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Existence of infinitely many solutions for generalized Schrödinger-Poisson system

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Abstract

We study the nonlinear generalized Schrödinger-Poisson system: $-\Delta u + V(x)u + K(x)\phi g(u) = f(x,u)$, in R^3, $-\Delta\phi = 2K(x)G(u)$, in R^3, where $V(x)$ and $K(x)$ are non-negative functions. The function $f(x,u)$ is superlinear. Under appropriate assumptions on $V(x)$, $K(x)$, and $g(u)$, we prove the existence and multiplicity of nontrivial solutions by the variant fountain theorem established by Zou. Some recent results due to different authors are extended.

MSC: 35J20; 35J65; 35J60

Keywords: generalized Schrödinger-Poisson system; variant fountain theorem; variational approaches

1 Introduction

Consider the existence and multiplicity of nontrivial solutions for the nonlinear generalized Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi g(u) = f(x,u), & \text{in } R^3, \\ -\Delta\phi = 2K(x)G(u), & \text{in } R^3, \end{cases} \tag{1.1}$$

where $G(u) := \int_0^u g(t)\,dt$. Setting $g(u) = u$, (1.1) represents the well-known Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & \text{in } R^3, \\ -\Delta\phi = K(x)u^2, & \text{in } R^3. \end{cases} \tag{1.2}$$

Such a system arises in an interesting physical context. If we look for solitary solutions of the Schrödinger equation for a particle in an electrostatic field, we just need to solve (1.2). We refer the interested readers to [1, 2] for more details of the physical aspects.

With the aid of variational methods, under various hypotheses on $V(x)$, $K(x)$, and $f(x,u)$, system (1.2) has been extensively investigated over the past several decades. See for example, Benci and Fortunato [1], D'Aprile and Mugnai [2], Ambrosetti and Ruiz [3], Ambrosetti [4], and the references therein.

In recent years, there has been a lot of research on the existence of solutions for system (1.2) with $f(x,u) = |u|^{p-1}u$ and the potential $V(x)$ being radially symmetric or nonradial.

In [3, 4], the authors proved the existence of infinitely many pairs of high energy radial solutions when $2 < p < 5$, and also obtained some existence results for $1 < p \leq 2$. Sun [5] studied the existence of infinitely many solutions when $p \in (0,1)$. The authors of [6] proved the existence of positive solutions without compactness conditions if $3 < p < 5$. Azzollini and Pomponio [7] proved the existence of a ground state solution to (1.2) for $3 < p < 5$.

Furthermore, Sun *et al.* [8] studied system (1.2) for $f(x,u)$ being asymptotically linear and obtained ground state solutions. In [9], Huang *et al.* obtained the existence of at least a pair of fixed sign solutions and a pair of sign-changing solutions to system (1.2) involving a critical nonlinearity. Ding *et al.* [10] studied (1.2) with a nonhomogeneous term, where $f(x,u) = f(u)$ is either asymptotically linear or asymptotically 3-linear with respect to u at infinity. Very recently, Liu and Guo [11] studied (1.2) with critical growth and obtained the existence of ground state solutions via variational methods.

The problem of finding infinitely many large energy solutions is a very classical problem. There is an extensive literature concerning the existence of infinitely many large energy solutions of (1.2). Chen and Tang [12] obtained infinitely many large energy solutions by the following variant 'Ambrosetti-Rabinowitz' type condition (AR for short)

$$\exists \mu > 4 \text{ such that } 0 \leq \mu F(x,u) \leq u f(x,u), \quad \text{for every } x \in R^3, |u| \geq 1,$$

where $F(x,u) := \int_0^u f(x,s)\,ds$. After that, in [13], the authors studied (1.2) without the (AR) condition by the variant fountain theorem established by Zou [14, Theorem 2.1]. Later, Li and Chen in [15] also obtained infinitely many large energy solutions of (1.2) with $K(x) = 1$ without the (AR) condition.

By using the method of a cut-off function and the variational arguments, the authors [16] studied the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + \varepsilon q \phi f(u) = \eta |u|^{p-1} u, & \text{in } \Omega, \\ -\Delta \phi = 2qF(u), & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p < 5$, $\varepsilon, \eta = \pm 1$, $f : R \to R$ is a continuous function and $F(t) := \int_0^t f(s)\,ds$. They proved the existence and multiplicity results assuming on f a subcritical growth condition and also they considered the existence and nonexistence results in the critical case. Lately, Li and Zhang [17] discussed (1.1) with $V(x)$ and $K(x)$ being constants and obtained the existence of a positive radially symmetric solution without compactness conditions.

Motivated by the above facts, in the present paper our aim is to study the existence of infinitely many solutions for system (1.1). To the best of our knowledge, the existence and multiplicity of nontrivial solutions to system (1.1) has never been studied by variational methods, where g is a more general function, $f(x,u) \neq |u|^{p-1}u$ is also general, and $V(x)$ and $K(x)$ may be non-radial symmetrical and non-periodic. Before we state our main result, we list some conditions as follows, which have a role to play.

(V) $V(x) \in C(R^3, R)$ satisfies $\inf_{x \in R^3} V(x) \geq a > 0$, where $a > 0$ is a constant. Moreover, for any $M > 0$, $\operatorname{meas}\{x \in R^3 : V(x) \leq M\} < \infty$, where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in R^3.

(K) $K(x) \in L^\infty(R^3, R)$, and $K(x) \geq 0$ for any $x \in R^3$.

(g1) $g \in C(R, R)$ and there exist constants $c > 0$, $\frac{5}{3} < \alpha < 2$ such that

$$\left|g(u)\right| \le c\left(|u| + |u|^{\alpha-1}\right) \quad \text{for all } u \in R.$$

(g2) $g(-u) = -g(u)$, $\forall u \in R$.

(g3) $G(u) \ge 0$ for all $u \in R$, and there exists a constant $\vartheta_1 \ge 1$ such that

$$\tilde{G}(su) \le \vartheta_1 \tilde{G}(u), \quad \forall s \in [0,1],$$

where $\tilde{G}(u) := G(u) - \frac{1}{4}g(u)u$.

(f1) $f \in C(R^3 \times R, R)$ and there exist constants $c_1 > 0$, $4 < \nu < 2^*$ such that

$$\left|f(x,u)\right| \le c_1\left(1 + |u|^{\nu-1}\right) \quad \text{for all } (x,u) \in R^3 \times R,$$

where $2^* = 6$ is the critical exponent for the Sobolev embedding in dimension 3.

(f2) $\lim_{|u|\to 0} \frac{f(x,u)}{u} = 0$ uniformly for $x \in R^3$.

(f3) $F(x,0) \equiv 0$, $F(x,u) \ge 0$ and $\lim_{|u|\to\infty} \frac{F(x,u)}{u^4} = +\infty$ uniformly for $x \in R^3$.

(f4) For a.e. $x \in R^3$, there exists a constant $\vartheta_2 \ge 1$ such that

$$\tilde{F}(x,su) \le \vartheta_2 \tilde{F}(x,u), \quad \forall (x,u) \in R^3 \times R, \text{ and } s \in [0,1],$$

where $\tilde{F}(x,u) = \frac{1}{4}f(x,u)u - F(x,u)$.

(f5) $f(x,-u) = -f(x,u)$, $\forall (x,u) \in R^3 \times R$.

Our main result reads as follows.

Theorem 1.1 *Assume that* (V), (K), (g1)-(g3), *and* (f1)-(f5) *hold*; *the problem* (1.1) *possesses infinitely many nontrivial solutions* $\{(u_k, \phi_k)\}$ *satisfying*

$$\begin{aligned} &\frac{1}{2}\int_{R^3} \left(|\nabla u_k|^2 + V(x)u_k^2\right) dx - \frac{1}{4}\int_{R^3} |\nabla \phi_k|^2\, dx \\ &\quad + \int_{R^3} K(x)\phi_k G(u_k)\, dx - \int_{R^3} F(x,u_k)\, dx \to +\infty, \quad \text{as } k \to \infty. \end{aligned}$$

Remark 1.1 It is well known that the (AR) condition is used to guarantee the boundedness of (P.S.) sequences of the corresponding functional. However, there are functions satisfying the assumptions (f1)-(f5), but not satisfying the (AR) condition, for instance, $f(x,t) = a(x)t^3 \ln(1 + |t|)$, where $0 < \inf_{R^3} a(x) \le \sup_{R^3} a(x) < \infty$.

Remark 1.2 Li *et al.* [13] used

(f′4) $\tilde{F}(x,s) \le \tilde{F}(x,t)$, $\forall (s,t) \in R^+ \times R^+$, $s \le t$.

to solve the problem (1.2). We claim that our condition (f4) is more general than (f′4). In fact, setting $\vartheta_2 = 1$, we find that $\tilde{F}(x,t)$ is increasing in R^+ with respect to t. Moreover, the function $f(x,t) = a(x)[4t^3 \ln(1 + t^4) + 4\sin t]$ satisfies (f4) but not satisfies (f′4), where $0 < \inf_{R^3} a(x) \le \sup_{R^3} a(x) < \infty$.

Remark 1.3 It is easy to present many functions satisfying (g1)-(g3), for example, $g(u) = u$, $g(u) = u^{\frac{1}{3}}$, $g(u) = u^{\frac{1}{5}}$, and so on. Moreover, setting $K(x) = 1$, $g(u) = u$ in (1.1), we can obtain similar results to the problem (1.1) in [15]. But our proof is different from [15]. For this reason, we use a small step.

Remark 1.4 Since we have the lack of the (AR) condition, in order to obtain the boundedness of (P.S.) sequences (see the proof of Theorem 1.1), we assume the range of α in (g1) is $\frac{5}{3} < \alpha < 2$, *i.e.* α is subquadratic, but that of reference [17] is superquadratic and there exist some functions which satisfy the condition (f) in [17] that do not satisfy the conditions (g1) and (g3), for example, $g(u) = u^3$.

The outline of the paper is as follows: in Section 2, we present some preliminary results, which are necessary for Section 3. In Section 3, we give the proof of Theorem 1.1. Throughout the paper we shall denote by $C_i > 0$ various positive constants.

2 Preliminaries

In this section we outline the variational framework for the problem (1.1) and give some preliminary lemmas. Define the function space

$$E = \left\{ u \in H^1(R^3) : \int_{R^3} \left(|\nabla u|^2 + V(x)u^2 \right) dx < +\infty \right\}.$$

Then E is a Hilbert space equipped with the inner product and norm

$$\langle u, v \rangle = \int_{R^3} \left(\nabla u \cdot \nabla v + V(x)uv \right) dx, \qquad \|u\| = \langle u, u \rangle^{1/2}.$$

Since $V(x)$ is bounded from below, the E is continuously embedded into $L^q(R^3)$ for all $q \in [2, 2^*]$. Therefore, there exists a positive constant $\eta_q > 0$ such that

$$\|u\|_{L^q} \le \eta_q \|u\|, \quad \forall u \in E, \tag{2.1}$$

where

$$\|u\|_{L^q} := \left(\int_{R^3} |u|^q \, dx \right)^{\frac{1}{q}}, \quad \text{for any } q \in [1, \infty)$$

is the norm of the usual Lebesgue space $L^q(R^3)$. Moreover, by (V), the embedding $E \hookrightarrow L^q(R^3)$ is also compact for any $q \in [2, 2^*)$ [18, Lemma 3.4]. Let $D^{1,2}(R^3)$ be the completion of $C_0^\infty(R^3)$ with respect to the norm

$$\|u\|_{D^{1,2}} = \left(\int_{R^3} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

It is well known that the embedding $D^{1,2}(R^3) \hookrightarrow L^6(R^3)$ is continuous (see [19]).

It is clear that system (1.1) is the Euler-Lagrange equations of the functional $J : E \times D^{1,2}(R^3) \to R$ defined by

$$J(u, \phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{R^3} |\nabla \phi|^2 \, dx + \int_{R^3} K(x)\phi G(u) \, dx - \int_{R^3} F(x, u) \, dx.$$

Obviously, the action functional J belongs to $C^1(E \times D^{1,2}(R^3), R)$ and its critical points are the solutions of (1.1); see for instance [1]. For any $u \in E$, by the Lax-Milgram theorem, we can obtain the result that the second equation in (1.1) has a unique solution ϕ_u. Substituting ϕ_u to the first equation of the problem (1.1), then the problem can be transformed to a one variable equation. In fact, we firstly get the following lemma.

Lemma 2.1 *For any $u \in E$, we have*

(1) $\phi_u \geq 0$;
(2) $\|\phi_u\|_{D^{1,2}} \leq C_4(\|u\|^2 + \|u\|^\alpha)$;
(3) $\int_{R^3} 2K(x)G(u)\phi_u \, dx \leq C_5(\|u\|^4 + \|u\|^{2\alpha})$.

Proof By the condition (g1), we find that there exists $C_1 > 0$ such that

$$G(u) \leq C_1\big(|u|^2 + |u|^\alpha\big), \quad \forall u \in E. \tag{2.2}$$

Then, by the Minkowski inequality and (2.1), we have

$$\begin{aligned} \big\|G(u)\big\|_{L^{\frac{6}{5}}} &\leq \left(\int_{R^3} \big[C_1\big(|u|^2 + |u|^\alpha\big)\big]^{\frac{6}{5}} \, dx\right)^{\frac{5}{6}} \\ &\leq C_1\big(\|u\|^2_{L^{\frac{12}{5}}} + \|u\|^\alpha_{L^{\frac{6\alpha}{5}}}\big) \\ &\leq C_2\big(\|u\|^2 + \|u\|^\alpha\big). \end{aligned} \tag{2.3}$$

For any $u \in E$, the linear functional $T_u : D^{1,2}(R^3) \to R$ is defined as

$$T_u(v) = \int_{R^3} 2K(x)G(u)v \, dx.$$

By the Sobolev embedding theorem, $K(x) \in L^\infty(R^3)$, $K(x) \geq 0$, and (2.3), we have

$$\begin{aligned} \int_{R^3} 2K(x)G(u)v \, dx &\leq \left(\int_{R^3} \big(2K(x)G(u)\big)^{\frac{6}{5}} \, dx\right)^{\frac{5}{6}} \left(\int_{R^3} |v|^6 \, dx\right)^{\frac{1}{6}} \\ &\leq C_3 \big\|G(u)\big\|_{L^{6/5}} \|v\|_{L^6} \\ &\leq C_4\big(\|u\|^2 + \|u\|^\alpha\big)\|v\|_{D^{1,2}}. \end{aligned} \tag{2.4}$$

So, T_u is continuous on $D^{1,2}(R^3)$. Hence, the Lax-Milgram theorem implies that, for every $u \in E$, there exists a unique $\phi_u \in D^{1,2}(R^3)$ such that

$$\int_{R^3} 2K(x)G(u)v \, dx = \int_{R^3} \nabla\phi_u \cdot \nabla v \, dx, \quad \text{for any } v \in D^{1,2}\big(R^3\big).$$

Using integration by parts, we get

$$\int_{R^3} \nabla\phi_u \cdot \nabla v \, dx = -\int_{R^3} v \triangle\phi_u \, dx, \quad \text{for any } v \in D^{1,2}\big(R^3\big),$$

therefore,

$$-\triangle\phi_u = 2K(x)G(u) \tag{2.5}$$

in a weak sense. We can write an integral expression for ϕ_u in the form

$$\phi_u = 2\int_{R^3} \frac{K(y)G(u(y))}{|x-y|}\,dy,$$

for any $u \in C_0^\infty(R^3)$ (see [20], Theorem 1 or Lemma 2.1 of [21]). It follows from $K(x) \geq 0$, (g2), and (g3) that $\phi_u \geq 0$, and $\phi_{-u} = \phi_u$ for any $u \in E$. By (2.4), (2.5), and $K(x) \in L^\infty(R^3)$, for any $u \in E$ we get

$$\|\phi_u\|_{D^{1,2}} \leq C_4\bigl(\|u\|^2 + \|u\|^\alpha\bigr) \quad \text{and} \quad \int_{R^3} 2K(x)G(u)\phi_u\,dx \leq C_5\bigl(\|u\|^4 + \|u\|^{2\alpha}\bigr).$$

The proof is complete. □

We consider the functional $I : E \to R$ defined by $I(u) = J(u, \phi_u)$. By (2.5), the reduced functional takes the form

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{R^3} K(x)\phi_u G(u)\,dx - \int_{R^3} F(x,u)\,dx. \tag{2.6}$$

By (f1) and (f2), for any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that, for all $x \in R^3$, $u \in R$,

$$\bigl|f(x,u)\bigr| \leq 2\varepsilon|u| + \nu c(\varepsilon)|u|^{\nu-1}$$

and

$$\bigl|F(x,u)\bigr| \leq \varepsilon|u|^2 + c(\varepsilon)|u|^\nu. \tag{2.7}$$

Therefore, by (f3), we obtain

$$0 \leq \int_{R^3} F(x,u)\,dx \leq \varepsilon\|u\|_{L^2}^2 + c(\varepsilon)\|u\|_{L^\nu}^\nu \leq C_6\|u\|^2 + C_7\|u\|^\nu. \tag{2.8}$$

Then, by Lemma 2.1, I is well defined and is a C^1 functional with derivative given by

$$\bigl\langle I'(u), v\bigr\rangle = \int_{R^3} \bigl(\nabla u \cdot \nabla v + V(x)uv + K(x)\phi_u g(u)v - f(x,u)v\bigr)\,dx. \tag{2.9}$$

Now, we can apply Theorem 2.3 of [22] to our functional I and obtain the following.

Lemma 2.2 *The following statements are equivalent*:
(1) $(u,\phi) \in E \times D^{1,2}(R^3)$ *is a solution of* (1.1);
(2) *u is a critical point of I and $\phi = \phi_u$.*

Since we do not assume the (AR) condition, the verification of the (P.S.) condition becomes complicated, so we use the following variant fountain theorem introduced in [14] without the (P.S.) condition to handle the problem (1.1).

Theorem 2.1 *Let E be a Banach space with $\|\cdot\|$ and $E = \overline{\bigoplus_{j\in N} X_j}$ with $\dim X_j < \infty$ for any $j \in N$. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k+1}^\infty X_j}$ and $B_k = \{u \in Y_k : \|u\| \leq r_k\}$.*

Consider the following C^1 functional $\psi_\lambda : E \to R$ defined by

$$\psi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2],$$

where $A, B : E \to R$ are two functionals. Suppose that

(F1) ψ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore,

$$\psi_\lambda(-u) = \psi_\lambda(u) \quad \text{for all } (\lambda, u) \in [1,2] \times E.$$

(F2) $B(u) \geq 0$ for all $u \in E$, and $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$.

(F3) There exist $r_k > \rho_k > 0$ such that

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \psi_\lambda(u) > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \psi_\lambda(u), \quad \forall \lambda \in [1,2].$$

Then

$$\alpha_k(\lambda) \leq \varsigma_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \psi_\lambda\big(\gamma(u)\big), \quad \forall \lambda \in [1,2],$$

where $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma \text{ is odd } \gamma|_{\partial B_k} = id\}$. Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \qquad \psi'_\lambda\big(u_m^k(\lambda)\big) \to 0 \quad \text{and} \quad \psi_\lambda\big(u_m^k(\lambda)\big) \to \varsigma_k(\lambda) \quad \text{as } m \to \infty.$$

3 Proof of Theorem 1.1

In order to apply Theorem 2.1 to prove our main result, we define the functional ψ_λ on our working space E by

$$\psi_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{R^3} K(x)\phi_u G(u)\,dx - \lambda \int_{R^3} F(x,u)\,dx := A(u) - \lambda B(u), \tag{3.1}$$

for all $u \in E$ and $\lambda \in [1,2]$. Then $B(u) \geq 0$ for all $u \in E$, $A(u) \to \infty$ as $\|u\| \to \infty$.

We choose a completely orthonormal basis $\{e_j : j \in N\}$ of E and let $X_j = \operatorname{span}\{e_j\}$ for all $j \in N$. Then Z_k, Y_k can be defined as those in Section 2. Note that $\psi_1 = I$, where I is the functional defined in (2.6). We further need the following lemmas.

Lemma 3.1 *Let* (V), (K), (g3), *and* (f1) *be satisfied, then there exist a positive integer* k_1 *and a sequence* $\rho_k \to +\infty$ *as* $k \to \infty$ *such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \psi_\lambda(u) > 0, \quad \forall k \geq k_1, \tag{3.2}$$

where $Z_k = \overline{\bigoplus_{j=k+1}^\infty X_j} = \overline{\operatorname{span}\{e_{k+1}, e_{k+2}, \ldots\}}$ *for all* $k \in N$.

Proof Set

$$l_2(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^2} \quad \text{and} \quad l_\nu(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^\nu}, \quad \forall k \in N. \tag{3.3}$$

Since E is compactly embedded into $L^2(R^3)$ and $L^\nu(R^3)$, we have (see [19, Lemma 3.8])

$$l_2(k) \to 0 \quad \text{and} \quad l_\nu(k) \to 0 \quad \text{as } k \to \infty. \tag{3.4}$$

By (V), (K), (f1), (3.1), (3.3), and the fact that $\int_{R^3} 2K(x)\phi_u G(u)\,dx \geq 0$, we have

$$\begin{aligned}
\psi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \lambda \int_{R^3} F(x,u)\,dx \\
&\geq \frac{1}{2}\|u\|^2 - 2\bigl(\varepsilon\|u\|_{L^2}^2 + c(\varepsilon)\|u\|_{L^\nu}^\nu\bigr) \\
&\geq \frac{1}{2}\|u\|^2 - 2\varepsilon l_2^2(k)\|u\|^2 - 2c(\varepsilon)l_\nu^\nu(k)\|u\|^\nu.
\end{aligned} \tag{3.5}$$

It follows from (3.4) that there exists a positive integer k_1 such that $2\varepsilon l_2^2(k) \leq \frac{1}{8}$, $\forall k \geq k_1$. Then we have

$$\begin{aligned}
\psi_\lambda(u) &\geq \left(\frac{1}{2} - \frac{1}{8}\right)\|u\|^2 - 2c(\varepsilon)l_\nu^\nu(k)\|u\|^\nu \\
&= \frac{3}{8}\|u\|^2 - 2c(\varepsilon)l_\nu^\nu(k)\|u\|^\nu.
\end{aligned} \tag{3.6}$$

For each $k \geq k_1$, choose

$$\rho_k := \bigl(16c(\varepsilon)l_\nu^\nu(k)\bigr)^{\frac{1}{2-\nu}}. \tag{3.7}$$

Then

$$\rho_k \to +\infty \quad \text{as } k \to \infty \tag{3.8}$$

since $4 < \nu < 2^*$. By (3.6) and (3.7), direct computation shows

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \psi_\lambda(u) \geq \frac{\rho_k^2}{4} > 0, \quad \forall k \geq k_1. \tag{3.9}$$

The proof is complete. □

Lemma 3.2 *Under the assumptions of* (V), (K) *and* (f1)-(f3), *then for the positive integer* k_1 *and the sequence* $\{\rho_k\}$ *obtained in Lemma* 3.1, *there exists* $r_k > \rho_k$ *for each* $k \geq k_1$ *such that*

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \psi_\lambda(u) < 0 \quad \textit{for } \lambda \in [1,2], \forall k \geq k_1,$$

where $Y_k = \bigoplus_{j=1}^k X_j = \operatorname{span}\{e_1, \ldots, e_k\}$ *for all* $k \in N$.

Proof It follows from (f3) that, for any $M > 0$, there exists $\delta = \delta(M) > 0$ such that, for all $x \in R^3$, $|u| > \delta$,

$$F(x,u) \geq M|u|^4. \tag{3.10}$$

From (f1) and (f2), there exists $C_8 = C_8(M) > 0$ such that, for all $x \in R^3$ and $0 < |u| \le \delta$,

$$\frac{|f(x,u)u|}{|u|^2} \le C_8.$$

Then, by the mean value theorem, for all $x \in R^3$, $0 < |u| \le \delta$, we obtain

$$\left|F(x,u)\right| \le \frac{C_8}{2}|u|^2. \tag{3.11}$$

Set $\tilde{C} = M\delta + \frac{C_8}{2}$, combining (3.10) with (3.11), we get

$$F(x,u) \ge M|u|^4 - \tilde{C}|u|^2, \quad \forall (x,u) \in R^3 \times R. \tag{3.12}$$

For $u \in Y_k$, by Lemma 2.1 and (3.12), we have

$$\begin{aligned}\psi_\lambda(u) &\le \frac{1}{2}\|u\|^2 + \frac{C_5}{4}\left(\|u\|^4 + \|u\|^{2\alpha}\right) - \int_{R^3} F(x,u)\,dx \\ &\le \frac{1}{2}\|u\|^2 + \frac{C_5}{4}\left(\|u\|^4 + \|u\|^{2\alpha}\right) - M\|u\|_{L^4}^4 + \tilde{C}\|u\|_{L^2}^2 \\ &\le \frac{1}{2}\|u\|^2 + \frac{C_5}{4}\left(\|u\|^4 + \|u\|^{2\alpha}\right) - MC_9\|u\|^4 + C_9\tilde{C}\|u\|^2,\end{aligned} \tag{3.13}$$

where in the last inequality we use the equivalence of all norms on the finite dimensional subspace Y_k. Let us choose M large enough such that $\frac{C_5}{4} - MC_9 < 0$. Then, when M is fixed, $\tilde{C}$ is also fixed. Since $\frac{5}{3} < \alpha < 2$, we can choose $\|u\| = r_k > \rho_k > 0$ such that

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \psi_\lambda(u) < 0 \quad \text{for } \lambda \in [1,2], \forall k \ge k_1.$$

The proof is complete. □

Proof of Theorem 1.1 It follows from Lemma 2.1 and (2.8) that ψ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. By (g2) and (f5), $\psi_\lambda(-u) = \psi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$. Thus, it follows from Lemmas 3.1 and 3.2 that the conditions of Theorem 2.1 are satisfied. Hence, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty \subset E$ such that

$$\begin{aligned}&\sup_m \left\|u_m^k(\lambda)\right\| < \infty, \qquad \psi_\lambda'\left(u_m^k(\lambda)\right) \to 0 \quad \text{and} \\ &\psi_\lambda\left(u_m^k(\lambda)\right) \to \varsigma_k(\lambda) \ge \alpha_k(\lambda) \ge \tilde{\alpha}_k \quad \text{as } m \to \infty,\end{aligned} \tag{3.14}$$

where

$$\varsigma_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \psi_\lambda\left(\gamma(u)\right), \quad \forall \lambda \in [1,2], \qquad \tilde{\alpha}_k = \rho_k^2/4 \to \infty, \quad \text{as } k \to \infty$$

with $B_k = \{u \in Y_k : \|u\| \le r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$.

Furthermore, it follows from the proof of Lemma 3.1 that

$$\varsigma_k(\lambda) \in [\tilde{\alpha}_k, \tilde{\zeta}_k] \quad \text{for all } \lambda \in [1,2], \tag{3.15}$$

where $\tilde{\zeta}_k := \max_{u \in B_k} \psi_1(u)$.

In view of (3.14), we can choose $\lambda_n \to 1$ with $\lambda_n \in [1,2]$ and obtain the corresponding sequences $\{u_m^k(\lambda_n)\}$ (denoted by $\{u_m\}$) satisfying

$$\sup_m \|u_m\| < \infty, \qquad \psi'_{\lambda_n}(u_m) \to 0 \quad \text{as } m \to \infty. \tag{3.16}$$

Claim 1. The sequence $\{u_m\}$ has a strong convergent subsequence.

Set $\Phi(u) = \int_{R^3} F(x,u)\,dx$. It follows from (2.8) and the Sobolev embedding theorem that Φ is well defined and $\Phi' : E \to E^*$ is compact, where $\langle \Phi'(u), v\rangle = \int_{R^3} f(x,u)v\,dx$ for $u, v \in E$. By (3.16), without loss of generality, we may assume

$$u_m \rightharpoonup u_0 \quad \text{as } m \to \infty, \tag{3.17}$$

for $u_0 \in E$. By virtue of the Riesz representation theorem, $\Phi' : E \to E^*$ can be viewed as $\Phi' : E \to E$. We obtain from (2.6) and (3.1) that

$$u_m = \psi'_{\lambda_n}(u_m) + \lambda_n \Phi'(u_m), \quad \forall m \in \mathrm{N}. \tag{3.18}$$

By (3.16), (3.17), and the compactness of $\Phi' : E \to E$, the right-hand side of (3.18) converges strongly in E. Hence $u_m \to u_0$ in E. We may suppose $u_m \to u^k(\lambda_n)$ (denoted by $\{u_n\}$) as $m \to \infty$; then we obtain from (3.15) and (3.16)

$$\psi'_{\lambda_n}(u_n) = 0, \qquad \psi_{\lambda_n}(u_n) \in [\tilde{\alpha}_k, \tilde{\varsigma}_k]. \tag{3.19}$$

Claim 2. The sequence $\{u_n\}$ is bounded.

If not, without loss of generality, we suppose that $\|u_n\| \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$, then, up to a sequence, in view of the compact embedding of E into $L^q(R^3)$, $2 \le q < 2^*$, we have

$$\begin{aligned} & w_n \rightharpoonup w_0 \quad \text{weakly in } E, \\ & w_n \to w_0 \quad \text{strongly in } L^q(R^3), 2 \le q < 2^*, \\ & w_n(x) \to w_0(x) \quad \text{a.e. } x \in R^3. \end{aligned} \tag{3.20}$$

Case 1. $w_0 = 0$ in E. As in [23], we choose $\{t_n\} \subset [0,1]$ such that

$$\psi_{\lambda_n}(t_n u_n) := \max_{t\in[0,1]} \psi_{\lambda_n}(t u_n).$$

For any $D > 0$, we set $\tilde{w}_n = \sqrt{4D} w_n$. By (2.8) and (f3), we have

$$0 \le \int_{R^3} F(x, \tilde{w}_n)\,dx \le \int_{R^3} \left(\varepsilon|\tilde{w}_n|^2 + c(\varepsilon)|\tilde{w}_n|^\nu\right) dx \to 0 \quad \text{as } n \to \infty.$$

Then, choosing n sufficiently large such that

$$\psi_{\lambda_n}(t_n u_n) \ge \psi_{\lambda_n}(\tilde{w}_n) = 2D + \frac{1}{2}\int_{R^3} K(x)\phi_{\tilde{w}_n} G(\tilde{w}_n)\,dx - \lambda_n \int_{R^3} F(x, \tilde{w}_n)\,dx \ge D.$$

Thus, $\lim_{n\to\infty}\psi_{\lambda_n}(t_nu_n)=\infty$. In view of the choice of t_n, we know that $\langle\psi'_{\lambda_n}(t_nu_n),t_nu_n\rangle=0$. Then, by (g3) and (f4), we have

$$\begin{aligned}\infty\leftarrow\psi_{\lambda_n}(t_nu_n)&=\psi_{\lambda_n}(t_nu_n)-\frac{1}{4}\bigl\langle\psi'_{\lambda_n}(t_nu_n),t_nu_n\bigr\rangle\\&=\frac{1}{4}\|t_nu_n\|^2+\frac{1}{2}\int_{R^3}K(x)\phi_{t_nu_n}\biggl[G(t_nu_n)-\frac{1}{4}g(t_nu_n)t_nu_n\biggr]\,dx\\&\quad+\lambda_n\int_{R^3}\biggl[\frac{1}{4}f(x,t_nu_n)t_nu_n-F(x,t_nu_n)\biggr]\,dx\\&\le\frac{1}{4}\|u_n\|^2+\frac{1}{2}\vartheta_1\int_{R^3}K(x)\phi_{u_n}\biggl[G(u_n)-\frac{1}{4}g(u_n)u_n\biggr]\,dx\\&\quad+\lambda_n\vartheta_2\int_{R^3}\biggl[\frac{1}{4}f(x,u_n)u_n-F(x,u_n)\biggr]\,dx\\&\le\frac{1}{4}\|u_n\|^2+\frac{1}{2}\vartheta\int_{R^3}K(x)\phi_{u_n}\biggl[G(u_n)-\frac{1}{4}g(u_n)u_n\biggr]\,dx\\&\quad+\lambda_n\vartheta\int_{R^3}\biggl[\frac{1}{4}f(x,u_n)u_n-F(x,u_n)\biggr]\,dx\\&=\psi_{\lambda_n}(u_n)-\frac{1}{4}\vartheta\bigl\langle\psi'_{\lambda_n}(u_n),u_n\bigr\rangle\in[\tilde{\alpha}_k,\tilde{\zeta}_k],\end{aligned}$$

where $\vartheta=\max\{\vartheta_1,\vartheta_2\}$. This is a contradiction according to (3.19).

Case 2. $w_0\neq0$ in E. On the subspace $\Omega:=\{x\in R^3:w_0(x)\neq0\}$, combining Lemma 2.1, (3.1), (3.20) with (f3), by Fatou's lemma, we have

$$\begin{aligned}&\frac{1}{2\|u_n\|^2}+\frac{\int_{R^3}K(x)G(u_n)\phi_{u_n}\,dx}{2\|u_n\|^4}-\frac{\psi_{\lambda_n}(u_n)}{\|u_n\|^4}\\&\quad=\lambda_n\int_{R^3}\frac{F(x,u_n)}{\|u_n\|^4}\,dx\\&\quad\ge\lambda_n\int_{\Omega}|w_n|^4\frac{F(x,u_n)}{|u_n|^4}\,dx\to+\infty\end{aligned}$$

as $n\to\infty$, a contradiction to (3.19) again. Then $\{u_n\}$ is bounded in E.

In view of Claim 2 and (3.19), using similar arguments to the proof of Claim 1, we can also show that the sequence $\{u_n\}$ has a strong convergent subsequence with the limit u^k being just a critical point of $I=\psi_1$. Obviously, $I(u^{(k)})\in[\tilde{\alpha}_k,\tilde{\zeta}_k]$. Since $\tilde{\alpha}_k\to+\infty$ as $k\to\infty$, we know that $\{u^{(k)}\}_{k=1}^{\infty}$ is an unbound sequence of critical points of functional I. Thus, the proof of Theorem 1.1 is complete. □

Competing interests
The authors declare to have no competing interests.

Authors' contributions
All authors, LX and HC, contributed to each part of this work equally and read and approved the final version of the manuscript.

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References

1. Benci, V, Fortunato, D: An eigenvalue problem for the Schrödinger-Maxwell equations. Topol. Methods Nonlinear Anal. **11**, 283-293 (1998)
2. D'Aprile, T, Mugnai, D: Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations. Proc. R. Soc. Edinb., Sect. A **134**, 893-906 (2004)
3. Ambrosetti, A, Ruiz, D: Multiple bound states for the Schrödinger-Poisson problem. Commun. Contemp. Math. **10**, 391-404 (2008)
4. Ambrosetti, A: On Schrödinger-Poisson systems. Milan J. Math. **76**, 257-274 (2008)
5. Sun, J: Infinitely many solutions for a class of sublinear Schrödinger-Maxwell equations. J. Math. Anal. Appl. **390**, 514-522 (2012)
6. Cerami, G, Vaira, G: Positive solutions for some non-autonomous Schrödinger-Poisson systems. J. Differ. Equ. **248**, 521-543 (2010)
7. Azzollini, A, Pomponio, A: Ground state solutions for the nonlinear Schrödinger-Maxwell equations. J. Math. Anal. Appl. **345**, 90-108 (2008)
8. Sun, J, Chen, H, Nieto, J: On ground state solutions for some non-autonomous Schrödinger-Poisson systems. J. Differ. Equ. **252**, 3365-3380 (2012)
9. Huang, L, Rocha, EM, Chen, J: Positive and sign-changing solutions of a Schrödinger-Poisson system involving a critical nonlinearity. J. Math. Anal. Appl. **408**, 55-69 (2013)
10. Ding, L, Li, L, Zhang, J: Multiple solutions for nonhomogeneous Schrödinger-Poisson systems with asymptotical nonlinearity in R^3. Taiwan. J. Math. **17**, 1627-1650 (2013)
11. Liu, Z, Guo, S: On ground state solutions for the Schrödinger-Poisson equations with critical growth. J. Math. Anal. Appl. (2014). doi:10.1016/j.jmaa.2013.10.066
12. Chen, S, Tang, C: High energy solutions for the Schrödinger-Maxwell equations. Nonlinear Anal. **71**, 4927-4934 (2009)
13. Li, Q, Su, H, Wei, Z: Existence of infinitely many large solutions for the nonlinear Schrödinger-Maxwell equations. Nonlinear Anal. **72**, 4264-4270 (2010)
14. Zou, W: Variant fountain theorems and their applications. Manuscr. Math. **104**, 343-358 (2001)
15. Li, L, Chen, S: Infinitely many large energy solutions of superlinear Schrödinger-Maxwell equations. Electron. J. Differ. Equ. **224**, 1-9 (2012)
16. Azzollini, A, d'Avenia, P, Luisi, V: Generalized Schrödinger-Poisson type system. arXiv:1009.2728v3
17. Li, F, Zhang, Q: Existence of positive solutions to the Schrödinger-Poisson system without compactness conditions. J. Math. Anal. Appl. **401**, 754-762 (2013)
18. Zou, W, Schechter, M: Critical Point Theory and Its Applications. Springer, New York (2006)
19. Willem, M: Minimax Theorems. Birkhäuser, Boston (1996)
20. Evans, LC: Partial Differential Equations. Am. Math. Soc., Providence (1998)
21. Benmlih, K: Stationary solutions for a Schrödinger-Poisson system in R^3. In: Proceedings of the 2002 Fez Conference on Partial Differential Equations. Electron. J. Differ. Equ. Conf., vol. 9, pp. 65-76 (2002)
22. Benci, V, Fortunato, D, Masiello, A, Pisani, L: Solitons and the electromagnetic field. Math. Z. **232**, 73-102 (1999)
23. Jeanjean, L: On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on R^N. Proc. R. Soc. Edinb., Sect. A **129**, 787-809 (1999)

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