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Boundary value problems for fractional differential equations

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Abstract

In this paper we study the existence of solutions of nonlinear fractional differential equations at resonance. By using the coincidence degree theory, some results on the existence of solutions are obtained.

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1 Introduction

In recent years, the fractional differential equations have received more and more attention. The fractional derivative has been occurring in many physical applications such as a non-Markovian diffusion process with memory [1], charge transport in amorphous semiconductors [2], propagations of mechanical waves in viscoelastic media [3], *etc.* Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, and material science are also described by differential equations of fractional order (see [4–9]).

Recently boundary value problems (BVPs for short) for fractional differential equations have been studied in many papers (see [10–33]).

In [10], by means of a fixed point theorem on a cone, Agarwal *et al.* considered two-point boundary value problem at nonresonance given by

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f(t, x(t), D_{0^+}^{\mu}x(t)) = 0, \\ x(0) = x(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $\mu > 0$ are real numbers, $\alpha - \mu \geq 1$ and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative.

Zhao *et al.* [18] studied the following two-point BVP of fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f(t, x(t)), \quad t \in (0,1), \\ x(0) = x'(0) = x'(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional differential operator of order α, $2 < \alpha \leq 3$. By using the lower and upper solution method and fixed point theorem, they obtained some new existence results.

Liang and Zhang [19] studied the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f(t, x(t)), \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = x''(1) = 0, \end{cases}$$

where $3 < \alpha \leq 4$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional differential operator of order α. By means of fixed point theorems, they obtained results on the existence of positive solutions for BVPs of fractional differential equations.

In [20], Bai considered the boundary value problem of the fractional order differential equation

$$\begin{cases} D_{0^+}^{\alpha}x(t) + a(t)f(t, x(t), x'(t)), \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = x''(1) = 0, \end{cases}$$

where $3 < \alpha \leq 4$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional differential operator of order α.

Motivated by the above works, in this paper, we consider the following BVP of fractional equation at resonance

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f(t, x(t), x'(t), x''(t), x'''(t)), \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = 0, \qquad x'''(0) = x'''(1), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ denotes the Caputo fractional differential operator of order α, $3 < \alpha \leq 4$. $f : [0,1] \times \mathbb{R}^4 \to \times\mathbb{R}$ is continuous.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, we establish a theorem on existence of solutions for BVP (1.1) under nonlinear growth restriction of f, basing on the coincidence degree theory due to Mawhin (see [34]). Finally, in Section 4, an example is given to illustrate the main result.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Let X and Y be real Banach spaces and let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, and $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L,$$
$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X, and $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact, where I is identity operator.

Lemma 2.1 ([34]) *If Ω is an open bounded set, let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L)] \cap \partial\Omega \times (0,1)$;
(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$;
(3) $\deg(QN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \operatorname{Ker} Q$.

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function x is given by

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ of a function x with $x^{(n-1)}$ absolutely continuous on $[0,1]$ is given by

$$D_{0^+}^{\alpha} x(t) = I_{0^+}^{n-\alpha} \frac{d^n x(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds,$$

where $n = -[-\alpha]$.

Lemma 2.2 ([35]) *Let* $\alpha > 0$ *and* $n = -[-\alpha]$. *If* $x^{(n-1)} \in AC[0,1]$, *then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

In this paper, we denote $X = C^3[0,1]$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\}$ and $Y = C[0,1]$ with the norm $\|y\|_Y = \|y\|_\infty$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Obviously, both X and Y are Banach spaces.

Define the operator $L : \operatorname{dom} L \subset X \to Y$ by

$$Lx = D_{0^+}^{\alpha} x, \tag{2.1}$$

where

$$\operatorname{dom} L = \left\{x \in X \mid D_{0^+}^{\alpha} x(t) \in Y, x(0) = x'(0) = x''(0) = 0, x'''(0) = x'''(1)\right\}.$$

Let $N : X \to Y$ be the operator

$$Nx(t) = f\bigl(t, x(t), x'(t), x''(t), x'''(t)\bigr), \quad \forall t \in [0,1].$$

Then BVP (1.1) is equivalent to the operator equation

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

3 Main result

In this section, a theorem on existence of solutions for BVP (1.1) will be given.

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous. Assume that*

(H_1) *there exist nonnegative functions $a, b, c, d, e \in C[0,1]$ with $\Gamma(\alpha-2) - 2(b_1 + c_1 + d_1 + e_1) > 0$ such that*

$$\bigl|f(t,u,v,w,x)\bigr| \le a(t) + b(t)|u| + c(t)|v| + d(t)|w| + e(t)|x|,$$
$$\forall t \in [0,1], (u,v,w,x) \in \mathbb{R}^4,$$

where $a_1 = \|a\|_\infty$, $b_1 = \|b\|_\infty$, $c_1 = \|c\|_\infty$, $d_1 = \|d\|_\infty$, $e_1 = \|e\|_\infty$;

(H_2) *there exists a constant $B > 0$ such that for all $x \in \mathbb{R}$ with $|x| > B$ either*

$$xf(t,u,v,w,x) > 0, \quad \forall t \in [0,1], (u,v,w) \in \mathbb{R}^3$$

or

$$xf(t,u,v,w,x) < 0, \quad \forall t \in [0,1], (u,v,w) \in \mathbb{R}^3.$$

Then BVP (1.1) *has at least one solution in X.*

Now, we begin with some lemmas below.

Lemma 3.1 *Let L be defined by* (2.1), *then*

$$\operatorname{Ker} L = \left\{ x \in X \;\middle|\; x(t) = \frac{x'''(0)}{6} t^3, \forall t \in [0,1] \right\}, \tag{3.1}$$

$$\operatorname{Im} L = \left\{ y \in Y \;\middle|\; \int_0^1 (1-s)^{\alpha-4} y(s)\, ds = 0 \right\}. \tag{3.2}$$

Proof By Lemma 2.2, $D_{0+}^{\alpha} x(t) = 0$ has solution

$$x(t) = x(0) + x'(0)t + \frac{x''(0)}{2} t^2 + \frac{x'''(0)}{6} t^3.$$

Combining with the boundary value condition of BVP (1.1), one sees that (3.1) holds.

For $y \in \operatorname{Im} L$, there exists $x \in \operatorname{dom} L$ such that $y = Lx \in Y$. By Lemma 2.2, we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + x(0) + x'(0)t + \frac{x''(0)}{2} t^2 + \frac{x'''(0)}{6} t^3.$$

Then we have

$$x'''(t) = \frac{1}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4} y(s)\, ds + x'''(0).$$

By the conditions of BVP (1.1), we see that y satisfies

$$\int_0^1 (1-s)^{\alpha-4} y(s)\, ds = 0.$$

Thus we get (3.2). On the other hand, suppose $y \in Y$ and satisfies $\int_0^1 (1-s)^{\alpha-4} y(s)\,ds = 0$. Let $x(t) = I_{0^+}^{\alpha} y(t)$, then $x \in \operatorname{dom} L$ and $D_{0^+}^{\alpha} x(t) = y(t)$. So $y \in \operatorname{Im} L$. The proof is complete. □

Lemma 3.2 *Let L be defined by* (2.1), *then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ can be defined as*

$$Px(t) = \frac{x'''(0)}{6} t^3, \quad \forall t \in [0,1],$$
$$Qy(t) = (\alpha - 3) \int_0^1 (1-s)^{\alpha-4} y(s)\,ds, \quad \forall t \in [0,1].$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by

$$K_P y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \quad \forall t \in [0,1].$$

Proof Obviously, $\operatorname{Im} P = \operatorname{Ker} L$ and $P^2 x = Px$. It follows from $x = (x - Px) + Px$ that $X = \operatorname{Ker} P + \operatorname{Ker} L$. By a simple calculation, we get $\operatorname{Ker} L \cap \operatorname{Ker} P = \{0\}$. Then we get

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

For $y \in Y$, we have

$$Q^2 y = Q(Qy) = Qy \cdot (\alpha - 3) \int_0^1 (1-s)^{\alpha-4}\,ds = Qy.$$

Let $y = (y - Qy) + Qy$, where $y - Qy \in \operatorname{Ker} Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2 y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Then we have

$$Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 1.$$

This means that L is a Fredholm operator of index zero.

From the definitions of P, K_P, it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in \operatorname{Im} L$, we have

$$LK_P y = D_{0^+}^{\alpha} I_{0^+}^{\alpha} y = y. \tag{3.3}$$

Moreover, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we get $x(0) = x'(0) = x''(0) = x'''(0) = 0$. By Lemma 2.2, we obtain

$$I_{0^+}^{\alpha} Lx(t) = I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + x(0) + x'(0)t + \frac{x''(0)}{2} t^2 + \frac{x'''(0)}{6} t^3,$$

which together with $x(0) = x'(0) = x''(0) = x'''(0) = 0$ yields

$$K_P L x = x. \tag{3.4}$$

Combining (3.3) with (3.4), we know that $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. The proof is complete. □

Lemma 3.3 *Assume $\Omega \subset X$ is an open bounded subset such that $\operatorname{dom} L \cap \overline{\Omega} \neq \varnothing$, then N is L-compact on $\overline{\Omega}$.*

Proof By the continuity of f, we can see that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. So, in view of the Arzelà-Ascoli theorem, we need only prove that $K_P(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

From the continuity of f, there exists constant $A > 0$ such that $|(I-Q)Nx| \leq A$, $\forall x \in \overline{\Omega}$, $t \in [0,1]$. Furthermore, denote $K_{P,Q} = K_P(I-Q)N$ and for $0 \leq t_1 < t_2 \leq 1$, $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&\bigl|(K_{P,Q}x)(t_2) - (K_{P,Q}x)(t_1)\bigr| \\
&\quad\leq \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nx(s)\,ds - \int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q)Nx(s)\,ds\right| \\
&\quad\leq \frac{A}{\Gamma(\alpha)}\left[\int_0^{t_1}(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\,ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\right] \\
&\quad= \frac{A}{\Gamma(\alpha+1)}\bigl(t_2^{\alpha} - t_1^{\alpha}\bigr), \\
&\bigl|(K_{P,Q}x)'(t_2) - (K_{P,Q}x)'(t_1)\bigr| \leq \frac{A}{\Gamma(\alpha)}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr), \\
&\bigl|(K_{P,Q}x)''(t_2) - (K_{P,Q}x)''(t_1)\bigr| \leq \frac{A}{\Gamma(\alpha-1)}\bigl(t_2^{\alpha-2} - t_1^{\alpha-2}\bigr),
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|(K_{P,Q}x)'''(t_2) - (K_{P,Q}x)'''(t_1)\bigr| \\
&\quad= \frac{1}{\Gamma(\alpha-3)}\left|\int_0^{t_2}(t_2-s)^{\alpha-4}(I-Q)Nx(s)\,ds - \int_0^{t_1}(t_1-s)^{\alpha-4}(I-Q)Nx(s)\,ds\right| \\
&\quad\leq \frac{A}{\Gamma(\alpha-3)}\left[\int_0^{t_1}(t_1-s)^{\alpha-4} - (t_2-s)^{\alpha-4}\,ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-4}\,ds\right] \\
&\quad\leq \frac{A}{\Gamma(\alpha-2)}\bigl[t_1^{\alpha-3} - t_2^{\alpha-3} + 2(t_2-t_1)^{\alpha-3}\bigr].
\end{aligned}$$

Since t^{α}, $t^{\alpha-1}$, $t^{\alpha-2}$, and $t^{\alpha-3}$ are uniformly continuous on $[0,1]$, we see that $K_{P,Q}(\overline{\Omega}) \subset C[0,1]$, $(K_{P,Q})'(\overline{\Omega}) \subset C[0,1]$, $(K_{P,Q})''(\overline{\Omega}) \subset C[0,1]$ and $(K_{P,Q})'''(\overline{\Omega}) \subset C[0,1]$ are equicontinuous. Thus, we find that $K_{P,Q} : \overline{\Omega} \to X$ is compact. The proof is completed. □

Lemma 3.4 *Suppose* (H_1), (H_2) *hold, then the set*

$$\Omega_1 = \bigl\{x \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid Lx = \lambda Nx, \lambda \in (0,1)\bigr\}$$

is bounded.

Proof Take $x \in \Omega_1$, then $Nx \in \operatorname{Im} L$. By (3.2), we have

$$\int_0^1 (1-s)^{\alpha-4} f\big(s, x(s), x'(s), x''(s), x'''(s)\big)\, ds = 0.$$

Then, by the integral mean value theorem, there exists a constant $\xi \in (0,1)$ such that $f(\xi, x(\xi), x'(\xi), x''(\xi), x'''(\xi)) = 0$. Then from ($H_2$), we have $|x'''(\xi)| \le B$.

From $x \in \operatorname{dom} L$, we get $x(0) = 0$, $x'(0) = 0$, and $x''(0) = 0$. Therefore

$$\begin{aligned}
\big|x''(t)\big| &= \left|x''(0) + \int_0^t x'''(s)\, ds\right| \le \big\|x'''\big\|_\infty, \\
\big|x'(t)\big| &= \left|x'(0) + \int_0^t x''(s)\, ds\right| \le \big\|x''\big\|_\infty,
\end{aligned}$$

and

$$\big|x(t)\big| = \left|x(0) + \int_0^t x'(s)\, ds\right| \le \big\|x'\big\|_\infty.$$

That is

$$\|x\|_\infty \le \big\|x'\big\|_\infty \le \big\|x''\big\|_\infty \le \big\|x'''\big\|_\infty. \tag{3.5}$$

By $Lx = \lambda Nx$ and $x \in \operatorname{dom} L$, we have

$$x(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s), x'(s), x''(s), x'''(s)\big)\, ds + \frac{1}{6} x'''(0) t^3.$$

Then we get

$$x'''(t) = \frac{\lambda}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4} f\big(s, x(s), x'(s), x''(s), x'''(s)\big)\, ds + x'''(0).$$

Take $t = \xi$, we get

$$x'''(\xi) = \frac{\lambda}{\Gamma(\alpha-3)} \int_0^\xi (\xi-s)^{\alpha-4} f\big(s, x(s), x'(s), x''(s), x'''(s)\big)\, ds + x'''(0).$$

Together with $|x'''(\xi)| \le B$, (H_1), and (3.5), we have

$$\begin{aligned}
\big|x'''(0)\big| &\le \big|x'''(\xi)\big| + \frac{\lambda}{\Gamma(\alpha-3)} \int_0^\xi (\xi-s)^{\alpha-4} \big|f\big(s, x(s), x'(s), x''(s), x'''(s)\big)\big|\, ds \\
&\le B + \frac{1}{\Gamma(\alpha-3)} \int_0^\xi (\xi-s)^{\alpha-4} \big[a(s) + b(s)\big|x(s)\big| + c(s)\big|x'(s)\big| \\
&\quad + d(s)\big|x''(s)\big| + e(s)\big|x'''(s)\big|\big]\, ds \\
&\le B + \frac{1}{\Gamma(\alpha-3)} \int_0^\xi (\xi-s)^{\alpha-4} \big(a_1 + b_1\|x\|_\infty + c_1\big\|x'\big\|_\infty \\
&\quad + d_1\big\|x''\big\|_\infty + e_1\big\|x'''\big\|_\infty\big)\, ds
\end{aligned}$$

$$\begin{aligned}
&\le B + \frac{1}{\Gamma(\alpha-3)} \int_0^\xi (\xi-s)^{\alpha-4}\bigl[a_1 + (b_1+c_1+d_1+e_1)\bigl\|x'''\bigr\|_\infty\bigr]\,ds \\
&\le B + \frac{1}{\Gamma(\alpha-2)}\bigl[a_1 + (b_1+c_1+d_1+e_1)\bigl\|x'''\bigr\|_\infty\bigr].
\end{aligned}$$

Then we have

$$\begin{aligned}
\bigl\|x'''\bigr\|_\infty &\le \frac{1}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4}\bigl|f\bigl(s,x(s),x'(s),x''(s),x'''(s)\bigr)\bigr|\,ds + \bigl|x'''(0)\bigr| \\
&\le \frac{1}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4}\bigl[a(s) + b(s)\bigl|x(s)\bigr| + c(s)\bigl|x'(s)\bigr| \\
&\quad + d(s)\bigl|x''(s)\bigr| + e(s)\bigl|x'''(s)\bigr|\bigr]\,ds + x'''(0) \\
&\le \frac{1}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4}\bigl(a_1 + b_1\|x\|_\infty + c_1\bigl\|x'\bigr\|_\infty \\
&\quad + d_1\bigl\|x''\bigr\|_\infty + e_1\bigl\|x'''\bigr\|_\infty\bigr)\,ds + \bigl|x'''(0)\bigr| \\
&\le \frac{1}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4}\bigl[a_1 + (b_1+c_1+d_1+e_1)\bigl\|x'''\bigr\|_\infty\bigr]\,ds + \bigl|x'''(0)\bigr| \\
&\le \frac{1}{\Gamma(\alpha-2)}\bigl[a_1 + (b_1+c_1+d_1+e_1)\bigl\|x'''\bigr\|_\infty\bigr] + \bigl|x'''(0)\bigr| \\
&\le B + \frac{2}{\Gamma(\alpha-2)}\bigl[a_1 + (b_1+c_1+d_1+e_1)\bigl\|x'''\bigr\|_\infty\bigr].
\end{aligned}$$

Thus, from $\Gamma(\alpha-2) - 2(b_1+c_1+d_1+e_1) > 0$, we obtain

$$\bigl\|x'''\bigr\|_\infty \le \frac{2a_1 + \Gamma(\alpha-2)B}{\Gamma(\alpha-2) - 2(b_1+c_1+d_1+e_1)} := M_1.$$

Thus, together with (3.5), we get

$$\|x\|_\infty \le \bigl\|x'\bigr\|_\infty \le \bigl\|x''\bigr\|_\infty \le \bigl\|x'''\bigr\|_\infty \le M_1.$$

Therefore,

$$\|x\|_X \le M_1.$$

So Ω_1 is bounded. The proof is complete. □

Lemma 3.5 *Suppose* (H_2) *holds, then the set*

$$\Omega_2 = \{x \mid x \in \operatorname{Ker} L, Nx \in \operatorname{Im} L\}$$

is bounded.

Proof For $x \in \Omega_2$, we have $x(t) = \frac{x'''(0)}{6}t^3$ and $Nx \in \operatorname{Im} L$. Then we get

$$\int_0^1 (1-s)^{\alpha-4} f\left(s, \frac{x'''(0)}{6}s^3, \frac{x'''(0)}{2}s^2, x'''(0)s, x'''(0)\right) ds = 0,$$

which together with (H_2) implies $|x'''(0)| \leq B$. Thus, we have

$$\|x\|_X \leq B.$$

Hence, Ω_2 is bounded. The proof is complete. $\square$

Lemma 3.6 *Suppose the first part of* (H_2) *holds, then the set*

$$\Omega_3 = \{x \mid x \in \operatorname{Ker} L, \lambda x + (1-\lambda)QNx = 0, \lambda \in [0,1]\}$$

is bounded.

Proof For $x \in \Omega_3$, we have $x(t) = \frac{x'''(0)}{6}t^3$ and

$$\begin{aligned}
&\lambda \frac{x'''(0)}{6}t^3 + (1-\lambda)(\alpha-3) \\
&\quad \times \int_0^1 (1-s)^{\alpha-4} f\left(s, \frac{x'''(0)}{6}s^3, \frac{x'''(0)}{2}s^2, x'''(0)s, x'''(0)\right) ds = 0.
\end{aligned} \tag{3.6}$$

If $\lambda = 0$, then $|x'''(0)| \leq B$ because of the first part of (H_2). If $\lambda \in (0,1]$, we can also obtain $|x'''(0)| \leq B$. Otherwise, if $|x'''(0)| > B$, in view of the first part of (H_2), one has

$$\begin{aligned}
&\lambda \frac{[x'''(0)]^2}{6}t^3 + (1-\lambda)(\alpha-3) \\
&\quad \times \int_0^1 (1-s)^{\alpha-4} x'''(0) f\left(s, \frac{x'''(0)}{6}s^3, \frac{x'''(0)}{2}s^2, x'''(0)s, x'''(0)\right) ds > 0,
\end{aligned}$$

which contradicts (3.6).

Therefore, Ω_3 is bounded. The proof is complete. $\square$

Remark 3.1 Suppose the second part of (H_2) hold, then the set

$$\Omega_3' = \{x \mid x \in \operatorname{Ker} L, -\lambda x + (1-\lambda)QNx = 0, \lambda \in [0,1]\}$$

is bounded.

Proof of Theorem 3.1 Set $\Omega = \{x \in X \mid \|x\|_X < \max\{M_1, B\} + 1\}$. It follows from Lemmas 3.2 and 3.3 that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By Lemmas 3.4 and 3.5, we see that the following two conditions are satisfied:

(1) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.

Take

$$H(x, \lambda) = \pm\lambda x + (1-\lambda)QNx.$$

According to Lemma 3.6 (or Remark 3.1), we know that $H(x,\lambda) \neq 0$ for $x \in \operatorname{Ker} L \cap \partial\Omega$. Therefore

$$\begin{aligned}\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg(\pm I, \Omega \cap \operatorname{Ker} L, 0) \neq 0.\end{aligned}$$

So the condition (3) of Lemma 2.1 is satisfied. By Lemma 2.1, we find that $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. Therefore, BVP (1.1) has at least one solution. The proof is complete. □

4 An example

Example 4.1 Consider the following BVP:

$$\begin{cases} D_{0^+}^{\frac{7}{2}} x(t) = \frac{1}{16}(x''' - 10) + \frac{t^2}{16} e^{-|x'|-|x''|} + \frac{t^3}{16}\sin(x^2), \quad t \in [0,1], \\ x(0) = x'(0) = x''(0) = 0, \qquad x'''(0) = x'''(1). \end{cases} \tag{4.1}$$

Here

$$f(t,u,v,w,x) = \frac{1}{16}(x - 10) + \frac{t^2}{16} e^{-|v|-|w|} + \frac{t^3}{16}\sin\bigl(u^2\bigr).$$

Choose $a(t) = \frac{3}{4}$, $b(t) = 0$, $c(t) = 0$, $d(t) = 0$, $e(t) = \frac{1}{16}$, $B = 10$. We get $b_1 = 0$, $c_1 = 0$, $d_1 = 0$, $e_1 = \frac{1}{16}$, and

$$\Gamma\left(\frac{7}{2} - 2\right) - 2(b_1 + c_1 + d_1 + e_1) > 0.$$

Then all conditions of Theorem 3.1 hold, so BVP (4.1) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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