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{s}\int_\Omega a(x)|u|^s\,dx - \int_\Omega F(x,u)\,dx \tag{1.2}$$

for all $u \in W_0^{1,p}(\Omega)$, where $F(x,t) = \int_0^t f(x,s)\,ds$.

For the case of $p > 2$, $a(x) \equiv 0$ and $\mu > 0$, there has been an increasing interest in looking for the existence of solutions of (1.1). Using the following conditions,

$$\mu_m < f'(x,0) < \mu_{m+1}, \qquad F(x,t) < \frac{\lambda_1}{p}|t|^p + C, \quad x \in \Omega,$$

where $m \geq 1$ and C is a constant, the authors in [1, 2] prove that (1.1) has at least two nontrivial solutions by the three critical point theorems. Here and in the sequel, $0 < \mu_1 <$

$\mu_2 < \cdots$ denote the eigenvalues of $-\triangle$ in $H_0^1(\Omega)$, and λ_1 is the first eigenvalue of $-\triangle_p$ in $W_0^{1,p}(\Omega)$ (see [3]). For Eq. (1.1) with the right-hand side having p-linear growth at infinity, *i.e.*, $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \lambda \notin \sigma(-\triangle_p)$, the spectrum of $-\triangle_p$ in $W_0^{1,p}(\Omega)$, the papers [4, 5] get the existence of a nontrivial solution. In [6], the author extends the results in [1, 2] under the general asymptotically linear condition.

The main purpose of this paper is to establish existence results of a nontrivial solution for problem (1.1) with $2 < p \leq N$ when the nonlinearity term $f(x,\cdot)$ exhibits an asymmetric behavior as $t \in \mathbb{R}$ approaches $+\infty$ and $-\infty$. More precisely, we assume that for a.e. $x \in \Omega$, $f(x,\cdot)$ grows superlinear at $+\infty$, while at $-\infty$ it has a linear growth. In case of $1 < p < N$, $\mu = 0$ and $a(x) \equiv 0$, equations with nonlinearities which are superlinear in one direction and linear in the other were investigated by Arcoya and Villegas [7], de Figueiredo and Ruf [8], Perera [9]. All three works express the superlinear growth at $+\infty$ using the Ambrosetti-Rabinowitz condition ((AR)-condition, for short). Recall that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the (AR)-condition in the positive direction if there exist $\mu > p$ and $M > 0$ such that

$$0 < \mu F(x,t) \leq tf(x,t) \quad \text{for all } t \geq M \text{ and a.e. } x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. Since Ambrosetti and Rabinowitz proposed the mountain pass theorem in their celebrated paper [10], critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. When we apply the mountain pass theorem, the (AR)-condition usually plays an important role in verifying that the functional I has a 'mountain pass' geometry and showing that a related $(\mathrm{PS})_c$ sequence is bounded.

By simple calculation, it is easy to see that the previous one side (AR)-condition implies that $\lim_{t\to+\infty} \frac{F(x,t)}{t^p} = +\infty$. That is, $f(x,t)$ must be superlinear with respect to $|t|^{p-2}t$ at positive infinity. Recently, Motreanu *et al.* [11], Papageorgiou and Papageorgiou [12] and Papageorgiou and Smyrlis [13] studied asymmetric problem (1.1) with nonlinearity f not satisfying the (AR)-condition on the positive semi-axis when $\mu = 0$ and $a(x) \equiv 0$. Nevertheless, all of the above-mentioned works involve the nonlinear term $f(x,u)$ of a subcritical (polynomial) growth, say,

(SCP): there exist positive constants c_1 and c_2 and $q_0 \in (p-1, p^*-1)$ such that

$$\left|f(x,t)\right| \leq c_1 + c_2|t|^{q_0} \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \Omega,$$

where $p^* = Np/(N-p)$ denotes the critical Sobolev exponent. One of the main reasons to assume this condition (SCP) is that they can use the Sobolev compact embedding $W_0^{1,p} \hookrightarrow L^q(\Omega)$, $1 \leq q < p^*$.

In this paper, *we always assume that* $\mu = 1$ *in* (1.1). Under the motivation of Lam and Lu [14], our first main results will be to study problem (1.1) in the improved subcritical polynomial growth

$$\text{(SCPI):} \quad \lim_{t\to\infty} \frac{f(x,t)}{t^{p^*-1}} = 0 \quad \text{uniformly on } x \in \Omega,$$

which is much weaker than (SCP). Note that in this case, we do not have the Sobolev compact embedding anymore. Our work again is to study asymmetric problem (1.1) without the (AR)-condition in the positive semi-axis. In fact, this condition was studied by Liu and Wang in [15] in the case of Laplacian (*i.e.*, $p = 2$) by the Nehari manifold approach. However, we will use the mountain pass theorem and a suitable version of the mountain pass theorem to get the nontrivial solution to problem (1.1) in the general case $2 < p < N$. Our results are different from those in [11–13] and our proof of the compactness condition is skillful.

Let us now state our results: Suppose that $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$ and satisfies:

(H_1) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = f_0$ uniformly for a.e. $x \in \Omega$, where $f_0 \in [0,+\infty)$;
(H_2) $\lim_{t\to -\infty} \frac{f(x,t)}{|t|^{p-2}t} = l$ uniformly for a.e. $x \in \Omega$, where $l \in [0,+\infty]$;
(H_3) $\lim_{t\to +\infty} \frac{f(x,t)}{t^{p-1}} = +\infty$ uniformly for a.e. $x \in \Omega$;
(H_4) $\frac{f(x,t)}{|t|^{p-2}t}$ is nonincreasing with respect to $t \leq 0$ for a.e. $x \in \Omega$.

Let λ_1 be the first eigenvalue of $(-\triangle_p, W_0^{1,p}(\Omega))$ and $\phi_1(x) > 0$ for every $x \in \Omega$ be the λ_1 eigenfunction. Throughout this paper, we denote by $|\cdot|_p$ the $L^p(\Omega)$ norm and the norm of u in $W_0^{1,p}(\Omega)$ will be defined by

$$\|u\| = \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

Theorem 1.1 *Let $2 < p < N$ and assume that f has the improved subcritical polynomial growth on Ω (condition (SCPI)) and satisfies (H_1)-(H_3). If $f_0 < \lambda_1 < l < \infty$, then there exists $m = m(f_0, s, f, N, \Omega)$ such that for all $a(x) \in L^\infty(\Omega)$ with $|a|_\infty < m$, problem (1.1) has at least a nontrivial solution if l is not any of the eigenvalues of $-\triangle_p$ on $W_0^{1,p}(\Omega)$.*

Remark 1.1 In view of conditions (H_2) and (H_3), problem (1.1) is called asymmetric. Hence, Theorem 1.1 is completely different from the results contained in [4–6].

Theorem 1.2 *Let $2 < p < N$ and assume that f has the improved subcritical polynomial growth on Ω (condition (SCPI)) and satisfies (H_1)-(H_3). If $f_0 < \lambda_1 = l$ and $\lim_{t\to -\infty}[f(x,t)t - pF(x,t)] = +\infty$ uniformly for a.e. $x \in \Omega$, then there exists $m = m(f_0, s, f, N, \Omega)$ such that for all $a(x) \in L^\infty(\Omega)$ and $a(x) < 0$ with $|a|_\infty < m$, problem (1.1) has at least a nontrivial solution.*

Remark 1.2 When $l = \lambda_1$, problem (1.1) is called resonant at negative infinity. This case is completely new. Here, we also give an example for $f(x,t)$. It satisfies our conditions (H_1)-(H_3) and (SCPI).

Example A Define

$$f(x,t) = \begin{cases} g(t)|t|^{p-2}t, & t \leq 0, \\ g(t)|t|^{p-2}t + h(t), & t > 0, \end{cases}$$

where $g(t) \in C(R)$, $g(0) = 0$; $g(t) \geq 0$, $t \in \mathbb{R}$; $h(t) \in C[0,+\infty)$; $\lim_{t\to +0} \frac{h(t)}{t^{p-1}} = 0$; $\lim_{t\to +\infty} \frac{h(t)}{t^{p^*-1}} = 0$; $\lim_{t\to +\infty} \frac{h(t)}{t^{p-1}} = +\infty$. Moreover, there exists $t_0 > 0$ such that $g(t) \equiv \lambda_1$ for all $|t| \geq t_0$.

Theorem 1.3 *Let $2 < p < N$ and assume that f has the improved subcritical polynomial growth on Ω (condition (SCPI)) and satisfies (H_1)-(H_4). If $f_0 < \lambda_1$ and $l = +\infty$, then there* *exists* $m = m(f_0, s, f, N, \Omega)$ *such that for all* $a(x) \in L^\infty(\Omega)$ *with* $|a|_\infty < m$, *problem* (1.1) *has at least a nontrivial solution.*

In case of $p = N$, we have $p^* = +\infty$. In this case, every polynomial growth is admitted, but one knows easy examples that $W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \to R^+$ with maximal growth such that

$$\sup_{u \in W_0^{1,N}, \|u\| \le 1} \int_\Omega g(u)\, dx < \infty.$$

It was shown by Trudinger [16] and Moser [17] that the maximal growth is of exponential type. So, we must redefine the subcritical (exponential) growth in this case as follows.

(SCE): f has subcritical (exponential) growth on Ω, *i.e.*, $\lim_{t\to\infty} \frac{|f(x,t)|}{\exp(\alpha|t|^{\frac{N}{N-1}})} = 0$ uniformly on $x \in \Omega$ for all $\alpha > 0$.

When $p = N$ and f has the subcritical (exponential) growth (SCE), our work is still to study asymmetric problem (1.1) without the (AR)-condition in the positive semi-axis. To our knowledge, this case is completely new. Our results are as follows.

Theorem 1.4 *Let* $p = N$ *and assume that* f *has the subcritical exponential growth on* Ω (*condition* (SCE)) *and satisfies* (H_1)-(H_3). *If* $f_0 < \lambda_1 < l < \infty$, *then there exists* $m = m(f_0, s, f, N, \Omega)$ *such that for all* $a(x) \in L^\infty(\Omega)$ *with* $|a|_\infty < m$, *problem* (1.1) *has at least a nontrivial solution if* l *is not any of the eigenvalues of* $-\triangle_N$ *on* $W_0^{1,N}(\Omega)$.

Remark 1.3 In view of conditions (H_2), (H_3) and (SCE), problem (1.1) is called asymmetric subcritical exponential problem. Hence, Theorem 1.4 is completely new.

Theorem 1.5 *Let* $p = N$ *and assume that* f *has the subcritical exponential growth on* Ω (*condition* (SCE)) *and satisfies* (H_1)-(H_3). *If* $f_0 < \lambda_1 = l$ *and* $\lim_{t\to-\infty}[f(x,t)t - NF(x,t)] = +\infty$ *uniformly for a.e.* $x \in \Omega$, *then there exists* $m = m(f_0, s, f, N, \Omega)$ *such that for all* $a(x) \in L^\infty(\Omega)$ *and* $a(x) < 0$ *with* $|a|_\infty < m$, *problem* (1.1) *has at least one nontrivial solution.*

Remark 1.4 When $l = \lambda_1$, problem (1.1) is called resonant at negative infinity. This case is new and completely different from the results contained in [14].

Theorem 1.6 *Let* $p = N$ *and assume that* f *has the subcritical exponential growth on* Ω (*condition* (SCE)) *and satisfies* (H_1)-(H_4). *If* $f_0 < \lambda_1$ *and* $l = +\infty$, *then there exists* $m = m(f_0, s, f, N, \Omega)$ *such that for all* $a(x) \in L^\infty(\Omega)$ *with* $|a|_\infty < m$, *problem* (1.1) *has at least one nontrivial solution.*

2 Preliminaries and some lemmas

Definition 2.1 Let $(E, \|\cdot\|_E)$ be a real Banach space with its dual space $(E^*, \|\cdot\|_{E^*})$ and $I \in C^1(E, \mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies the $(\mathrm{PS})_c$ condition if for any sequence $\{x_n\} \subset E$ with

$$I(x_n) \to c, \qquad DI(x_n) \to 0 \quad \text{in } E^*,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in E. Also, we say that I satisfies the $(\mathrm{C})_{\mathrm{c}}$ condition if for any sequence $\{x_n\} \subset E$ with

$$I(x_n) \to c, \qquad \|DI(x_n)\|_{E^*}\left(1 + \|x_n\|_E\right) \to 0,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in E.

We have the following version of the mountain pass theorem (see [18]).

Proposition 2.1 *Let E be a real Banach space and suppose that $I \in C^1(E,R)$ satisfies the condition*

$$\max\{I(0), I(u_1)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} I(u)$$

for some $\alpha < \beta$, $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$. Let $c \geq \beta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],E), \gamma(0) = 0, \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and u_1. Then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geq \beta \quad \text{and} \quad \left(1 + \|u_n\|\right)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

Lemma 2.1 *Let $2 < p < N$ and $\phi_1 > 0$ be a λ_1-eigenfunction with $\|\phi_1\| = 1$ and assume that (H_1)-(H_3) and (SCPI) hold. If $f_0 < \lambda_1 < l \leq +\infty$, then there exists $m = m(f_0, s, f, N, \Omega)$ such that for all $a(x) \in L^\infty(\Omega)$ with $|a|_\infty < m$, we have*:

(i) *There exist $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho$.*
(ii) *$I(t\phi_1) \to -\infty$ as $t \to +\infty$.*

Proof By (SCPI) and (H_1)-(H_3), if $l \in (\lambda_1, +\infty)$, for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$ such that for all $(x,s) \in \Omega \times \mathbb{R}$,

$$F(x,s) \leq \frac{1}{p}(f_0 + \varepsilon)|s|^p + A_1|s|^{p^*}, \tag{2.1}$$

$$F(x,s) \geq \frac{1}{p}(l - \varepsilon)|s|^p - B_1 \quad \text{if } l \in (\lambda_1, +\infty). \tag{2.2}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \lambda_1$. By (2.1), the Poincaré inequality and the Sobolev inequality, $|u|_{p^*}^{p^*} \leq K\|u\|^{p^*}$,

$$\begin{aligned} I(u) &\geq \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\|a\|_\infty}{s}\int_\Omega |u|^s\,dx - \int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\|a\|_\infty}{s}\int_\Omega |u|^s\,dx - \frac{1}{p}\int_\Omega \left[(f_0 + \varepsilon)|u|^p + A_1|u|^{p^*}\right]dx \\ &\geq \frac{1}{p}\left(1 - \frac{f_0 + \varepsilon}{\lambda_1}\right)\|u\|^p - \frac{\|a\|_\infty K^s}{s}\|u\|^s - A_1 K^{p^*}\|u\|^{p^*}. \end{aligned}$$

Set

$$\rho = \left(\frac{\|a\|_\infty (p-s) K^s}{s A_1 K^{p^*}} \right)^{\frac{1}{p^*-s}},$$

$$m = \left(\frac{\lambda_1 - f_0 - \varepsilon}{p\lambda_1} \right)^{\frac{p^*-s}{p^*-p}} s K^{-s} \left(A_1 K^{p^*} \right)^{\frac{s-p}{p^*-p}} \left[\left(\frac{p-s}{p^*-p} \right)^{\frac{s-p}{p^*-s}} + \left(\frac{p-s}{p^*-p} \right)^{\frac{p^*-p}{p^*-s}} \right]^{\frac{s-p^*}{p^*-p}}.$$

So, part (i) holds if we choose $\|u\| = \rho > 0$ and $\|a\|_\infty < m$.

On the other hand, if $l \in (\lambda_1, +\infty)$, take $\varepsilon > 0$ such that $l - \varepsilon > \lambda_1$. By (2.2), we have

$$I(t\phi_1) \le \frac{t^p}{p} \|\phi_1\|^p + t^2 \int_\Omega |\nabla \phi_1|^2 \, dx - \frac{t^s}{s} \int_\Omega a(x)(\phi_1)^s \, dx - \frac{l-\varepsilon}{p} |\phi_1|_p^p + B|\Omega|.$$

Since $l - \varepsilon > \lambda_1$ and $\|\phi_1\| = 1$, it is easy to see that

$$\begin{aligned} I(t\phi_1) &\le \frac{1}{p}\left(1 - \frac{l-\varepsilon}{\lambda_1}\right) t^p + t^2 \int_\Omega |\nabla \phi_1|^2 \, dx - \frac{t^s}{s} \int_\Omega a(x)(\phi_1)^s \, dx + B_1|\Omega| \\ &\to -\infty \quad \text{as } t \to +\infty, \end{aligned}$$

and part (ii) is proved. □

Lemma 2.2 (see [16, 17]) *Let $u \in W_0^{1,N}(\Omega)$, then $\exp(|u|^{\frac{N}{N-1}}) \in L^q(\Omega)$ for all $1 \le q < \infty$. Moreover,*

$$\sup_{u \in W_0^{1,N}(\Omega), \|u\| \le 1} \int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \le C(\Omega) \quad \textit{for } \alpha \le \alpha_N.$$

The inequality is optimal: for any growth $\exp(\alpha|u|^{\frac{N}{N-1}})$ with $\alpha > \alpha_N$, the corresponding supremum is $+\infty$.

Lemma 2.3 *Let $p = N$ and $\phi_1 > 0$ be a λ_1-eigenfunction with $\|\phi_1\| = 1$ and assume that* (H_1)-(H_3) *and* (SCE) *hold. If $f_0 < \lambda_1 < l \le +\infty$, then there exists $m = m(f_0, s, f, N, \Omega)$ such that for all $a(x) \in L^\infty(\Omega)$ with $|a|_\infty < m$, we have:*

(i) *There exist $\rho, \alpha > 0$ such that $I(u) \ge \alpha$ for all $u \in W_0^{1,N}(\Omega)$ with $\|u\| = \rho$.*
(ii) *$I(t\phi_1) \to -\infty$ as $t \to +\infty$.*

Proof By (SCE) and (H_1)-(H_3), if $l \in (\lambda_1, +\infty)$, for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$, $\kappa > 0$ and $q > N$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F(x,s) \le \frac{1}{N}(f_0 + \varepsilon)|s|^N + A_1 \exp\left(\kappa |s|^{\frac{N}{N-1}}\right)|s|^q, \tag{2.3}$$

$$F(x,s) \ge \frac{1}{N}(l - \varepsilon)|s|^N - B_1 \quad \text{if } l \in (\lambda_1, +\infty). \tag{2.4}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \lambda_1$. By (2.3), the Holder inequality and the Moser-Trudinger embedding inequality, we get

$$\begin{aligned} I(u) &\ge \frac{1}{N}\|u\|^N - \frac{f_0+\varepsilon}{N}|u|_N^N - \frac{\|a\|_\infty}{s}\int_\Omega |u|^s \, dx - A_1 \int_\Omega \exp\left(\kappa |u|^{\frac{N}{N-1}}\right)|u|^q \, dx \\ &\ge \frac{1}{N}\left(1 - \frac{f_0+\varepsilon}{\lambda_1}\right)\|u\|^N - \frac{\|a\|_\infty}{s}\int_\Omega |u|^s \, dx \end{aligned}$$

$$
\begin{aligned}
&- A_1\left(\int_\Omega \exp\left(\kappa r\|u\|^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|}\right)^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{r}}\left(\int_\Omega |u|^{r'q}\,dx\right)^{\frac{1}{r'}} \\
&\geq \frac{1}{N}\left(1-\frac{f_0+\varepsilon}{\lambda_1}\right)\|u\|^N - \frac{\|a\|_\infty K^s}{s}\|u\|^s - C\|u\|^q,
\end{aligned}
$$

where $r > 1$ sufficiently close to 1, $\|u\| \leq \sigma$ and $\kappa r\sigma^{\frac{N}{N-1}} < \alpha_N$. Set

$$
\begin{aligned}
\rho &= \left(\frac{\|a\|_\infty (N-s)K^s}{sC}\right)^{\frac{1}{q-s}}, \\
m &= \left(\frac{\lambda_1 - f_0 - \varepsilon}{N\lambda_1}\right)^{\frac{q-s}{q-N}} sK^{-s}(C)^{\frac{s-N}{q-N}}\left[\left(\frac{N-s}{q-N}\right)^{\frac{s-N}{q-s}} + \left(\frac{N-s}{q-N}\right)^{\frac{q-N}{q-s}}\right]^{\frac{s-q}{q-N}}.
\end{aligned}
$$

So, part (i) holds if we choose $\|u\| = \rho > 0$ and $\|a\|_\infty < m$.

On the other hand, if $l \in (\lambda_1, +\infty)$, taking $\varepsilon > 0$ such that $l - \varepsilon > \lambda_1$ and using (2.4), we have

$$
\begin{aligned}
I(t\phi_1) &\leq \frac{1}{N}\left(1-\frac{l-\varepsilon}{\lambda_1}\right)|t|^N + t^2\int_\Omega |\nabla\phi_1|^2\,dx - \frac{t^s}{s}\int_\Omega a(x)(\phi_1)^s\,dx + B_1|\Omega| \\
&\to -\infty \quad \text{as } t \to +\infty.
\end{aligned}
$$

Thus part (ii) is proved. By exactly slight modification to the proof above, we can prove (ii) if $l = +\infty$. □

Lemma 2.4 *For the functional I defined by* (1.2), *if $u_n(x) \leq 0$ a.e. $x \in \Omega$, $n \in \mathbb{N}$ and*

$$
\langle I'(u_n), u_n\rangle \to 0 \quad \text{as } n \to \infty,
$$

then there exists a subsequence, still denoted by $\{u_n\}$, such that

$$
I(tu_n) \leq \frac{1+t^p}{np} + \left(\frac{t^p}{p} - \frac{t^s}{s}\right)\int_\Omega a(x)|u_n|^s\,dx + I(u_n) \quad \text{for all } t \geq 0 \text{ and } n \in \mathbb{N}.
$$

Proof By $\langle I'(u_n), u_n\rangle \to 0$ as $n \to \infty$, for a suitable subsequence, we may assume that

$$
-\frac{1}{n} < \langle I'(u_n), u_n\rangle = \|u_n\|^p + \|u_n\|_*^2 - \int_\Omega a(x)|u_n|^s\,dx - \int_\Omega f\bigl(x, u_n(x)\bigr)u_n\,dx < \frac{1}{n} \tag{2.5}
$$

for all n, where $\|\cdot\|_*$ denotes the norm of $H_0^1(\Omega)$.

We claim that for any $0 \leq t$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
I(tu_n) \leq{}& \left(\frac{1}{2}\|u_n\|_*^2 - \frac{1}{p}\|u_n\|_*^2\right) + \left(\frac{t^p}{p} - \frac{t^s}{s}\right)\int_\Omega a(x)|u_n|^s\,dx \\
&+ \frac{t^p}{pn} + \int_\Omega \left\{\frac{1}{p}f\bigl(x, u_n(x)\bigr)u_n - F\bigl(x, u_n(x)\bigr)\right\}dx.
\end{aligned} \tag{2.6}
$$

Indeed, for any $t \geq 0$, at fixed $x \in \Omega$ and $n \in \mathbb{N}$, if we set

$$
h(t) = \frac{1}{p}t^p f(x, u_n)u_n(x) - F\bigl(x, tu_n(x)\bigr),
$$

then

$$\begin{aligned}
h'(t) &= t^{p-1}f(x,u_n)u_n(x) - f(x,tu_n)u_n(x) \\
&= t^{p-1}u_n(x)\{f(x,u_n) - f(x,tu_n(x))/t^{p-1}\} \\
&= \begin{cases} \geq 0 & \text{for } 0 < t \leq 1, \\ \leq 0 & \text{for } t \geq 1 \end{cases} \quad \text{by } (\mathrm{H}_4),
\end{aligned}$$

hence

$$h(t) \leq h(1) \quad \text{for all } t \geq 0.$$

Therefore,

$$\begin{aligned}
I(tu_n) &= \frac{1}{p}t^p\|u_n\|^p + \frac{1}{2}t^2\|u_n\|_*^2 - \frac{t^s}{s}\int_\Omega a(x)|u_n|^s\,dx - \int_\Omega F(x,tu_n(x))\,dx \\
&< \frac{1}{2}t^2\|u_n\|_*^2 + \frac{1}{p}t^p\left\{\frac{1}{n} - \|u_n\|_*^2 + \int_\Omega f(x,u_n(x))u_n(x)\,dx\right\} \\
&\quad - \frac{t^s}{s}\int_\Omega a(x)|u_n|^s\,dx - \int_\Omega F(x,tu_n(x))\,dx \\
&\leq \left(\frac{1}{2}t^2\|u_n\|_*^2 - \frac{t^p}{p}\|u_n\|_*^2\right) + \frac{t^p}{pn} + \left(\frac{t^p}{p} - \frac{t^s}{s}\right)\int_\Omega a(x)|u_n|^s\,dx \\
&\quad + \int_\Omega\left\{\frac{1}{p}t^pf(x,u_n(x))u_n(x) - F(x,tu_n(x))\right\}dx \\
&\leq \left(\frac{1}{2}\|u_n\|_*^2 - \frac{1}{p}\|u_n\|_*^2\right) + \frac{t^p}{pn} + \left(\frac{t^p}{p} - \frac{t^s}{s}\right)\int_\Omega a(x)|u_n|^s\,dx \\
&\quad + \int_\Omega\left\{\frac{1}{p}f(x,u_n(x))u_n(x) - F(x,u_n(x))\right\}dx,
\end{aligned}$$

and our claim (2.6) is proved.

On the other hand,

$$\begin{aligned}
I(u_n) &= \frac{1}{p}\|u_n\|^p + \frac{1}{2}\|u_n\|_*^2 - \frac{1}{s}\int_\Omega a(x)|u_n|^s\,dx - \int_\Omega F(x,u_n(x))\,dx \\
&\geq \frac{1}{2}\|u_n\|_*^2 + \frac{1}{p}\left\{-\frac{1}{n} - \|u_n\|_*^2 + \int_\Omega f(x,u_n(x))u_n(x)\,dx\right\} - \int_\Omega F(x,u_n(x))\,dx,
\end{aligned}$$

that is,

$$\int_\Omega\left\{\frac{1}{p}f(x,u_n(x))u_n(x) - F(x,u_n(x))\right\}dx \leq \frac{1}{pn} + \frac{1}{p}\|u_n\|_*^2 - \frac{1}{2}\|u_n\|_*^2 + I(u_n). \tag{2.7}$$

Combining (2.6) and (2.7), we find that

$$I(tu_n) \leq \frac{1+t^p}{np} + \left(\frac{t^p}{p} - \frac{t^s}{s}\right)\int_\Omega a(x)|u_n|^s\,dx + I(u_n) \quad \text{for all } t \geq 0 \text{ and } n \in \mathbb{N}. \tag{2.8}$$

□

3 Proofs of the main results

Here, we only prove Theorems 1.1-1.4. Others follow these results.

Proof of Theorem 1.1 By Lemma 2.1, the geometry conditions of mountain pass theorem hold. So, we only need to verify condition (PS). Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a (PS) sequence such that for every $n \in \mathbb{N}$,

$$\left|\frac{1}{p}\int_\Omega |\nabla u_n|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - \frac{t^s}{s}\int_\Omega a(x)|u_n|^s\,dx - \int_\Omega F(x,u_n)\,dx\right| \le c \tag{3.1}$$

and

$$\begin{aligned}&\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v\,dx + \int_\Omega \nabla u_n \nabla v\,dx \right.\\ &\quad\left. - \int_\Omega a(x)|u_n|^{s-2}u_n v - \int_\Omega f(x,u_n)v\,dx\right| \le \varepsilon_n \|v\|\end{aligned} \tag{3.2}$$

for all $v \in W_0^{1,p}(\Omega)$, where $c > 0$ is a positive constant and $\{\varepsilon_n\} \subset \mathbb{R}^+$ is a sequence which converges to zero.

Step 1. In order to prove that $\{u_n\}$ has a convergent subsequence, we first show that it is a bounded sequence. To do this, we argue by contradiction assuming that for a subsequence, which we denote by $\{u_n\}$, we have

$$\|u_n\| \to +\infty \quad \text{as } n \to \infty.$$

Without loss of generality, we can assume $\|u_n\| > 1$ for all $n \in \mathbb{N}$ and define $z_n = \frac{u_n}{\|u_n\|}$. Obviously, $\|z_n\| = 1$, $\forall n \in \mathbb{N}$, and then it is possible to extract a subsequence (denoted also by $\{z_n\}$) such that

$$z_n \rightharpoonup z_0 \quad \text{in } W_0^{1,p}(\Omega), \tag{3.3}$$

$$z_n \to z_0 \quad \text{in } L^p(\Omega), \tag{3.4}$$

$$z_n(x) \to z_0(x) \quad \text{a.e. } x \in \Omega, \tag{3.5}$$

$$\left|z_n(x)\right| \le q(x) \quad \text{a.e. } x \in \Omega, \tag{3.6}$$

where $z_0 \in W_0^{1,p}(\Omega)$ and $q \in L^p(\Omega)$. Dividing both sides of (3.2) by $\|u_n\|^{p-1}$, we obtain

$$\begin{aligned}&\left|\int_\Omega |\nabla z_n|^{p-2}\nabla z_n \nabla v\,dx + \|u_n\|^{2-p}\int_\Omega \nabla z_n \nabla v\,dx \right.\\ &\quad\left. - \int_\Omega \frac{a(x)|u_n|^{s-2}u_n v}{\|u_n\|^{p-1}} - \int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}} v\,dx\right| \le \frac{\varepsilon_n}{\|u_n\|^{p-1}}\|v\|\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. Passing to the limit we deduce from (3.3) that

$$\lim_{n\to\infty}\int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}} v\,dx = \int_\Omega |\nabla z_0|^{p-2}\nabla z_0 \nabla v\,dx \tag{3.7}$$

for all $v \in W_0^{1,p}(\Omega)$.

Now we claim that $z_0(x) \leq 0$ for a.e. $x \in \Omega$. To verify this, let us observe that by choosing $v = z_0^+ = \max\{z_0, 0\}$ in (3.7), we have

$$\lim_{n\to\infty} \int_{\Omega^+} \frac{f(x,u_n)}{\|u_n\|^{p-1}} z_0\,dx = \int_{\Omega^+} |\nabla z_0|^p\,dx < +\infty, \tag{3.8}$$

where $\Omega^+ = \{x \in \Omega | z_0(x) > 0\}$. On the other hand, from (H_2) and (H_3), it implies

$$\frac{f(x,u_n(x))}{\|u_n\|^{p-1}} z_0(x) \geq \bigl(-l\bigl|q(x)\bigr|^{p-2} q(x) - K_1\bigr) z_0(x) \quad \text{a.e. } x \in \Omega$$

for some positive constant $K_1 > 0$. Moreover, using $\lim_{n\to\infty} u_n(x) = +\infty$ for a.e. $x \in \Omega^+$, (3.5) and the superlinearity of f (see (H_3)), we also deduce

$$\lim_{n\to\infty} \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} z_0(x) = \lim_{n\to\infty} \frac{f(x,u_n(x))}{u_n^{p-1}} z_n(x)^{p-1} z_0(x) = +\infty \quad \text{a.e. } x \in \Omega^+.$$

Therefore, if $|\Omega^+| > 0$, by Fatou's lemma, we will obtain that

$$\lim_{n\to\infty} \int_{\Omega^+} \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} z_0(x)\,dx = +\infty,$$

which contradicts (3.8). Thus $|\Omega^+| = 0$ and the claim is proved.

Clearly, $z_0(x) \not\equiv 0$. By (H_2), there exists $c > 0$ such that $\frac{|f(x,u_n)|}{|u_n|^{p-1}} \leq c$ for a.e. $x \in \Omega$. By using the Lebesgue dominated convergence theorem in (3.7), we have

$$\int_\Omega |\nabla z_0|^{p-2} \nabla z_0 \nabla v\,dx - \int_\Omega l|z_0|^{p-2} z_0 v\,dx = 0 \tag{3.9}$$

for all $v \in W_0^{1,p}(\Omega)$. This contradicts our assumption, *i.e.*, l is not any of the eigenvalues of $-\triangle_p$ on $W_0^{1,p}(\Omega)$.

Step 2. Now, we prove that $\{u_n\}$ has a convergent subsequence. In fact, we can suppose that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \\
&u_n \to u \quad \text{in } L^q(\Omega), \forall 1 \leq q < p^*, \\
&u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.
\end{aligned}$$

Now, since f has the subcritical growth on Ω, for every $\varepsilon > 0$, we can find a constant $C(\varepsilon) > 0$ such that

$$f(x,s) \leq C(\varepsilon) + \varepsilon|s|^{p^*-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

then

$$\begin{aligned}
&\left|\int_\Omega f(x,u_n)(u_n - u)\,dx\right| \\
&\quad \leq C(\varepsilon) \int_\Omega |u_n - u|\,dx + \varepsilon \int_\Omega |u_n - u||u_n|^{p^*-1}\,dx
\end{aligned}$$

$$\begin{aligned}&\le C(\varepsilon)\int_\Omega |u_n-u|\,dx+\varepsilon\left(\int_\Omega \left(|u_n|^{p^*-1}\right)^{\frac{p^*}{p^*-1}}dx\right)^{\frac{p^*-1}{p^*}}\left(\int_\Omega |u_n-u|^{p^*}\right)^{\frac{1}{p^*}}\\&\le C(\varepsilon)\int_\Omega |u_n-u|\,dx+\varepsilon C(\Omega).\end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $\int_\Omega |u_n-u|\,dx\to 0$. Since $\varepsilon>0$ is arbitrary, we can conclude that

$$\int_\Omega \bigl(f(x,u_n)-f(x,u)\bigr)(u_n-u)\,dx\to 0\quad \text{as } n\to\infty. \tag{3.10}$$

By (3.2), we have

$$\bigl\langle I'(u_n)-I'(u),(u_n-u)\bigr\rangle\to 0\quad \text{as } n\to\infty. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\int_\Omega \bigl(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\bigr)(\nabla u_n-\nabla u)\to 0\quad \text{as } n\to\infty.$$

Using an elementary inequality

$$2^{2-p}|b-a|^p\le \bigl\langle |b|^{p-2}b-|a|^{p-2}a,b-a\bigr\rangle,\quad \forall a,b\in\mathbb{R}^N,$$

we can imply that

$$\nabla u_n\to\nabla u\quad \text{in } L^p(\Omega).$$

So we have $u_n\to u$ in $W_0^{1,p}(\Omega)$, which means that I satisfies (PS). □

Proof of Theorem 1.2 Since $l=\lambda_1$, obviously, Lemma 2.1(i) holds. We only need to show that Lemma 2.1(ii) holds. Let $u=t\phi_1$. Using condition (H_3), we have

$$\begin{aligned}I(t\phi_1)&=\frac{1}{p}t^p\int_\Omega|\nabla\phi_1|^p\,dx+\frac{t^2}{2}\int_\Omega|\nabla\phi_1|^2\,dx-\frac{t^s}{s}\int_\Omega a(x)|\phi_1|^s\,dx\\&\quad-\int_\Omega F(x,t\phi_1)\,dx\\&=\frac{1}{p}t^p\int_\Omega|\nabla\phi_1|^p\,dx+\frac{t^2}{2}\int_\Omega|\nabla\phi_1|^2\,dx-\frac{t^s}{s}\int_\Omega a(x)|\phi_1|^s\,dx\\&\quad-t^pM\int_\Omega|\phi_1|^p\,dx+C\to-\infty\end{aligned}$$

as $t\to+\infty$, where M is a positive constant large enough. By Proposition 2.1, there exists a sequence $\{u_n\}\subset W_0^{1,p}(\Omega)$ such that

$$I(u_n)=\frac{1}{p}\|u_n\|^p+\frac{1}{2}\|u_n\|_*^2-\frac{1}{s}\int_\Omega a(x)|u_n|^s\,dx-\int_\Omega F(x,u_n)\,dx=c+\circ(1), \tag{3.12}$$

$$\bigl(1+\|u_n\|\bigr)\bigl\|I'(u_n)\bigr\|_{W_0^{-1,p}}\to 0\quad \text{as } n\to\infty. \tag{3.13}$$

Clearly, (3.13) implies that

$$\begin{aligned}\langle I'(u_n), u_n\rangle = \|u_n\|^p + \|u_n\|_*^2 - \int_\Omega a(x)|u_n|^s\,dx \\ - \int_\Omega f\bigl(x, u_n(x)\bigr)u_n\,dx = \circ(1).\end{aligned} \tag{3.14}$$

To complete our proof, we first need to verify that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Similar to the proof of Theorem 1.1, we have $z_0(x) \le 0$, $x \in \Omega$, $z_0(x) \not\equiv 0$ and

$$\int_\Omega |\nabla z_0|^{p-2}\nabla z_0 \nabla v\,dx - \int_\Omega l|z_0|^{p-2}z_0 v\,dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$. By the maximum principle (see [19]), $z_0 < 0$ is an eigenfunction of λ_1, then $|u_n(x)| \to \infty$ for a.e. $x \in \Omega$. By our assumptions, we have

$$\lim_{n\to\infty}\bigl(f\bigl(x, u_n(x)\bigr)u_n(x) - pF\bigl(x, u_n(x)\bigr)\bigr) = +\infty$$

uniformly in $x \in \Omega$, which implies that

$$\int_\Omega \bigl(f\bigl(x, u_n(x)\bigr)u_n(x) - pF\bigl(x, u_n(x)\bigr)\bigr)\,dx \to +\infty \quad \text{as } n \to \infty. \tag{3.15}$$

On the other hand, (3.14) implies that

$$pI(u_n) - \langle I'(u_n), u_n\rangle \to pc \quad \text{as } n \to \infty.$$

Thus

$$\int_\Omega \bigl(f(x, u_n)u_n - pF(x, u_n)\bigr)\,dx \to -\infty \quad \text{as } n \to \infty,$$

which contradicts (3.15). Hence $\{u_n\}$ is bounded. According to Step 2 of the proof of Theorem 1.1, we have $u_n \to u$ in $W_0^{1,p}(\Omega)$, which means that I satisfies $(\mathrm{C})_c$. □

Proof of Theorem 1.3 By Lemma 2.1 and Proposition 2.1, (3.12)-(3.14) hold. We still can prove that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Assume $\|u_n\| \to +\infty$ as $n \to \infty$. Similar to the proof of Theorem 1.1, we have $z_0(x) \le 0$ and when $z_0(x) < 0$, $u_n = z_n\|u_n\| \to -\infty$ as $n \to \infty$. Let

$$s_n = \frac{\sqrt[p]{2pc}}{\|u_n\|}, \qquad w_n = s_n u_n = \frac{\sqrt[p]{2pc}\,u_n}{\|u_n\|}. \tag{3.16}$$

Since $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$, it is possible to extract a subsequence (denoted also by $\{w_n\}$) such that

$$\begin{aligned}&w_n \rightharpoonup w_0 \quad \text{in } W_0^{1,p}(\Omega),\\ &w_n \to w_0 \quad \text{in } L^p(\Omega),\end{aligned}$$

$$w_n(x) \to w_0(x) \quad \text{a.e. } x \in \Omega,$$

$$\left|w_n(x)\right| \le h(x) \quad \text{a.e. } x \in \Omega,$$

where $w_0 \in W_0^{1,p}(\Omega)$ and $h \in L^p(\Omega)$.

If $\|u_n\| \to +\infty$ as $n \to \infty$, then $w_0(x) \equiv 0$. In fact, letting $\Omega^- = \{x \in \Omega : w_0(x) < 0\}$ and noticing $l = +\infty$, from (H_3) we have that

$$\frac{f(x,u_n)}{|u_n|^{p-2}u_n} \ge M \quad \text{uniformly for all } x \in \Omega^-,$$

where M is a large enough constant. Therefore, by (3.14) and (3.16), we have

$$\begin{aligned} 2pc &= \lim_{n\to\infty} \|w_n\|^p \\ &= \lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{|u_n|^{p-2}u_n}|w_n|^p\,dx \\ &\ge \lim_{n\to\infty} \int_{\Omega^-} \frac{f(x,u_n)}{|u_n|^{p-2}u_n}|w_n|^p\,dx \\ &\ge M \lim_{n\to\infty} \int_{\Omega^-} |w_0|^p\,dx. \end{aligned}$$

So $w_0 \equiv 0$ for a.e. $x \in \Omega$. But, if $w_0 \equiv 0$, then $\int_\Omega F(x,w_n)\,dx \to 0$. Hence

$$I(w_n) = \frac{1}{p}\|w_n\|^p + \frac{1}{2}\|w_n\|_*^2 + \circ(1) \ge 2c + \circ(1). \tag{3.17}$$

On the other hand, by $\|u_n\| \to \infty$ as $n \to \infty$, we have $s_n \to 0$ as $n \to \infty$. From Lemma 2.4 and (3.12), we get

$$\begin{aligned} I(w_n) &= I(s_n u_n) \\ &\le \frac{1 + (s_n)^p}{np} + I(u_n) \\ &\le c \quad \text{as } n \to \infty. \end{aligned}$$

Obviously, it contradicts (3.17). So $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. According to Step 2 of the proof of Theorem 1.1, we have $u_n \to u$ in $W_0^{1,p}(\Omega)$, which means that I satisfies (C_c). □

Proof of Theorem 1.4 By Lemma 2.3, the geometry conditions of mountain pass theorem hold. So, we only need to verify condition (PS). Similar to Step 1 of the proof of Theorem 1.1, we easily know that the (PS) sequence $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. Next, we prove that $\{u_n\}$ has a convergent subsequence. Without loss of generality, suppose that

$$\|u_n\| \le \beta,$$

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,N}(\Omega),$$

$$u_n \to u \quad \text{in } L^q(\Omega), \forall q \ge 1,$$

$$u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.$$

Now, since f has the subcritical exponential growth (SCE) on Ω, we can find a constant $C_\beta > 0$ such that

$$\bigl|f(x,t)\bigr| \le C_\beta \exp\biggl(\frac{\alpha_N}{2\beta^{\frac{N}{N-1}}}|t|^{\frac{N}{N-1}}\biggr), \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Thus, by the Moser-Trudinger inequality (see Lemma 2.2),

$$\begin{aligned}
&\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| \\
&\quad \le C\biggl(\int_\Omega \exp\biggl(\frac{\alpha_N}{\beta^{\frac{N}{N-1}}}|u_n|^{\frac{N}{N-1}}\biggr)\,dx\biggr)^{\frac{1}{2}} |u_n-u|_2 \\
&\quad \le C\biggl(\int_\Omega \exp\biggl(\frac{\alpha_N}{\beta^{\frac{N}{N-1}}}\|u_n\|^{\frac{N}{N-1}}\left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-1}}\biggr)\,dx\biggr)^{\frac{1}{2}} |u_n-u|_2 \\
&\quad \le C|u_n-u|_2 \to 0.
\end{aligned}$$

Similar to the last proof of Theorem 1.1, we have $u_n \to u$ in $W_0^{1,N}(\Omega)$, which means that I satisfies (PS). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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