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The Cauchy problem for the modified Novikov equation

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Abstract

In this paper, we are concerned with the Cauchy problem for the modified Novikov equation. By using the transport equation theory and Littlewood-Paley decomposition as well as nonhomogeneous Besov spaces, we prove that the Cauchy problem for the modified Novikov equation is locally well posed in the Besov space $B^s_{p,r}$ with $1 \leq p, r \leq +\infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and show that the Cauchy problem for the modified Novikov equation is locally well posed in the Besov space $B^{3/2}_{2,1}$ with the aid of Osgood lemma.

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1 Introduction

Recently, Zhao and Zhou [1] considered the exact traveling wave solution to the following modified Novikov equation:

$$u_t - u_{txx} + 4u^4 u_x = 3uu_x u_{xx} + u^2 u_{xxx}. \tag{1.1}$$

We recall that the Novikov equation

$$u_t - u_{txx} = u^2 u_{xxx} + 3uu_x u_{xx} - 4u^2 u_x \tag{1.2}$$

was discovered by Vladimir Novikov [2] and it possesses the bi-Hamiltonian structure, infinite conservation laws. The well-posedness and blow-up of the Cauchy problem for the Novikov equation in Sobolev spaces and Besov spaces have been investigated by some authors [3–7]. The weak solution of the Cauchy problem for the Novikov equation has been investigated by some authors [4, 5, 8]. Recently, Li and Yan [9] considered the Cauchy problem for the KdV equation with higher dispersion.

We define $P_1(D) = \partial_x(1 - \partial_x^2)^{-1}$ and $P_2(D) = (1 - \partial_x^2)^{-1}$. By using the fact that $G(x) = \frac{1}{2}e^{-|x|}$ and $G(x) * f = (1 - \partial_x^2)^{-1} f$ for all $f \in L^2(\mathbf{R})$ and $G * y = u$, we can rewrite (1.1) as follows:

$$u_t + u^2 u_x + P_1(D)\left[\frac{4}{5}u^5 - \frac{1}{3}u^3 + \frac{3}{2}uu_x^2\right] + P_2(D)\left[\frac{1}{2}u_x^3\right] = 0, \quad t > 0.$$

Now we consider the following problem:

$$u_t + u^2u_x + P_1(D)\left[\frac{4}{5}u^5 - \frac{1}{3}u^3 + \frac{3}{2}uu_x^2\right] + P_2(D)\left[\frac{1}{2}u_x^3\right] = 0, \quad t > 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}. \tag{1.4}$$

To the best of our knowledge, the well-posedness and blow-up of the Cauchy problem for (1.3) and (1.4) in Besov spaces are open up to now. More precisely, in this paper, motivated by [10, 11], using Littlewood-Paley decomposition and nonhomogeneous Besov spaces, we prove that the Cauchy problem for (1.4) is locally well posed in the Besov space $B^s_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and we give a blow-up criterion.

To introduce the main results, we define

$$E^s_{p,r}(T) = C\big([0,T]; B^s_{p,r}\big) \cap C^1\big([0,T]; B^{s-1}_{p,r}\big) \quad \text{if } r < \infty,$$
$$E^s_{p,\infty}(T) = L^\infty\big(0,T; B^s_{p,\infty}\big) \cap \operatorname{Lip}\big([0,T]; B^{s-1}_{p,r}\big).$$

The main results of this paper are as follows.

Theorem 1.1 *Let $1 \le p, r \le \infty$ and $s > \max(\frac{3}{2}, 1 + \frac{1}{p})$ and $u_0 \in B^s_{p,r}$. Then there exists a time $T > 0$ such that problem* (1.3) *and* (1.4) *has a unique solution u in $E^s_{p,r}(T)$. The map $u_0 \longrightarrow u$ is continuous from a neighborhood of u_0 in $B^s_{p,r}$ into $C([0,T]; B^{s'}_{p,r}) \cap C^1([0,T]; B^{s'-1}_{p,r})$ for every $s' < s$. When $r < \infty$, the solution to problem* (1.3) *and* (1.4) *is continuous in $E^s_{p,r}(T)$.*

Theorem 1.2 *When $u_0 \in B^{3/2}_{2,1}$,* (1.3) *and* (1.4) *is locally well posed in the sense of Hadamard.*

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish local well-posedness of the Cauchy problem for the generalized Camassa-Holm equation in Besov spaces. In Section 4, we prove Theorem 1.2.

2 Preliminaries

In this section, the nonhomogeneous Besov spaces and the theory of transport equation which can be seen in [10–13] are presented.

Lemma 2.1 (Littlewood-Paley decomposition) *There exists a couple of smooth radial functions (χ, ϕ) valued in $[0,1]$ such that χ is supported in the ball $B = \{\xi \in \mathbf{R}^n, |\xi| \le \frac{4}{3}\}$ and ϕ is supported in the ring $C = \{\xi \in \mathbf{R}^n, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Moreover,*

$$\forall \xi \in \mathbf{R}^n, \quad \chi(\xi) + \sum_{q \in \mathbf{N}} \phi\big(2^{-q}\xi\big) = 1$$

and

$$\operatorname{Supp}\phi\big(2^{-q}\cdot\big) \cap \operatorname{Supp}\phi\big(2^{-q'}\cdot\big) = \emptyset \quad \text{if } \big|q - q'\big| \ge 2,$$
$$\operatorname{Supp}\chi(\cdot) \cap \operatorname{Supp}\phi\big(2^{-q}\cdot\big) = \emptyset \quad \text{if } |q| \ge 1.$$

Then, for $u \in \mathcal{S}'(\mathbf{R})$, the nonhomogeneous dyadic blocks are defined as follows:

$$\Delta_q u = 0 \quad \text{if } q \leq -2,$$
$$\Delta_{-1} u = \chi(D)u = \mathcal{F}_x^{-1}\chi\mathcal{F}_x u,$$
$$\Delta_q u = \phi\left(2^{-q}D\right) = \mathcal{F}_x^{-1}\phi\left(2^{-q}\xi\right)\mathcal{F}_x u \quad \text{if } q \geq 0.$$

Thus $u = \sum_{q\in\mathbf{Z}} \Delta_q u$ *in* $\mathcal{S}'(\mathbf{R})$.

Remark The low frequency cut-off S_q is defined by

$$S_q u = \sum_{p=-1}^{q-1} \Delta u = \chi\left(2^{-q}D\right)u = \mathcal{F}_x^{-1}\chi\left(2^{-q}\xi\right)\mathcal{F}_x u, \quad \forall q \in N.$$

It is easily checked that

$$\Delta_p \Delta_q u \equiv 0 \quad \text{if } |p-q| \geq 2,$$
$$\Delta_q(S_{p-1}u\Delta_p v) \equiv 0 \quad \text{if } |p-q| \geq 5, \forall u, v \in \mathcal{S}'(\mathbf{R})$$

as well as

$$\|\Delta_q u\|_{L^p} \leq \|u\|_{L^p}, \qquad \|S_q u\|_{L^p} \leq C\|u\|_{L^p}, \quad \forall 1 \leq p \leq +\infty$$

with the aid of Young's inequality, where C is a positive constant independent of q.

Definition (Besov spaces) Let $s \in \mathbf{R}$, $1 \leq p \leq +\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbf{R}^n)$ is defined by

$$B^s_{p,r}\left(\mathbf{R}^n\right) = \left\{f \in \mathcal{S}'(\mathbf{R}) : \|f\|_{B^s_{p,r}} = \left\|2^{qs}\Delta_q f\right\|_{l^r(L^p)} = \left\|\left(2^{qs}\|\Delta_q f\|_{L^p}\right)_{q\geq -1}\right\|_{l^r} < \infty\right\}.$$

In particular, $B^{\infty}_{p,r} = \bigcap_{s\in\mathbf{R}} B^s_{p,r}$. Let $T > 0$, $s \in \mathbf{R}$ and $1 \leq p \leq \infty$. Define $E^s_{p,r} = \bigcap_{T>0} E^s_{p,r}(T)$.

Lemma 2.2 *Let $s \in \mathbf{R}$, $1 \leq p, r, p_j, r_j \leq \infty$, $j = 1, 2$, then*:

(1) *Topological properties*: $B^s_{p,r}$ *is a Banach space which is continuously embedded in* $\mathcal{S}'(\mathbf{R})$.

(2) *Density*: C_c^{∞} *is dense in* $B^s_{p,r} \Leftrightarrow 1 \leq p, r < \infty$.

(3) *Embedding*: $B^s_{p_1,r_1} \hookrightarrow B^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$ *if* $p_1 \leq p_2$ *and* $r_1 \leq r_2$.

$$B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1} \text{ locally compact} \quad \text{if } s_1 < s_2.$$

(4) *Algebraic properties*: $\forall s > 0$, $B^s_{p,r} \cap L^{\infty}$ *is a Banach algebra*. $B^s_{p,r}$ *is a Banach algebra* $\Leftrightarrow B^s_{p,r} \hookrightarrow L^{\infty} \Leftrightarrow s > \frac{1}{p}$ *or* ($s \geq \frac{1}{p}$ *and* $r = 1$). *In particular*, $B^{1/2}_{2,1}$ *is continuously embedded in* $B^{1/2}_{2,\infty} \cap L^{\infty}$ *and* $B^{1/2}_{2,\infty} \cap L^{\infty}$ *is a Banach algebra*.

(5) 1-*D Moser-type estimates*:

 (i) *For* $s > 0$,

$$\|fg\|_{B^s_{p,r}} \leq C\left(\|f\|_{B^s_{p,r}}\|g\|_{L^{\infty}} + \|f\|_{L^{\infty}}\|g\|_{B^s_{p,r}}\right).$$

(ii) $\forall s_1 \leq \frac{1}{p} < s_2$ ($s_2 \geq \frac{1}{p}$ *if* $r = 1$) *and* $s_1 + s_2 > 0$, *we have*

$$\|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

(6) *Complex interpolation*:

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|f\|^{\theta}_{B^{s_1}_{p,r}}\|g\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}, \forall \theta \in [0,1].$$

(7) *Real interpolation*: $\forall \theta \in (0,1)$, $s_1 > s_2$, $s = \theta s_1 + (1-\theta)s_2$, *there exists a constant* C *such that*

$$\|u\|_{B^s_{p,1}} \leq \frac{C(\theta)}{s_1 - s_2}\|u\|^{\theta}_{B^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}, \quad \forall u \in B^{s_1}_{p,\infty}.$$

In particular, for any $0 < \theta < 1$, *we have that*

$$\|u\|_{B^{1/2}_{2,1}} \leq \|u\|_{B^{\frac{3}{2}-\theta}_{2,1}} \leq C(\theta)\|u\|^{\theta}_{B^{1/2}_{2,\infty}}\|u\|^{1-\theta}_{B^{3/2}_{2,\infty}}. \tag{2.1}$$

(8) *Fatou lemma*: *if* $(u_n)_{n\in\mathbf{N}}$ *is bounded in* $B^s_{p,r}$ *and* $u_n \longrightarrow u$ *in* $\mathcal{S}'(\mathbf{R})$, *then* $u \in B^s_{p,r}$ *and*

$$\|u\|_{B^s_{p,r}} \leq \liminf_{n\longrightarrow\infty}\|u_n\|_{B^s_{p,r}}.$$

(9) *Let* $m \in \mathbf{R}$ *and* f *be an* S^m*-multiplier* (*i.e.*, $f : \mathbf{R}^n \to \mathbf{R}$ *is smooth and satisfies that* $\forall \alpha \in N^n$, $\ni$ *a constant* C_α, *s.t.* $|\partial_\alpha f(\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}$ *for all* $\xi \in \mathbf{R}^n$). *Then the operator* $f(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-m}_{p,r}$. *Notice that* $P_1(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-1}_{p,r}$ *and* $P_2(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-2}_{p,r}$.

(10) *The usual product is continuous from* $B^{-1/2}_{2,1} \times (B^{1/2}_{2,\infty} \cap L^\infty)$ *to* $B^{-1/2}_{2,\infty}$.

(11) *There exists a constant* $C > 0$ *such that the following interpolation inequality holds*:

$$\|f\|_{B^{1/2}_{2,1}} \leq C\|f\|_{B^{1/2}_{2,\infty}} \ln\left(e + \frac{\|f\|_{B^{3/2}_{2,\infty}}}{\|f\|_{B^{1/2}_{2,\infty}}}\right).$$

Lemma 2.3 (*A priori* estimates in Besov spaces) *Let* $1 \leq p, r \leq \infty$ *and* $s > -\min(\frac{1}{p}, 1 - \frac{1}{p})$. *Assume that* $f_0 \in B^s_{p,r}$, $F \in L^1(0,T;B^s_{p,r})$ *and* $\partial_x v$ *belongs to* $L^1(0,T;B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or to* $L^1(0,T;B^{1/p}_{p,r} \cap L^\infty)$ *otherwise. If* $f \in L^\infty(0,T;B^s_{p,r}) \cap C([0,T];\mathcal{S}'(\mathbf{R}))$ *solves the following* 1*-D linear transport equation*:

$$f_t + vf_x = F, \tag{2.2}$$

$$f(x,0) = f_0, \tag{2.3}$$

then there exists a constant C *depending only on* s, p, r *such that the following statements hold*:

(1) *If* $r = 1$ *or* $s \neq 1 + \frac{1}{p}$, *then*

$$\|f\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \big\|F(\tau)\big\|_{B^s_{p,r}}\,d\tau + C\int_0^t V'(\tau)\big\|f(\tau)\big\|_{B^s_{p,r}}\,d\tau$$

or hence,

$$\|f\|_{B^s_{p,r}} \le e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\big\|F(\tau)\big\|_{B^s_{p,r}}\,d\tau\right) \tag{2.4}$$

with $V(t) = \int_0^t \|v_x(\tau)\|_{B^{1/p}_{p,r}\cap L^\infty}\,d\tau$ *if* $s < 1 + \frac{1}{p}$ *and* $V(t) = \int_0^t \|v_x(\tau)\|_{B^{s-1}_{p,r}}\,d\tau$ *else.*

(2) *If* $s \le 1 + \frac{1}{p}$, $f_0' \in L^\infty$ *and* $f_x \in L^\infty((0,T)\times \mathbf{R})$ *and* $F_x \in L^1(0,T;L^\infty)$, *then*

$$\begin{aligned}&\big\|f(t)\big\|_{B^s_{p,r}} + \big\|f_x(t)\big\|_{L^\infty}\\ &\quad\le e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \|f_{0x}\|_{L^\infty} + \int_0^t e^{-CV(\tau)}\big[\big\|F(\tau)\big\|_{B^s_{p,r}} + \big\|F_x(\tau)\big\|_{L^\infty}\big]\,d\tau\right)\end{aligned}$$

with

$$V(t) = \int_0^t \big\|\partial_x v(\tau)\big\|_{B^{1/p}_{p,r}\cap L^\infty}.$$

(3) *If* $f = v$, *then for all* $s > 0$, (1) *holds true when* $V(t) = \int_0^t \|v_x(\tau)\|_{L^\infty}\,d\tau$.

(4) *If* $r < \infty$, *then* $f \in C([0,T];B^s_{p,r})$. *If* $r = \infty$, *then* $f \in C([0,T];B^{s'}_{p,1})$ *for all* $s' < s$.

Lemma 2.4 (Existence and uniqueness) *Let* p, r, s, f_0 *and* F *be as in the statement of Lemma* 2.3. *Assume that* $v \in L^\rho(0,T;B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$ *and* $M > 0$ *and* $v_x \in L^1(0,T;B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or* $s = 1 + \frac{1}{p}$ *and* $r = 1$ *and* $v_x \in L^1(0,T;B^{1/p}_{p,\infty}\cap L^\infty)$ *if* $s < 1 + \frac{1}{p}$. *Then problem* (2.2) *and* (2.3) *has a unique solution* $f \in L^\infty(0,T;B^s_{p,r}) \cap (\bigcap_{s'<s} C([0,T];B^{s'}_{p,1}))$ *and the inequalities of Lemma* 2.3 *can hold true. Moreover, if* $r < \infty$, *then* $f \in C([0,T];B^s_{p,r})$.

3 Proof of Theorem 1.1

By using the following six steps, we will complete the proof of Theorem 1.1.

First step: Approximate solution. We will construct a solution with the aid of a standard iterative process. Starting from $u^{(0)} := 0$, by the inductive method and solving the following linear transport equation (3.1) and (3.2), we derive a sequence of smooth functions $(u^{(n)})_{n\in\mathbf{N}}$

$$\begin{aligned}\big[\partial_t + \big(u^{(n)}\big)^2\partial_x\big]u^{(n+1)} &= -P_1(D)\left[\frac{4}{5}\big(u^{(n)}\big)^5 - \frac{1}{3}\big(u^{(n)}\big)^3 + \frac{3}{2}u^{(n)}\big(u^{(n)}_x\big)^2\right]\\ &\quad - P_2(D)\left[\frac{1}{2}\big(u^{(n)}_x\big)^3\right],\end{aligned} \tag{3.1}$$

$$u^{(n+1)}(x,0) = u^{(n+1)}_0 = S_{n+1}u_0. \tag{3.2}$$

It is easily checked that $S_{n+1}u_0 \in B^\infty_{p,r}$, by using Lemma 2.4 and the inductive method, for all $n \in N$, we have that (3.1) and (3.2) has a global solution which belongs to $C(\mathbf{R}^+, B^\infty_{p,r})$.

Second step: Uniform bounds. We will prove

$$\begin{aligned}\big\|u^{(n+1)}(t)\big\|_{B^s_{p,r}} &\le e^{C\int_0^t \|u^{(n)}\|^2_{B^s_{p,r}}\,d\tau}\|u_0\|_{B^s_{p,r}} + \frac{C}{2}\int_0^t e^{C\int_\tau^t \|u^{(n)}\|^2_{B^s_{p,r}}\,d\tau}\big\|u^{(n)}\big\|^3_{B^s_{p,r}}\,d\tau\\ &\quad + \frac{C}{2}\int_0^t e^{C\int_\tau^t \|u^{(n)}\|^2_{B^s_{p,r}}\,d\tau}\big\|u^{(n)}\big\|^5_{B^s_{p,r}}\,d\tau\end{aligned} \tag{3.3}$$

for all $n \in \mathbf{N}$.

Combining (2.4) of Lemma 2.3 with (3.1), we have

$$\begin{aligned}\|u^{(n+1)}(t)\|_{B^s_{p,r}} &\le e^{C\int_0^t \|((u^{(n)})^2)(t')\|_{B^s_{p,r}}\,dt'}\|u_0\|_{B^s_{p,r}}\\ &\quad+\int_0^t e^{C\int_\tau^t \|((u^{(n)})^2)(t')\|_{B^s_{p,r}}\,dt'}\|F(u^{(n)},u^{(n)}_x)\|_{B^s_{p,r}}\,d\tau,\end{aligned}\tag{3.4}$$

where

$$F(u^n,u^n_x)=P_1(D)\left[\frac{4}{5}(u^{(n)})^5-\frac{1}{3}(u^{(n)})^3+\frac{3}{2}u^{(n)}(u^{(n)}_x)^2\right]+P_2(D)\left[\frac{1}{2}(u^{(n)}_x)^3\right].\tag{3.5}$$

When $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$, by using (4) in Lemma 2.2, we have

$$\|((u^{(n)})^2)(t')\|_{B^s_{p,r}}\le C\|(u^{(n)})(t')\|^2_{B^s_{p,r}},\tag{3.6}$$

$$\|F(u^{(n)},u^{(n)}_x)\|_{B^s_{p,r}}\le C\left[\|u^{(n)}\|^3_{B^s_{p,r}}+\|u^{(n)}\|^5_{B^s_{p,r}}\right].\tag{3.7}$$

Combining (3.6)-(3.7) with (3.4), we have (3.3).
Let $T>0$ satisfy

$$T<\min\left\{\frac{1}{16C\|u_0\|^2_{B^s_{p,r}}},\frac{1}{C}\right\},\tag{3.8}$$

$$\|u^{(n)}(t)\|_{B^s_{p,r}}\le\frac{2\|u_0\|_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{1/2}}.\tag{3.9}$$

By using (3.9), we have

$$e^{C\int_\tau^t \|u^{(n)}\|^2_{B^s_{p,r}}(t')\,dt'}\le e^{-\frac{1}{4}\int_\tau^t \frac{d(1-16C\|u_0\|^2_{B^s_{p,r}}t')}{1-16C\|u_0\|^2_{B^s_{p,r}}t'}}=\left(\frac{1-16C\|u_0\|^2_{B^s_{p,r}}\tau}{1-16C\|u_0\|^2_{B^s_{p,r}}t}\right)^{\frac{1}{4}}.\tag{3.10}$$

When $\tau=0$ in (3.10), we have

$$e^{CU^n(t)}\le\left(\frac{1}{1-16C\|u_0\|^2_{B^s_{p,r}}t}\right)^{1/4}.\tag{3.11}$$

By using (3.10) and (3.9), we have

$$\begin{aligned}&\frac{C}{2}\int_0^t e^{C\int_\tau^t \|u^{(n)}(t')\|^2_{B^s_{p,r}}\,dt'}\|u^{(n)}(\tau)\|^5_{B^s_{p,r}}\,d\tau\\ &\quad\le(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{-\frac{1}{4}}\int_0^t\frac{16C\|u_0\|^5_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}\tau)^{\frac{9}{4}}}\,d\tau\\ &\quad=\frac{4}{5}\|u_0\|^3_{B^s_{p,r}}(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{-\frac{1}{4}}\left[(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{-\frac{5}{4}}-1\right].\end{aligned}\tag{3.12}$$

With the aid of the mean value theorem, we have

$$\left[(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{-\frac{5}{4}}-1\right]=20C\|u_0\|^2_{B^s_{p,r}}t(1-\xi)^{-9/4},\tag{3.13}$$

where

$$16C\|u_0\|^2_{B^s_{p,r}}t < \xi < 1.$$

Combining (3.13) with (3.12), we have that

$$\frac{C}{2}\int_0^t e^{C\int_\tau^t \|u^{(n)}(t')\|^2_{B^s_{p,r}}\,dt'}\big\|u^{(n)}(\tau)\big\|^5_{B^s_{p,r}}\,d\tau \le \frac{16\|u_0\|^5_{B^s_{p,r}}Ct}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{5/2}}. \tag{3.14}$$

Inserting (3.10)-(3.14) into (3.3) leads to

$$\big\|u^{(n+1)}(t)\big\|_{B^s_{p,r}} \le \frac{2\|u_0\|_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{1/2}}. \tag{3.15}$$

Consequently, $(u^{(n)})_{n\in N}$ is uniformly bounded in $C([0,T];B^s_{p,r})$. By using the fact that $B^{s-1}_{p,r}$ with $s > \max\{1+\frac{1}{p},\frac{3}{2}\}$ is an algebra and $B^s_{p,r}\hookrightarrow B^{s-1}_{p,r}$ as well as the definition of the Besov spaces $B^s_{p,r}$, we derive that

$$\begin{aligned}\big\|\big(u^{(n)}\big)^2u_x^{(n+1)}\big\|_{B^{s-1}_{p,r}} &\le C\big\|\big(u^{(n)}\big)^2\big\|_{B^{s-1}_{p,r}}\big\|u_x^{(n+1)}\big\|_{B^{s-1}_{p,r}}\\ &\le C\big\|u^{(n)}\big\|^2_{B^s_{p,r}}\big\|u^{(n+1)}\big\|_{B^s_{p,r}}\\ &\le \frac{8C\|u_0\|^3_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{3}{2}}}.\end{aligned} \tag{3.16}$$

Since $s > \max\{1+\frac{1}{p},\frac{3}{2}\}$, which leads to that $B^{s-1}_{p,r}$ is an algebra, by using the S^{-1}-multiplier property of $P_1(D)$ and the S^{-2}-multiplier property of $P_2(D)$ as well as (3.8), we have

$$\begin{aligned}&\big\|F\big(u^{(n)},u_x^{(n)}\big)\big\|_{B^{s-1}_{p,r}} \le \frac{C}{2}\big[\big\|u^{(n)}\big\|^3_{B^s_{p,r}}+\big\|u^{(n)}\big\|^5_{B^s_{p,r}}\big]\\ &\quad\le 4C\biggl[\frac{\|u_0\|^3_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{3}{2}}}+\frac{4\|u_0\|^5_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{5}{2}}}\biggr].\end{aligned} \tag{3.17}$$

Consequently, combining (3.1) with (3.16) and (3.17), we derive that

$$\big\|u_t^{(n+1)}\big\|_{B^{s-1}_{p,r}} \le 4C\biggl[\frac{3\|u_0\|^3_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{3}{2}}}+\frac{4\|u_0\|^5_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{5}{2}}}\biggr], \tag{3.18}$$

which yields $(u^{(n)})_n \in C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$.

Third step: Convergence. We will derive that $(u^{(n)})_n$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$.

For $m,n\in N$, from (3.1), we have

$$\big(u^{(n+m+1)}-u^{(n+1)}\big)_t+\big(u^{(n+m)}\big)^2\big(u^{(n+m+1)}-u^{(n+1)}\big)_x = \sum_{k=1}^5 T_k, \tag{3.19}$$

where

$$
\begin{aligned}
T_1 &= -\frac{4}{5}P_1(D)\big[\big(u^{(n+m)}\big)^5 - \big(u^{(n)}\big)^5\big],\\
T_2 &= \frac{1}{3}P_1(D)\big[\big(u^{(n+m)}\big)^3 - \big(u^{(n)}\big)^3\big],\\
T_3 &= -\frac{3}{2}P_1(D)\big[\big(u^{n+m} - u^{n}\big)\big(u_x^{(n+m)}\big)^2\big],\\
T_4 &= -\frac{3}{2}P_1(D)\big[\big(u_x^{(n+m)}\big)^2 - \big(u_x^{(n)}\big)^2\big],\\
T_5 &= -\frac{1}{2}P_2(D)\big[\big(u^{(n+m)}\big)_x^3 - \big(u^{(n)}\big)_x^3\big],\\
T_6 &= \big[\big(u^{(n+m)}\big)^2 - \big(u^{(n)}\big)^2\big]u_x^{(n+1)}.
\end{aligned}
$$

When $s > \max\{1+\frac{1}{p}, \frac{3}{2}\}$, by using the S^{-1} multiplier property of $P_1(D)$, the S^{-2} multiplier property of $P_2(D)$ and $B^{s-1}_{p,r} \hookrightarrow B^{s-2}_{p,r}$, we have

$$
\|T_j\|_{B^{s-1}_{p,r}} \le C\big\|u^{(n+m)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}, \tag{3.20}
$$

where $1 \le j \le 6$, $j \in \mathbf{N}$. Since $\forall n \in N$, we have

$$
\big\|u^{(n)}\big\|_{B^s_{p,r}} \le \frac{2\|u_0\|_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{1}{2}}}.
$$

By using (3.20), we have

$$
\begin{aligned}
\left\|\sum_{j=1}^{6} T_j\right\|_{B^{s-1}_{p,r}} &\le \sum_{j=1}^{6}\|T_j\|_{B^{s-1}_{p,r}}\\
&\le C\big\|u^{(n+m)} - u^{(n)}\big\|_{B^{s-1}_{p,r}}\frac{4\|u_0\|^2_{B^s_{p,r}}}{1-16C\|u_0\|^2_{B^s_{p,r}}t}.
\end{aligned} \tag{3.21}
$$

When $s \neq 2+\frac{1}{p}$, from Lemma 2.4 and (3.20), we have

$$
\begin{aligned}
&\big\|(u^{(n+m+1)} - u^{(n+1)}(t)\big\|_{B^{s-1}_{p,r}}\\
&\quad\le e^{CW^{(n+m)}(t)}\big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(\cdot,0)\big\|_{B^{s-1}_{p,r}}\\
&\qquad + C\int_0^t e^{CW^{(n+m)}(t)-CW^{(n+m)}(\tau)}\sum_{j=1}^{6}\|T_j\|_{B^{s-1}_{p,r}}\,d\tau,
\end{aligned} \tag{3.22}
$$

where

$$
W^{n+m}(t) = \int_0^t \big\|\partial_x\big(u^{(n+m)}\big)^2(\tau)\big\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty}\,d\tau \tag{3.23}
$$

if $s-1<1+\frac{1}{p}$ and

$$W^{n+m}(t)=\int_0^t \big\|\partial_x\big(u^{(n+m)}\big)^2(\tau)\big\|_{B^{s-2}_{p,r}}\,d\tau \tag{3.24}$$

if $s-1>1+\frac{1}{p}$. From (3.23), if $s-1<1+\frac{1}{p}$, by using $B^{s-1}_{p,r}\hookrightarrow L^\infty$ with $s>1+\frac{1}{p}$, we have

$$W^{n+m}(t)\le C\int_0^t \big\|u^{(n+m)}(\tau)\big\|^2_{B^s_{p,r}}\,d\tau. \tag{3.25}$$

From (3.25), if $s-1>1+\frac{1}{p}$, we have

$$W^{n+m}(t)\le C\int_0^t \big\|u^{(n+m)}(\tau)\big\|^2_{B^s_{p,r}}\,d\tau. \tag{3.26}$$

It is easily showed that

$$\left\|\sum_{q=n+1}^{n+m}\Delta_q u_0\right\|_{B^{s-1}_{p,r}}\le C2^{-n}\|u_0\|_{B^{s-1}_{p,r}}. \tag{3.27}$$

Inserting (3.25)-(3.27) into (3.22), we have

$$\begin{aligned}&\big\|\big(u^{(n+m+1)}-u^{(n+1)}\big)(t)\big\|_{B^{s-1}_{p,r}}\\&\quad\le C_T\left(2^{-n}+\int_0^t\big\|\big(u^{(n+m)}-u^{(n)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau\right).\end{aligned} \tag{3.28}$$

We define

$$W_{n,k}(t)=\big\|\big(u^{(n+m)}-u^{(n)}\big)(t)\big\|_{B^{s-1}_{p,r}}, \tag{3.29}$$

$$W_n(t)=\sup_{m\in\mathbf{N}}W_{n,m}(t), \tag{3.30}$$

$$\widetilde{W}(t)=\limsup_{n\longrightarrow\infty}W_n(t). \tag{3.31}$$

Combining (3.28) with (3.29)-(3.30), we have that

$$W_{n+1}(t)\le C\int_0^t W_n(\tau)\,d\tau. \tag{3.32}$$

From (3.31) and (3.32), by using the Fatou lemma, we have that

$$\widetilde{W}(t)\le C\int_0^t \widetilde{W}(\tau)\,d\tau. \tag{3.33}$$

Applying the Gronwall inequality to (3.33), we have that

$$\widetilde{W}(t)\le e^C\,\widetilde{W}(0). \tag{3.34}$$

From (3.31), we have that $\widetilde{W}(0) = 0$. Thus, $\widetilde{W}(t) = 0$. Consequently, $(u^{(n)})_n$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$; moreover, $(u^{(n)})_n$ is convergent to some limit function $u \in C([0,T];B^{s-1}_{p,r})$.

When $s = 2 + \frac{1}{p}$, by using (6) of Lemma 2.2, we derive that

$$\begin{aligned} &\big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(t)\big\|_{L^\infty_T B^{s-1}_{p,r}} \\ &\quad = \big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(t)\big\|_{L^\infty_T B^{1+\frac{1}{p}}_{p,r}} \\ &\quad \le \big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(t)\big\|^{\theta}_{L^\infty_T B^{s_1}_{p,r}} \big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(t)\big\|^{1-\theta}_{L^\infty_T B^{s_2}_{p,r}} \\ &\quad \le \big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(t)\big\|^{\theta}_{B^{1+\frac{1}{p}}_{p,r}} \Big[\big\|u^{(n+m+1)}\big\|_{B^{2+\frac{1}{p}}_{p,r}} + \big\|u^{(n+1)}(t)\big\|_{B^{2+\frac{1}{p}}_{p,r}}\Big]^{1-\theta} \\ &\quad \le \big(C'_T\big)^{\theta} 2^{-\theta n}\Big[\big\|u^{(n+m+1)}\big\|_{B^{2+\frac{1}{p}}_{p,r}} + \big\|u^{(n+1)}(t)\big\|_{B^{2+\frac{1}{p}}_{p,r}}\Big]^{1-\theta}, \end{aligned} \tag{3.35}$$

where

$$s_1 \in \left(\max\left(1 + \frac{1}{p}, \frac{3}{2}\right) - 1, 1 + \frac{1}{p}\right), \qquad s_2 \in \left(1 + \frac{1}{p}, 2 + \frac{1}{p}\right).$$

Consequently, $(u^{(n)})_n$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$ and $(u^{(n)})_n$ converges to some limit function $u \in C([0,T];B^{s-1}_{p,r})$.

Fourth step: Existence of solution in $E^s_{p,r}(T)$. Existence of solution $E^s_{p,r}(T)$ can be proved similarly to [11].

Fifth step: Uniqueness of solution. We consider case $s \neq 2 + \frac{1}{p}$ and case $s = 2 + \frac{1}{p}$, respectively. In fact, this can be proved similarly to [14].

Sixth step: Continuity with respect to the initial data. Continuity with respect to the initial data can be proved similarly to [6].

4 Proof of Theorem 1.2

Since $B^{3/2}_{2,1}$ and $B^{1/2}_{2,1}$ are Banach algebras, by using a proof similar to (3.3), we can prove that

$$\begin{aligned} \big\|u^{(n+1)}(t)\big\|_{B^{3/2}_{2,1}} \le{}& e^{C\int_0^t \|u^{(n)}\|^2_{B^{3/2}_{2,1}}\,d\tau} \|u_0\|_{B^{3/2}_{2,1}} + \frac{C}{2}\int_0^t e^{C\int_\tau^t \|u^{(n)}\|^2_{B^{3/2}_{2,1}}\,d\tau} \big\|u^{(n)}\big\|^3_{B^{3/2}_{2,1}}\,d\tau \\ &+ \frac{C}{2}\int_0^t e^{C\int_\tau^t \|u^{(n)}\|^2_{B^{3/2}_{2,1}}\,d\tau} \big\|u^{(n)}\big\|^5_{B^{3/2}_{2,1}}\,d\tau \end{aligned} \tag{4.1}$$

for all $n \in \mathbf{N}$. We assume that

$$T < \min\left\{\frac{1}{16C\|u_0\|^2_{B^{3/2}_{2,1}}}, \frac{1}{C}\right\}, \tag{4.2}$$

$$\big\|u^{(n)}(t)\big\|_{B^{3/2}_{2,1}} \le \frac{2\|u_0\|_{B^{3/2}_{2,1}}}{(1 - 16C\|u_0\|^2_{B^{3/2}_{2,1}} t)^{1/2}}. \tag{4.3}$$

By a proof similar to (3.15), we can prove that

$$\big\|u^{(n+1)}(t)\big\|_{B^{3/2}_{2,1}} \le \frac{2\|u_0\|_{B^{3/2}_{2,1}}}{(1 - 16C\|u_0\|^2_{B^{3/2}_{2,1}} t)^{1/2}}. \tag{4.4}$$

Thus, $(u^{(n)})_n$ is uniformly bounded in $B^{3/2}_{2,1}$. From (3.1), by using (4) in Lemma 2.2, we can prove that $(u_t^{(n)})_n$ is uniformly bounded with respect to n in $B^{1/2}_{2,1}$.

Consequently, $(u^{(n)})_n \in C([0,T];B^{3/2}_{2,1}) \cap C^1([0,T];B^{1/2}_{2,1})$.

We define

$$\rho_{n,k}(t) = \left\| \left(u^{(n+m)} - u^{(n)}\right)(t) \right\|_{B^{1/2}_{2,\infty}}, \tag{4.5}$$

$$\rho_n(t) = \sup_{m\in\mathbf{N}} \rho_{n,m}(t), \tag{4.6}$$

$$\widetilde{\rho}(t) = \limsup_{n\longrightarrow\infty} \rho_n(t). \tag{4.7}$$

By a proof similar to [15], we derive that

$$\widetilde{\rho}(t) = 0, \tag{4.8}$$

and with the aid of (2.1), we can prove that $(u^{(n)})_n$ is a Cauchy sequence in $C([0,T];B^{1/2}_{2,1})$.

The rest of Theorem 1.2 can be proved similarly to [13, 15].

Competing interests
We declare that we have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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