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Existence and multiplicity of solutions for fourth-order elliptic equations of Kirchhoff type via genus theory

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Abstract

In this paper, we study the following fourth-order elliptic equations of Kirchhoff type: $\triangle^2 u - (a + b\int_{\mathrm{R}^3} |\nabla u|^2\,dx)\Delta u + V(x)u = f(x,u)$, in R^3, $u \in H^2(\mathrm{R}^3)$, where $a, b > 0$ are constants, we have the potential $V(x) : \mathrm{R}^3 \to \mathrm{R}$ and the nonlinearity $f(x,u) : \mathrm{R}^3 \times \mathrm{R} \to \mathrm{R}$. Under certain assumptions on $V(x)$ and $f(x,u)$, we show the existence and multiplicity of negative energy solutions for the above system based on the genus properties in critical point theory.

MSC: 35J20; 35J65; 35J60

Keywords: fourth-order elliptic equations of Kirchhoff type; genus theory; variational methods

1 Introduction and main results

Consider the following fourth-order elliptic equations of Kirchhoff type:

$$\begin{cases} \triangle^2 u - (a + b\int_{\mathrm{R}^3} |\nabla u|^2\,dx)\Delta u + V(x)u = f(x,u) & \text{in } \mathrm{R}^3, \\ u \in H^2(\mathrm{R}^3), \end{cases} \tag{1.1}$$

where a, b are positive constants. We assume that the functions $V(x)$, $f(x,u)$, and its primitive $F(x,u) := \int_0^u f(x,s)\,ds$ satisfy the following hypotheses.

(V) $V(x) \in C(\mathrm{R}^3, \mathrm{R})$ satisfies $\inf_{x\in\mathrm{R}^3} V(x) = a > 0$.

(f1) $f(x,u) \in C(\mathrm{R}^3 \times \mathrm{R}, \mathrm{R})$, and there exist $0 < p < q < 1$ and positive functions $b_1 \in L^{\frac{2}{1-p}}(\mathrm{R}^3, \mathrm{R})$, $b_2 \in L^{\frac{2}{1-q}}(\mathrm{R}^3, \mathrm{R})$ such that

$$\bigl|f(x,u)\bigr| \le (p+1)b_1(x)|u|^p + (q+1)b_2(x)|u|^q.$$

(f2) There exist a nonzero measure open set $J \subset \mathrm{R}^3$ and three constants $\alpha, \beta > 0$, and $1 < \nu < 2$ such that

$$F(x,u) \ge \beta|u|^\nu, \quad \forall (x,u) \in J \times [-\alpha, \alpha].$$

(f3) $f(x,-u) = -f(x,u)$, $\forall (x,u) \in \mathrm{R}^3 \times \mathrm{R}$.

(f4) There exist a nonzero measure open set $J \subset \mathrm{R}^3$ and three constants $\alpha, \beta > 0$ and $1 < \nu < 2$ such that

$$uf(x,u) \geq \beta |u|^{\nu}, \quad \forall (x,u) \in J \times [-\alpha, \alpha].$$

The problem (1.1) is a nonlocal problem because of the appearance of the term $\int_{\mathrm{R}^3} |\nabla u|^2\,dx$ which provokes some mathematical difficulties. This makes the study of (1.1) particularly interesting.

Let $V(x) = 0$, replace R^3 by a bounded smooth domain $\Omega \subset \mathrm{R}^N$ and set $u = \nabla u = 0$ on Ω, then problem (1.1) is reduced to the following fourth-order elliptic equations of Kirchhoff type:

$$\begin{cases} \triangle^2 u - (a + b\int_\Omega |\nabla u|^2\,dx)\triangle u = f(x,u), & x \in \Omega, \\ u = 0, \quad \nabla u = 0 & \text{on } \Omega, \end{cases} \tag{1.2}$$

which is related to the following stationary analogue of the equation of Kirchhoff type:

$$u_{tt} + \triangle^2 u - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\triangle u = f(x,u) \quad \text{in } \Omega, \tag{1.3}$$

where $\triangle^2$ is the biharmonic operator. In one and two dimensions, (1.3) is used to describe some phenomena appearing in different physical, engineering and other sciences because it is regarded as a good approximation for describing nonlinear vibrations of beams or plates (see [1, 2]). By using the fixed point theorems in cones of ordered Banach spaces, Ma [3] considered the existence and multiplicity of positive solutions for the fourth-order equation:

$$\begin{cases} u'''' - M(\int_0^1 |u'|^2\,dx)u'' = q(x)f(x,u,u'), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.4}$$

Recently, by the variational methods, Ma and Wang *etc.* studied (1.4) and the following fourth-order equation of Kirchhoff type:

$$\begin{cases} \triangle^2 u - M(\int_\Omega |\nabla u|^2\,dx)\nabla u = f(x,u) & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega \end{cases}$$

and obtained the existence and multiplicity of solutions; see [4–6]. Very recently, Wang *et al.* considered the existence of nontrivial solutions of (1.2) with one parameter λ in [7] by using the mountain pass techniques and the truncation method.

We note that problem (1.1) with $a = 1$, $b = 0$, and R^3 being replaced by R^N, reduces the well-known fourth-order elliptic equations

$$\begin{cases} \triangle^2 u - \triangle u + V(x)u = f(x,u), & x \in \mathrm{R}^N, \\ u \in H^2(\mathrm{R}^N). \end{cases} \tag{1.5}$$

There are some results for (1.5). For example, see [8–10]. By the mountain pass theorem and symmetric mountain pass theorem, Yin and Wu [8] obtained infinitely many high energy solutions for problem (1.5) under the condition that $f(x,u)$ is superlinear at infinity in u. In order to overcome lack of compactness for the Sobolev's embedding theorem in the whole space R^N case, they assumed that the potential $V(x)$ satisfies

(V_0) $V(x) \in C(\mathrm{R}^N, \mathrm{R})$ such that $\inf_{x\in\mathrm{R}^N} V(x) > 0$ and for any $M > 0$, $\operatorname{meas}\{x \in \mathrm{R}^N : V(x) \leq M\} < \infty$.

Later, under the condition (V_0), Ye and Tang [9] obtained the existence of infinitely many large-energy and small-energy solutions, which unifies and improves the results in [8]. They also considered the sublinear case. Very recently, Zhang *et al.* [10] established the existence of infinitely many solutions by using the genus properties. The solvability of (1.1) without $\triangle^2$ has also been well studied by various authors (see [11] and the references therein).

Motivated by the above works described, the object of this paper is to study the existence and multiplicity solutions for a class of sublinear fourth-order elliptic equation of Kirchhoff type by using the genus properties in critical theory. Our spirit is similar to [12, 13]. Our main results are the following.

Theorem 1.1 *Assume that* (V), *and* (f1)-(f2) *hold*, *then the problem* (1.1) *possesses at least a nontrivial solution*.

Theorem 1.2 *Assume that* (V), *and* (f1)-(f3) *hold*, *then the problem* (1.1) *possesses infinitely many negative energy solutions*.

Obviously, we see that (f4) implies (f2). Then we have the following corollary.

Corollary 1.1 *Assume that* (V), (f1), *and* (f4) *hold*, *then the problem* (1.1) *possesses at least a nontrivial solution*. *If additionally* (f3) *holds*, *then the problem* (1.1) *possesses infinitely many negative nontrivial solutions*.

Remark 1.1 It is well known that assumption (V_0) implies a coercive condition on the potential $V(x)$, which was firstly introduced in [14] and is used to overcome the lack of compactness of embedding of the working space. In other words, under the weaker condition (V), the Sobolev embedding is not compact, which is a difficulty we must overcome.

Remark 1.2 The conditions (V) and (f1)-(f4) were introduced in [10, 13] to obtain the existence of infinitely many solutions for fourth-order elliptic equations and sublinear Schrödinger-Maxwell equations. An interesting question now is whether the same existence results occur to the nonlocal problem (1.1). We now give a positive answer. Moreover, let $a = 1$, $b = 0$ and R^3 be replaced by R^N in problem (1.1); we will get the main results in [10].

Remark 1.3 According to Theorem 1.2, the nonlinearity can be allowed to indefinite sign-changing. For example, let $V(x) = 1 + \frac{1}{1+e^{|x|}}$ and $f(x,u) = \frac{6\cos x_i}{5(1+e^{|x|})}|u|^{-\frac{4}{5}}u + \frac{5\sin x_j}{4(1+e^{|x|})}|u|^{-\frac{3}{4}}u$, $i,j = 1,2,3$, $\forall(x,u) \in \mathrm{R}^3 \times \mathrm{R}$, where $x = (x_1, x_2, x_3)^T$. Obviously,

$$\bigl|f(x,u)\bigr| \leq \frac{6}{5(1+e^{|x|})}|u|^{\frac{1}{5}} + \frac{5}{4(1+e^{|x|})}|u|^{\frac{1}{4}}, \quad \forall(x,u) \in \mathrm{R}^3 \times \mathrm{R},$$

$$F(x,u) = \frac{\cos x_i}{1+e^{|x|}}|u|^{\frac{6}{5}} + \frac{\sin x_j}{1+e^{|x|}}|u|^{\frac{5}{4}} \geq \frac{\cos 1}{1+e}|u|^{\frac{6}{5}}, \quad i,j = 1,2,3, \forall(x,u) \in J \times [-1,1],$$

where $J = (0,1) \times (0,1) \times (0,1)$, then (f1), (f2) and (f3) are satisfied by choosing

$$p = \frac{1}{5}, \qquad q = \frac{1}{4}, \qquad b_1(x) = \frac{6}{5(1+e^{|x|})}, \qquad b_2(x) = \frac{5}{4(1+e^{|x|})},$$
$$\alpha = 1, \qquad \beta = \frac{\cos 1}{1+e}, \qquad \nu = \frac{6}{5}.$$

To the best of our knowledge, little has been done for the existence of infinitely many nontrivial solutions to problem (1.1) by using the genus properties in critical theory.

The outline of the paper is given as follows: in Section 2, we present some preliminary results. The proofs of our main results are given in Section 3. Throughout this paper, C denotes various positive constants.

2 Preliminaries

Let

$$H := H^2(\mathrm{R}^3) = \{u \in L^2(\mathrm{R}^3) : |\nabla u|, \triangle u \in L^2(\mathrm{R}^3)\}$$

with the inner product and the norm

$$\langle u,v\rangle_H = \int_{\mathrm{R}^3} (\triangle u \triangle v + \nabla u \nabla v + uv)\,dx, \qquad \|u\|_H = \langle u,u\rangle_H^{\frac{1}{2}}.$$

Define our working space

$$E = \left\{ u \in H : \int_{\mathrm{R}^3} \left(|\triangle u|^2 + |\nabla u|^2 + V(x)u^2\right) dx < +\infty \right\}$$

with the inner product and norm

$$\langle u,v\rangle = \int_{\mathrm{R}^3} \left(\triangle u \triangle v + a\nabla u \nabla v + V(x)uv\right) dx, \qquad \|u\| = \langle u,u\rangle^{\frac{1}{2}},$$

where $\|\cdot\|$ is an equivalent to the norm $\|\cdot\|_H$. Since the embedding $E \hookrightarrow L^s(\mathrm{R}^3)$ $(2 \le s < 2_*)$ is continuous, there exists $\eta_s > 0$ such that $\|u\|_{L^s} \le \eta_s \|u\|_E$, $\forall u \in E$.

We define the functional

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathrm{R}^3} |\nabla u|^2\,dx\right)^2 - \int_{\mathrm{R}^3} F(x,u)\,dx, \tag{2.1}$$

where $F(x,u) = \int_0^u f(x,s)\,ds$. Then we have the following lemma.

Lemma 2.1 *Under the conditions* (V) *and* (f1), *I is of class $C^1(E,\mathrm{R})$ and*

$$\begin{aligned}\langle I'(u),v\rangle = {}& \int_{\mathrm{R}^3} \left(\triangle u \triangle v + a\nabla u \nabla v + V(x)uv\right) dx + b\int_{\mathrm{R}^3} |\nabla u|^2\,dx \int_{\mathrm{R}^3} \nabla u \nabla v\,dx \\ & - \int_{\mathrm{R}^3} f(x,u)v\,dx \end{aligned} \tag{2.2}$$

for all $u, v \in E$. Furthermore, if $u \in E$ is a critical point of the functional I, then $u \in E$ is a solution of the problem (1.1).

Proof By (f1), one gets

$$\begin{aligned}\left|\int_{\mathrm{R}^3} F(x,u)\,dx\right| &\le \int_{\mathrm{R}^3} \big|b_1(x)\big||u|^{p+1}\,dx + \int_{\mathrm{R}^3} \big|b_2(x)\big||u|^{q+1}\,dx \\ &\le a^{-\frac{p+1}{2}} \left(\int_{\mathrm{R}^3} \big|b_1(x)\big|^{\frac{2}{1-p}}\,dx\right)^{\frac{1-p}{2}} \left(\int_{\mathrm{R}^3} V(x)u^2\,dx\right)^{\frac{1+p}{2}} \\ &\quad + a^{-\frac{q+1}{2}} \left(\int_{\mathrm{R}^3} \big|b_2(x)\big|^{\frac{2}{1-q}}\,dx\right)^{\frac{1-q}{2}} \left(\int_{\mathrm{R}^3} V(x)u^2\,dx\right)^{\frac{1+q}{2}} \\ &\le a^{-\frac{p+1}{2}} \|b_1\|_{L^{\frac{2}{1-p}}} \|u\|_{L^2}^{1+p} + a^{-\frac{q+1}{2}} \|b_2\|_{L^{\frac{2}{1-q}}} \|u\|_{L^2}^{1+q}.\end{aligned}$$

Thus, I is well defined on E. We now prove that (2.2) holds. For any $u, v \in E$ and the function $\theta : \mathrm{R}^3 \to (0,1)$, it follows from (f1) and the Hölder inequality that

$$\begin{aligned}&\int_{\mathrm{R}^3} \max_{t\in[0,1]} \big|f\big(x, u(x) + t\theta(x)v(x)\big)v(x)\big|\,dx \\ &\quad = \int_{\mathrm{R}^3} \max_{t\in[0,1]} \big|f\big(x, u(x) + t\theta(x)v(x)\big)\big|\big|v(x)\big|\,dx \\ &\quad \le C \int_{\mathrm{R}^3} \big[b_1(x)\big(|u| + |v|\big)^p + b_2(x)\big(|u| + |v|\big)^q\big]|v|\,dx \\ &\quad \le C\bigg[a^{-\frac{p+1}{2}} \left(\int_{\mathrm{R}^3} \big|b_1(x)\big|^{\frac{2}{1-p}}\,dx\right)^{\frac{1-p}{2}} \left(\int_{\mathrm{R}^3} V(x)|u|^2\,dx\right)^{\frac{p}{2}} \left(\int_{\mathrm{R}^3} V(x)|v|^2\,dx\right)^{\frac{1}{2}} \\ &\qquad + a^{-\frac{p+1}{2}} \left(\int_{\mathrm{R}^3} \big|b_1(x)\big|^{\frac{2}{1-p}}\,dx\right)^{\frac{1-p}{2}} \left(\int_{\mathrm{R}^3} V(x)|v|^2\,dx\right)^{\frac{p+1}{2}} \\ &\qquad + a^{-\frac{q+1}{2}} \left(\int_{\mathrm{R}^3} \big|b_2(x)\big|^{\frac{2}{1-q}}\,dx\right)^{\frac{1-q}{2}} \left(\int_{\mathrm{R}^3} V(x)|u|^2\,dx\right)^{\frac{q}{2}} \left(\int_{\mathrm{R}^3} V(x)|v|^2\,dx\right)^{\frac{1}{2}} \\ &\qquad + a^{-\frac{q+1}{2}} \left(\int_{\mathrm{R}^3} \big|b_2(x)\big|^{\frac{2}{1-q}}\,dx\right)^{\frac{1-q}{2}} \left(\int_{\mathrm{R}^3} V(x)|v|^2\,dx\right)^{\frac{q+1}{2}}\bigg] \\ &\quad \le C\big[\|b_1\|_{L^{\frac{2}{1-p}}} \big(\|u\|^p + \|v\|^p\big)\|v\| + \|b_2\|_{L^{\frac{2}{1-q}}} \big(\|u\|^q + \|v\|^q\big)\|v\|\big] < +\infty. \qquad (2.3)\end{aligned}$$

Combining (2.3) with (2.1) and the Lebesgue dominated convergence theorem, then

$$\begin{aligned}\big\langle I'(u), v\big\rangle &= \lim_{t\to 0^+} \frac{I(u+tv) - I(u)}{t} \\ &= \lim_{t\to 0^+} \bigg\{\langle u,v\rangle + \frac{t}{2}\|v\|^2 + \frac{b}{4}\bigg[t^3 \int_{\mathrm{R}^3} |\nabla v|^2\,dx + 4\int_{\mathrm{R}^3} |\nabla u|^2\,dx \int_{\mathrm{R}^3} \nabla u \nabla v\,dx \\ &\quad + 2t\int_{\mathrm{R}^3} |\nabla u|^2\,dx \int_{\mathrm{R}^3} |\nabla v|^2\,dx + 4t^2 \int_{\mathrm{R}^3} |\nabla v|^2\,dx \int_{\mathrm{R}^3} \nabla u \nabla v\,dx \\ &\quad + 4t\left(\int_{\mathrm{R}^3} \nabla u \nabla v\,dx\right)^2\bigg] - \int_{\mathrm{R}^3} f(x, u + \theta t v)v\,dx\bigg\} \\ &= \langle u,v\rangle + b\int_{\mathrm{R}^3} |\nabla u|^2\,dx \int_{\mathrm{R}^3} \nabla u \nabla v\,dx - \int_{\mathrm{R}^3} f(x,u)v\,dx. \qquad (2.4)\end{aligned}$$

This implies that (2.2) holds. Next, we prove that I' is continuous. Let $u_n \to u$ in E. By the embedding theorem, one has

$$u_n \to u \quad \text{in } L^s(\mathrm{R}^3) \text{ for any } s \in [2, 2_*) \quad \text{and} \quad u_n \to u \quad \text{a.e. in } \mathrm{R}^3. \tag{2.5}$$

Firstly, we show that

$$\lim_{n\to\infty} \int_{\mathrm{R}^3} \left|f(x,u_n) - f(x,u)\right|^2 dx = 0. \tag{2.6}$$

Otherwise, there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_i}\}$ such that

$$\int_{\mathrm{R}^3} \left|f(x,u_{n_i}) - f(x,u)\right|^2 dx \geq \varepsilon_0, \quad \forall i \in N. \tag{2.7}$$

In fact, by (2.5), passing to a subsequence if necessary, it can be assumed that $\sum_{i=1}^{\infty} \|u_{n_i} - u\|^2 < \infty$. Set $\omega(x) = (\sum_{i=1}^{\infty} \|u_{n_i} - u\|^2)^{\frac{1}{2}}$, $x \in \mathrm{R}^3$, then $\omega \in L^2(\mathrm{R}^3)$. Obviously,

$$\begin{aligned} \left|f(x,u_{n_i}) - f(x,u)\right|^2 &\leq 2\left|f(x,u_{n_i})\right|^2 + 2\left|f(x,u)\right|^2 \\ &\leq 4(p+1)^2 b_1^2(x)\left(|u_{n_i}|^{2p} + |u|^{2p}\right) + 4(q+1)^2 b_2^2(x)\left(|u_{n_i}|^{2q} + |u|^{2q}\right) \\ &\leq Cb_1^2(x)\left(|u_{n_i} - u|^{2p} + |u|^{2p}\right) + Cb_2^2(x)\left(|u_{n_i} - u|^{2q} + |u|^{2q}\right) \\ &\leq Cb_1^2(x)\left(\left|\omega(x)\right|^{2p} + |u|^{2p}\right) + Cb_2^2(x)\left(\left|\omega(x)\right|^{2q} + |u|^{2q}\right) \\ &:= h(x) \quad \text{a.e. in } \mathrm{R}^3 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \int_{\mathrm{R}^3} h(x)\,dx &= \int_{\mathrm{R}^3} Cb_1^2(x)\left(\left|\omega(x)\right|^{2p} + |u|^{2p}\right) dx + \int_{\mathrm{R}^3} Cb_2^2(x)\left(\left|\omega(x)\right|^{2q} + |u|^{2q}\right) dx \\ &\leq C\left[\|b_1\|^2_{L^{\frac{2}{1-p}}} \left(\|\omega\|^{2p}_{L^2} + \|u\|^{2p}_{L^2}\right) + \|b_2\|^2_{L^{\frac{2}{1-q}}} \left(\|\omega\|^{2q}_{L^2} + \|u\|^{2q}_{L^2}\right)\right] < +\infty. \end{aligned} \tag{2.9}$$

It follows from (2.8), (2.9), and the Lebesgue dominated convergence theorem that (2.6) holds. Secondly, set $\tilde{E} = \{u \in L^2(\mathrm{R}^3) : \nabla u \in L^2(\mathrm{R}^3)\}$ with the norm $\|u\|^2_{\tilde{E}} = \int_{\mathrm{R}^3} |\nabla u|^2\,dx$. Then the embedding $E \hookrightarrow \tilde{E}$ is continuous. By the continuity of the embedding $E \hookrightarrow \tilde{E}$ and the boundedness of $\{u_n\}$, one has

$$b\left(\int_{\mathrm{R}^3} |\nabla u_n|^2\,dx - \int_{\mathrm{R}^3} |\nabla u|^2\,dx\right) \int_{\mathrm{R}^3} \nabla u \nabla v\,dx \to 0, \quad \text{as } n \to \infty. \tag{2.10}$$

From (2.2), (2.6), and (2.10), we have

$$\begin{aligned} &\left|\langle I'(u_n) - I'(u), v\rangle\right| \\ &\quad = \left|\int_{\mathrm{R}^3} \triangle(u_n - u)\triangle v\,dx + \left(a + b\int_{\mathrm{R}^3} |\nabla u_n|^2\,dx\right) \int_{\mathrm{R}^3} \nabla(u_n - u)\nabla v\,dx \right. \\ &\qquad + \int_{\mathrm{R}^3} V(x)(u_n - u)v\,dx + b\left(\int_{\mathrm{R}^3} |\nabla u_n|^2\,dx - \int_{\mathrm{R}^3} |\nabla u|^2\,dx\right) \int_{\mathrm{R}^3} \nabla u \nabla v\,dx \end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^3} \big(f(x,u_n) - f(x,u)\big) v\,dx \bigg| \\
&\le C\|u_n - u\|\|v\| + \left| b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx - \int_{\mathbb{R}^3} |\nabla u|^2\,dx \right) \int_{\mathbb{R}^3} \nabla u \nabla v\,dx \right| \\
&\quad + \int_{\mathbb{R}^3} \big|f(x,u_n) - f(x,u)\big||v|\,dx = o(1),
\end{aligned}
$$

as $n \to \infty$, which implies the continuity of I'. Furthermore, by standard arguments, we can prove that $u \in E$ is a solution of (1.1) if and only if u is a critical point of the functional I. The proof is complete. □

In order to deduce our results, we need to quote a few results.

Theorem 2.1 ([14]) *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfy the (PS)-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

Let E be a Banach space, $c \in \mathbb{R}$ and $I \in C^1(E,\mathbb{R})$. Set

$$
\begin{aligned}
&\Sigma = \big\{ A \subset E \setminus \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0 \big\}, \\
&K_c = \big\{ u \in E : I(u) = c, I'(u) = 0 \big\}, \qquad I^c = \big\{ u \in E : I(u) \le c \big\}.
\end{aligned}
$$

Definition 2.1 ([15]) For $A \in \Sigma$, we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.

Theorem 2.2 ([16]) *Let I be an even C^1 functional on E and satisfy the (PS)-condition. For any $n \in N$, set*

$$
\Sigma_n = \big\{ A \in \Sigma : \gamma(A) \ge n \big\}, \qquad c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u).
$$

(1) *If $\Sigma_n \neq \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of I.*
(2) *If there exists $r \in N$ such that $c_n = c_{n+1} = \cdots = c_{n+r} = c \in \mathbb{R}$ and $c \neq I(0)$, then $\gamma(K_c) \ge r + 1$.*

3 Proofs of main results

According to Theorem 2.1, we need the following lemma.

Lemma 3.1 *Assume that* (V) *and* (f1)-(f2) *hold, then I is bounded from below and satisfies the (PS)-condition.*

Proof By (2.1), (f1), the Sobolev embedding theorem, and the Hölder inequality, one has

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx \right)^2 - \int_{\mathbb{R}^3} F(x,u)\,dx \\
&\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} b_1(x)|u|^{p+1}\,dx - \int_{\mathbb{R}^3} b_2(x)|u|^{q+1}\,dx \\
&\ge \frac{1}{2}\|u\|^2 - a^{-\frac{p+1}{2}}\|b_1\|_{L^{\frac{2}{1-p}}}\|u\|_{L^2}^{1+p} - a^{-\frac{q+1}{2}}\|b_2\|_{L^{\frac{2}{1-q}}}\|u\|_{L^2}^{1+q} \\
&\ge \frac{1}{2}\|u\|^2 - C\big(\|u\|^{1+p} + \|u\|^{1+q}\big).
\end{aligned}
\tag{3.1}
$$

Since $0 < p < q < 1$, (3.1) suggests $I(u) \to +\infty$ as $\|u\| \to +\infty$. Thus, I is bounded from below. We now prove that I satisfies the (PS)-condition. Assume that $\{u_n\} \subset E$ is a sequence such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$. It follows from (3.1) that there exists a constant $C_1 > 0$ such that

$$\|u_n\| \le C_1. \tag{3.2}$$

So it can be assumed that

$$u_n \rightharpoonup u_0 \quad \text{in } E, \qquad u_n \to u_0 \quad \text{in } L^s_{\mathrm{loc}}(\mathrm{R}^3), 2 \le s < 2_*. \tag{3.3}$$

For any given number $\varepsilon > 0$, by (f1), let us choose $\varrho > 0$ such that

$$\left(\int_{|x|\ge\varrho} \left|b_1(x)\right|^{\frac{2}{1-p}}\,dx\right)^{\frac{1-p}{2}} < \varepsilon, \qquad \left(\int_{|x|\ge\varrho} \left|b_2(x)\right|^{\frac{2}{1-q}}\,dx\right)^{\frac{1-q}{2}} < \varepsilon. \tag{3.4}$$

Then, by the Sobolev theorem, (2.8), (3.2), and (3.3), there exists $n_0 \in N$, for $n > n_0$, such that

$$\begin{aligned}
&\int_{|x|\le\varrho} \left|f(x,u_n) - f(x,u)\right||u_n - u|\,dx \\
&\quad\le \left(\int_{|x|\le\varrho} \left|f(x,u_n) - f(x,u)\right|^2\,dx\right)^{\frac{1}{2}} \left(\int_{|x|\le\varrho} |u_n - u|^2\,dx\right)^{\frac{1}{2}} \\
&\quad\le \varepsilon \left(\int_{|x|\le\varrho} 2\left(\left|f(x,u_n)\right|^2 + \left|f(x,u)\right|^2\right)dx\right)^{\frac{1}{2}} \\
&\quad\le \varepsilon \left[\int_{|x|\le\varrho} \left(4(p+1)^2 b_1^2(x)\left(|u_n|^{2p} + |u|^{2p}\right) + 4(q+1)^2 b_2^2(x)\left(|u_n|^{2q} + |u|^{2q}\right)\right)dx\right]^{\frac{1}{2}} \\
&\quad\le C\varepsilon \left[\|b_1\|^2_{L^{\frac{2}{1-p}}}\left(\|u_n\|^{2p}_{L^2} + \|u\|^{2p}_{L^2}\right) + \|b_2\|^2_{L^{\frac{2}{1-q}}}\left(\|u_n\|^{2q}_{L^2} + \|u\|^{2q}_{L^2}\right)\right]^{\frac{1}{2}} \\
&\quad\le C\varepsilon \left[\|b_1\|^2_{L^{\frac{2}{1-p}}}\left(C_1^{2p} + \|u\|^{2p}_{L^2}\right) + \|b_2\|^2_{L^{\frac{2}{1-q}}}\left(C_1^{2q} + \|u\|^{2q}_{L^2}\right)\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.5}$$

For another, we obtain from (f1), (3.2), the Sobolev embedding theorem and the Hölder inequality that

$$\begin{aligned}
&\int_{|x|>\varrho} \left|f(x,u_n) - f(x,u)\right||u_n - u|\,dx \\
&\quad\le \int_{|x|>\varrho} \left[(p+1)b_1(x)\left(|u_n|^p + |u|^p\right) + (q+1)b_2(x)\left(|u_n|^q + |u|^q\right)\right]\left(|u_n| + |u|\right)dx \\
&\quad\le 2(p+1)\int_{|x|>\varrho} b_1(x)\left(|u_n|^{p+1} + |u|^{p+1}\right)dx + 2(q+1)\int_{|x|>\varrho} b_2(x)\left(|u_n|^{q+1} + |u|^{q+1}\right)dx \\
&\quad\le 2(p+1)\|b_1\|_{L^{\frac{2}{1-p}}}\left(\|u_n\|^{p+1}_{L^2} + \|u\|^{p+1}_{L^2}\right) + 2(q+1)\|b_2\|_{L^{\frac{2}{1-q}}}\left(\|u_n\|^{q+1}_{L^2} + \|u\|^{q+1}_{L^2}\right) \\
&\quad\le C\varepsilon\left(C_1^{p+1} + \|u\|^{p+1} + C_1^{q+1} + \|u\|^{q+1}\right).
\end{aligned} \tag{3.6}$$

Since ε is arbitrary, (3.5) and (3.6) imply

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\big(f(x,u_n)-f(x,u)\big)(u_n-u)\,dx=0. \tag{3.7}$$

By (2.2), we get

$$\begin{aligned}
&\big\langle I'(u_n)-I'(u),u_n-u\big\rangle\\
&\quad=\int_{\mathbb{R}^3}\big|\triangle(u_n-u)\big|^2\,dx+\left(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\right)\int_{\mathbb{R}^3}\big|\nabla(u_n-u)\big|^2\,dx\\
&\qquad+\int_{\mathbb{R}^3}V(x)|u_n-u|^2\,dx-b\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx-\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)\,dx\\
&\qquad-\int_{\mathbb{R}^3}\big(f(x,u_n)-f(x,u)\big)(u_n-u)\,dx\\
&\quad\geq\|u_n-u\|^2-b\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx-\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)\,dx\\
&\qquad-\int_{\mathbb{R}^3}\big(f(x,u_n)-f(x,u)\big)(u_n-u)\,dx,
\end{aligned}$$

then

$$\begin{aligned}
\|u_n-u\|^2\leq{}&\big\langle I'(u_n)-I'(u),u_n-u\big\rangle\\
&+b\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx-\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)\,dx\\
&+\int_{\mathbb{R}^3}\big(f(x,u_n)-f(x,u)\big)(u_n-u)\,dx.
\end{aligned} \tag{3.8}$$

Clearly,

$$\big\langle I'(u_n)-I'(u),u_n-u\big\rangle\to 0,\quad \text{as } n\to\infty. \tag{3.9}$$

It follows from (2.10), (3.7), (3.8), and (3.9) that $\|u_n-u\|\to 0$. Hence, I satisfies the (PS)-condition. This completes the proof. □

Proof of Theorem 1.1 By Lemmas 2.1 and 3.1, the conditions of Theorem 2.1 are satisfied. Thus, $c=\inf_E I(u)$ is a critical value of I, that is, there exists a critical point u^* such that $I(u^*)=c$. Now, we show $u^*\neq 0$. Choose $\psi\in(W_0^{1,2}(J)\cap E)\setminus\{0\}$, then by (2.1) and (f2), for $0<t<1$, we have

$$\begin{aligned}
I(t\psi)&=\frac{t^2}{2}\|\psi\|^2+\frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\,dx\right)^2-\int_{\mathbb{R}^3}F(x,t\psi)\,dx\\
&\leq\frac{t^2}{2}\|\psi\|^2+\frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\,dx\right)^2-\beta t^\nu\int_{\mathbb{R}^3}|\psi|^\nu\,dx.
\end{aligned} \tag{3.10}$$

Since $1<\nu<2$, it follows from (3.10) that $I(t\psi)<0$ for $0<t<1$ small enough. Hence $I(u^*)=c<0$, which implies u^* being a nontrivial critical point of I with $I(u^*)=c$. That is to say, that u^* is a nontrivial solution of (1.1). The proof is completed. □

Proof of Theorem 1.2 By Lemma 3.1, $I \in C^1(E, \mathrm{R})$ is bounded from below and satisfies the (PS)-condition. It follows from (f3) that I is even and $I(0) = 0$. In order to apply Theorem 2.2, we now show that for any $n \in N$ there exists ε such that

$$\gamma\left(I^{-\varepsilon}\right) \geq n. \tag{3.11}$$

For any $n \in N$, we take n disjoint open sets J_i such that $\bigcup_{i=1}^n J_i = J$. For each $i \in \{1, 2, \ldots, n\}$, choose $u_i \in (W_0^{1,2}(J_i) \cap E) \setminus \{0\}$ such that $\|u_i\| = 1$ and

$$E_n = \operatorname{span}\{u_1, u_2, \ldots, u_n\}, \qquad S_n = \left\{u \in E_n : \|u\| = 1\right\}.$$

Then, for each $u \in E_n$, there exist λ_i, $i = 1, 2, \ldots, n$, such that

$$u(x) = \sum_{i=1}^n \lambda_i u_i(x) \quad \text{for } x \in \mathrm{R}^3. \tag{3.12}$$

Then we get

$$\|u\|_{L^\nu} = \left(\int_{\mathrm{R}^3} |u|^\nu \, dx\right)^{\frac{1}{\nu}} = \left(\sum_{i=1}^n |\lambda_i|^\nu \int_{J_i} |u_i|^\nu \, dx\right)^{\frac{1}{\nu}} \tag{3.13}$$

and

$$\begin{aligned} \|u\|^2 &= \int_{\mathrm{R}^3} \left(|\triangle u|^2 + a|\nabla u|^2 + V(x)|u|^2\right) dx \\ &= \sum_{i=1}^n \lambda_i^2 \int_{J_i} \left(|\triangle u_i|^2 + a|\nabla u_i|^2 + V(x)|u_i|^2\right) dx = \sum_{i=1}^n \lambda_i^2 \|u_i\|^2 = \sum_{i=1}^n \lambda_i^2. \end{aligned} \tag{3.14}$$

Since all norms are equivalent in a finite dimensional normed space, there exists a positive constant C_2 such that

$$C_2 \|u\| \leq \|u\|_{L^\nu} \quad \text{for } u \in E_n. \tag{3.15}$$

Then, by (f2), (2.1), (3.13), and (3.15), for any $u \in S_n$, one has

$$\begin{aligned} I(su) &= \frac{s^2}{2}\|u\|^2 + \frac{b}{4}s^4\left(\int_{\mathrm{R}^3} |\nabla u|^2 \, dx\right)^2 - \int_{\mathrm{R}^3} F(x, su) \, dx \\ &\leq \frac{s^2}{2}\|u\|^2 + \frac{Cb}{4}s^4\|u\|^4 - \beta s^\nu \int_{\mathrm{R}^3} |u|^\nu \, dx \\ &\leq \frac{s^2}{2}\|u\|^2 + \frac{Cb}{4}s^4\|u\|^4 - \beta s^\nu \sum_{i=1}^n \int_{J_i} |\lambda_i u_i|^\nu \, dx \\ &= \frac{s^2}{2}\|u\|^2 + \frac{Cb}{4}s^4\|u\|^4 - \beta s^\nu \sum_{i=1}^n |\lambda_i|^\nu \int_{J_i} |u_i|^\nu \, dx \\ &= \frac{s^2}{2}\|u\|^2 + \frac{Cb}{4}s^4\|u\|^4 - \beta s^\nu \|u\|_{L^\nu}^\nu \end{aligned}$$

$$\begin{aligned} &\leq \frac{s^2}{2}\|u\|^2 + \frac{Cb}{4}s^4\|u\|^4 - \beta s^\nu C_2^\nu\|u\|^\nu \\ &= \frac{s^2}{2} + \frac{Cb}{4}s^4 - \beta s^\nu C_2^\nu. \end{aligned} \tag{3.16}$$

Since $1 < \nu < 2$, it follows from (3.16) that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$I(\delta u) < -\varepsilon \quad \text{for } u \in S_n. \tag{3.17}$$

Set

$$S_n^\delta = \{\delta u : u \in S_n\}, \qquad \Omega = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_n) : \sum_{i=1}^n \lambda_i^2 < \delta^2\right\}.$$

By (3.17), we know that $I(u) < -\varepsilon$ for $u \in S_n^\delta$, which, together with the fact that $I \in C^1(E, \mathbb{R})$ and is even, implies that

$$S_n^\delta \subset I^{-\varepsilon} \in \Sigma.$$

On the other side, by (3.12) and (3.14), then there exists an odd homeomorphism mapping $\psi \in C(S_n^\delta, J)$. By some properties of the genus (see 3^0 of Propositions 7.5 and 7.7 in [15]), we get

$$\gamma\left(I^{-\varepsilon}\right) \geq \gamma\left(S_n^\delta\right) = n. \tag{3.18}$$

Thus, (3.11) holds. Let $c_n = \inf_{A\in\Sigma_n}\sup_{u\in A} I(u)$. By (3.18) and I being bounded from below on E, then $-\infty < c_n \leq -\varepsilon < 0$, that is to say, for any $n \in N$, c_n is a real negative number. It follows from Theorem 2.2 that I has infinitely many nontrivial critical points. Thus, problem (1.1) possesses infinitely many nontrivial negative energy solutions. □

Competing interests
The authors declare to have no competing interests.

Authors' contributions
All authors, LX and HC, contributed to each part of this work equally and read and approved the final version of the manuscript.

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