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Chen inequalities on warped product Legendrian submanifolds in Kenmotsu space forms and applications

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Abstract

In the current work, we study the geometry and topology of warped product Legendrian submanifolds in Kenmotsu space forms $\mathbb{F}^{2n+1}(\epsilon)$ and derive the first Chen inequality, including extrinsic invariants such as the mean curvature and the length of the warping functions. Additionally, sectional curvature and the δ-invariant are intrinsic invariants related to this inequality. An integral bound is also given in terms of the gradient Ricci curvature for the Bochner operator formula of compact warped product submanifolds. Our primary technique is applying geometry to number structures and solving problems such as problems with Dirichlet eigenvalues.

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1 Introduction and main motivations

According to Nash's embedding theorem [21], every warped product can be immersed as a Riemannian submanifold in some Euclidean space with a sufficiently high co-dimension. Based on Nash's theorem, geometric obstructions, so-called *intrinsic* and *extrinsic invaraints*, for warped products have been obtained in various ambient space forms [5, 9, 14]. Chen [11, 13] provided the optimizations for the second fundamental form as a main intrinsic invariant and constant holomorphic sectional curvature and the Laplacian of the warping function as a main extrinsic invariant for CR-warped products in complex space form and complex projective spaces. He also completely classified the equality case of these inequalities. Several excellent papers on warped product submanifolds were devoted to different ambient space forms in [1, 16, 19], as well as Chen's work [6, 14]. Some applications are also derived on a compact Riemannian submanifold considering equality cases with empty boundaries. Chen [11] developed a novel technique to find the relationship between extrinsic and intrinsic invariants for warped product submanifolds of Kaehler manifolds and space forms using the Codazzi equation. There are not many studies on the δ-invariant for warped products other than Chen's optimal inequality for

CR-warped products in complex space [6]. The first Chen invariant for warped product submanifolds in real space forms was derived, and minimality conditions for submanifolds were recently discussed by Mustafa et al. [20]. Taking this into consideration, Gauss equations were used instead of Codazzi equations [13]. The squared norm of the mean curvature can be calculated using a warping function and the constant holomorphic sectional curvature [1, 16] and in light of the historical development in the study of a warping function of a warped product submanifold [12]. As the main objective of our study, we present a novel method for establishing inequalities for δ-invariant curvature inequalities for warped product Legendrian submanifolds isometrically immersed in Kenmotsu space forms. This has been discussed in [16, 20, 22]. We generalized a number of inequalities for areas of hyperbolic spaces based on the main results discussed in this paper. These products also include another important group of Riemannian products. Our first contribution is to calculate a sharp estimate of the squared norm of the mean curvature using a warping function and the constant holomorphic sectional curvature.

2 Preliminaries

A $(2m+1)$-dimensional manifold $\widetilde{\mathcal{F}}^{2m+1}$ endowed with almost contact structure (φ,ξ,η,g) is called an almost contact metric manifold when satisfies the following properties:

$$\varphi^2 = -I + \eta\otimes\xi, \qquad \eta(\xi) = 1, \qquad \varphi(\xi) = 0, \qquad \eta\circ\varphi = 0, \tag{2.1}$$

$$g(\varphi\mathcal{Y}_1, \qquad \varphi\mathcal{Y}_2) = g(\mathcal{Y}_1,\mathcal{Y}_2) - \eta(\mathcal{Y}_1)\eta(\mathcal{Y}_2) \quad \text{and} \quad \eta(\mathcal{Y}_1) = g(\mathcal{Y}_1,\xi), \tag{2.2}$$

for any $\mathcal{Y}_1,\mathcal{Y}_2 \in \mathfrak{X}(T\widetilde{\mathcal{F}}^{2m+1})$, the Lie algebra of vector fields on a manifold $\widetilde{\mathcal{F}}^{2m+1}$. In this case, φ, g, ξ and η are called $(1,1)$-tensor fields, a structure vector field and dual 1-form, respectively. Furthermore, an almost contact metric manifold is known to be a *Kenmotsu manifold* (cf. [4, 17]) if

$$(\widetilde{\nabla}_{\mathcal{Y}_1}\varphi)\mathcal{Y}_2 = g(\mathcal{Y}_1,\varphi\mathcal{Y}_2)\xi - \eta(\mathcal{Y}_2)\varphi\mathcal{Y}_1, \qquad \widetilde{\nabla}_{\mathcal{Y}_1}\xi = \mathcal{Y}_1 - \eta(\mathcal{Y}_1)\xi, \tag{2.3}$$

for any vector fields $\mathcal{Y}_1,\mathcal{Y}_2$ on $\widetilde{\mathcal{F}}^{2m+1}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. An n-dimensional Riemannian submanifold $\mathbb{F}^n$ of $\widetilde{\mathcal{F}}^{2m+1}$ is referred to as totally real if the standard almost contact structure φ of $\widetilde{\mathcal{F}}^{2m+1}$ maps any tangent space of $\mathbb{F}^n$ into its corresponding normal space [1, 19, 22]. Now, let $\mathbb{F}^n$ be an isometric immersed submanifold of dimension n in $\widetilde{\mathcal{F}}^{2m+1}$, then $\mathbb{F}^n$ is referred to as a Legendrian submanifold if ξ is a normal vector field on $\mathbb{F}^n$ (i.e., $\mathbb{F}^n$ is a C-totally real submanifold) and $m = n$ [1, 18, 19, 22]. Legendrian submanifolds play a substantial role in contact geometry. From the Riemannian geometric perspective, studying the Legendrian submanifolds of Kenmotsu manifolds was initiated in the 1970s [8].

Let $\mathbb{F}$ be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian $\widetilde{\mathcal{F}}^{2m+1}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are induced connections on the tangent bundle $T\mathbb{F}$ and normal bundle $T^{\perp}\mathbb{F}$ of $\mathbb{F}^n$, respectively. Then the Gauss and Weingarten formulas are given by

$$\text{(i)} \quad \widetilde{\nabla}_{\mathcal{Y}_1}\mathcal{Y}_2 = \nabla_{\mathcal{Y}_1}\mathcal{Y}_2 + \mathcal{B}(\mathcal{Y}_1,\mathcal{Y}_2), \qquad \text{(ii)} \quad \widetilde{\nabla}_{\mathcal{Y}_1}N = -A_N\mathcal{Y}_1 + \nabla^{\perp}_{\mathcal{Y}_1}N, \tag{2.4}$$

for each $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{X}(T\mathbb{F})$ and $N \in \mathfrak{X}(T^{\perp}\mathbb{F})$, where $\mathcal{B}$ and A_N are the second fundamental form and shape operator (corresponding to the normal vector field N), respectively, for the immersion of $\mathbb{F}^n$ into $\widetilde{\mathcal{F}}^{2m+1}$, and they are related as:

$$g\big(\mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_2), N\big) = g(A_N \mathcal{Y}_1, \mathcal{Y}_2). \tag{2.5}$$

Similarly, Gauss and Codazzi equations are given by

$$\begin{aligned} \text{(i)} \quad & R(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4) \\ & \quad = \widetilde{R}(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4) + g\big(\mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_4), \mathcal{B}(\mathcal{Y}_2, \mathcal{Y}_3)\big) - g\big(\mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_3), \mathcal{B}(\mathcal{Y}_2, \mathcal{Y}_4)\big), \qquad (2.6) \\ \text{(ii)} \quad & \big(\widetilde{R}(\mathcal{Y}_1, \mathcal{Y}_2)\mathcal{Y}_3\big)^{\perp} = (\widetilde{\nabla}_{\mathcal{Y}_1}\mathcal{B})(\mathcal{Y}_2, \mathcal{Y}_3) - (\widetilde{\nabla}_{\mathcal{Y}_2}\mathcal{B})(\mathcal{Y}_1, \mathcal{Y}_3), \qquad (2.7) \end{aligned}$$

for all $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4 \in \mathfrak{X}(T\widetilde{M})$, where $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are the curvature tensor of $\widetilde{\mathcal{F}}^{2n+1}$ and $\mathbb{F}^n$, respectively. The mean curvature $\mathbb{H}$ of the Riemannian submanifold $\mathbb{F}^n$ is given by

$$\mathbb{H} = \frac{1}{n} \operatorname{trace}(\mathcal{B}). \tag{2.8}$$

A submanifold $\mathbb{F}^n$ of Riemannian manifold $\widetilde{\mathcal{F}}^{2n+1}$ is said to be *totally umbilical* and *totally geodesic* if $\mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_2) = g(\mathcal{Y}_1, \mathcal{Y}_1)\mathbb{H}$ and $\mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_2) = 0$, for any $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{X}(TM)$, respectively, where $\mathbb{H}$ is the mean curvature vector of $\mathbb{F}^n$. Furthermore, if $\mathbb{H} = 0$, then $\mathbb{F}^n$ is *minimal* in $\widetilde{\mathbb{F}}^{2m+1}$. Moreover, the related null space or kernel of the second fundamental form of $\mathbb{F}^n$ at x is defined by

$$\mathbb{F}_x = \big\{\mathcal{Y}_1 \in T_x\mathbb{F} : \mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_2) = 0, \text{ for all } \mathcal{Y}_2 \in T_x\mathbb{F}\big\}. \tag{2.9}$$

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\widetilde{\mathcal{F}}^{2m+1}$ and denoted at $\widetilde{\tau}(T_x\widetilde{\mathcal{F}}^{2m+1})$, which, at some x in $\widetilde{\mathcal{F}}^{2m+1}$, is given

$$\widetilde{\tau}\big(T_x\widetilde{\mathcal{F}}^{2m+1}\big) = \sum_{1 \le i < j \le 2m+1} \widetilde{\mathcal{K}}_{ij}, \tag{2.10}$$

where $\widetilde{\mathcal{K}}_{ij} = \widetilde{\mathcal{K}}(e_i \wedge e_j)$. It is clear that first equality (2.10) is congruent to the following equation, which will be frequently used in a subsequent proof,

$$2\widetilde{\tau}\big(T_x\widetilde{\mathcal{F}}^{2m+1}\big) = \sum_{1 \le i < j \le 2m+1} \widetilde{\mathcal{K}}_{ij}. \tag{2.11}$$

Similarly, scalar curvature $\widetilde{\tau}(L_x)$ of L-plan is given by

$$\widetilde{\tau}(L_x) = \sum_{1 \le i < j \le m} \widetilde{\mathcal{K}}_{ij}. \tag{2.12}$$

Let $\{e_1, \dots e_n\}$ be an orthonormal basis of the tangent space $T_x\mathbb{F}$ and
$e_r = (e_{n+1}, \dots e_{2m+1})$ belong to an orthogonal basis of the normal space $T^{\perp}\mathbb{F}$, then we have

$$\mathcal{B}_{ij}^r = g\big(\mathcal{B}(e_i, e_j), e_r\big) \quad \text{and} \quad \|\mathcal{B}\|^2 = \sum_{i,j=1}^{n} g\big(\mathcal{B}(e_i, e_j), \mathcal{B}(e_i, e_j)\big). \tag{2.13}$$

Let $\mathcal{K}_{ij}$ and $\widetilde{\mathcal{K}}_{ij}$ denote the sectional curvature of the plane section spanned and e_i at x in the submanifold $\mathbb{F}^n$ and in the Riemannian space form $\widetilde{\mathcal{F}}^{2n+1}(c)$, respectively. Thus, $\mathcal{K}_{ij}$ and $\widetilde{\mathcal{K}}_{ij}$ are the intrinsic and extrinsic sectional curvature of the span $\{e_i, e_j\}$ at x. From the Gauss equation (2.6)(i), we have

$$2\widetilde{\tau}\big(T_x\widetilde{\mathbb{F}}^n\big) = \mathcal{K}_{ij} = 2\widetilde{\tau}\big(T_x\widetilde{\mathcal{F}}^{2m+1}\big) = \widetilde{\mathcal{K}}_{ij} + \sum_{r=n+1}^{2m+1}\big(\mathcal{B}^r_{ii}\mathcal{B}^r_{jj} - \big(\mathcal{B}^r_{ij}\big)^2\big). \tag{2.14}$$

The second invariant is called the *Chen first invariant*, which is defined as

$$\delta_{\widetilde{\mathcal{F}}^{2m+1}}(x) = \widetilde{\tau}\big(T_x\widetilde{\mathcal{F}}^{2m+1}\big) - \inf\big\{\widetilde{\mathcal{K}}(\pi) : \pi \subset T_x\widetilde{\mathcal{F}}^{2m+1}, x \in \widetilde{\mathcal{F}}^{2m+1}, \dim\mathcal{B} = 2\big\}. \tag{2.15}$$

Assume that $\mathbb{F}_1^{d_1}$ and $\mathbb{F}_2^{d_2}$ are two Riemannian manifolds with their Riemannain metrics g_1 and g_2, respectively. Let f be a smooth function defined on d_1. Then warped product manifold $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ is the manifold $\mathbb{F}_1^{d_1} \times \mathbb{F}_2^{d_2}$ furnished by the Riemannian metric $g = g_1 + f^2 g_2$. Assume that the $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ is warped product manifold, then for any $\mathcal{Y}_1 \in \Gamma(T\mathbb{F}_1^{d_1})$ and $\mathcal{Y}_3 \in \Gamma(T\mathbb{F}_2^{d_2})$, we find that

$$\nabla_{\mathcal{Y}_3}\mathcal{Y}_1 = \nabla_{\mathcal{Y}_1}\mathcal{Y}_3 = (\mathcal{Y}_1 \ln f)\mathcal{Y}_3. \tag{2.16}$$

Similarly, knowing that units vector fields $\mathcal{Y}_1$ and $\mathcal{Y}_3$ are tangent to $\mathbb{F}_1^{d_1}$ and $\mathbb{F}_2^{d_2}$, respectively, we derive

$$\begin{aligned}
\mathcal{K}(\mathcal{Y}_1 \wedge \mathcal{Y}_3) &= g\big(R(\mathcal{Y}_1, \mathcal{Y}_3)\mathcal{Y}_1, \mathcal{Y}_3\big) \\
&= (\nabla_{\mathcal{Y}_1}\mathcal{Y}_1)\ln f g(\mathcal{Y}_3, \mathcal{Y}_3) - g\big(\nabla_{\mathcal{Y}_1}\big((\mathcal{Y}_1 \ln f)\mathcal{Y}_3\big), \mathcal{Y}_3\big) \\
&= (\nabla_{\mathcal{Y}_1}\mathcal{Y}_1)\ln f g(\mathcal{Y}_3, \mathcal{Y}_3) - g(\nabla_{\mathcal{Y}_1}(\mathcal{Y}_1 \ln f)\mathcal{Y}_3 + (\mathcal{Y}_1 \ln f) g(\nabla_{\mathcal{Y}_1}\mathcal{Y}_3, \mathcal{Y}_3) \\
&= (\nabla_{\mathcal{Y}_1}\mathcal{Y}_1)\ln f g(\mathcal{Y}_3, \mathcal{Y}_3) - (\mathcal{Y}_1 \ln f)^2 - \mathcal{Y}_1(\mathcal{Y}_1 \ln f).
\end{aligned} \tag{2.17}$$

Assuming that $\{e_1, \dots e_n\}$ is an orthonormal frame for $\mathbb{F}^n$, we sum over the vector fields such that

$$\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\mathcal{K}(e_i \wedge e_j) = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\big((\nabla_{e_i}e_i)\ln f - e_i(e_i \ln f) - (e_i \ln f)^2\big),$$

which implies that

$$\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\mathcal{K}(e_i \wedge e_j) = d_2\big(\Delta(\ln f) - \|\nabla(\ln f)\|^2\big). \tag{2.18}$$

However, it was proved [10] that for arbitrary warped product submanifold,

$$\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\mathcal{K}(e_i \wedge e_j) = \frac{d_2\Delta f}{f}. \tag{2.19}$$

Thus, from (2.18) and (2.19), we get

$$\frac{\Delta f}{f} = \Delta(\ln f) - \|\nabla(\ln f)\|^2. \tag{2.20}$$

The following remarks are consequences of warped product submanifold:

Remark 2.1 A warped product manifold $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ is said to be *trivial* if the warping function f is constant or simply a Riemannian product manifold.

Remark 2.2 If $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ is a warped product manifold, then $\mathbb{F}_1$ is totally geodesic and $\mathbb{F}_2$ is totally umbilical submanifold of $\mathbb{F}^n$.

A Kenmotsu manifold is said to be Kenmotsu space form with constant φ-sectional curvature ϵ if and only if the Riemannian curvature tensor $\widetilde{R}$ can be written as [1, 19]:

$$\begin{aligned}\widetilde{R}(\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\mathcal{Y}_4) = {}& \left(\frac{\epsilon-3}{4}\right)\{g(\mathcal{Y}_2,\mathcal{Y}_3)g(\mathcal{Y}_1,\mathcal{Y}_4) - g(\mathcal{Y}_1,\mathcal{Y}_3)g(\mathcal{Y}_2,\mathcal{Y}_4)\} \\ & + \left(\frac{\epsilon+1}{4}\right)\{\eta(\mathcal{Y}_1)\eta(\mathcal{Y}_3)g(\mathcal{Y}_2,\mathcal{Y}_4) + \eta(\mathcal{Y}_4)\eta(\mathcal{Y}_2)g(\mathcal{Y}_1,\mathcal{Y}_3) \\ & - \eta(\mathcal{Y}_2)\eta(\mathcal{Y}_3)g(\mathcal{Y}_1,\mathcal{Y}_4) - \eta(\mathcal{Y}_1)g(\mathcal{Y}_2,\mathcal{Y}_3)\eta(\mathcal{Y}_4) \\ & + g(\varphi\mathcal{Y}_2,\mathcal{Y}_3)g(\varphi\mathcal{Y}_1,\mathcal{Y}_4) - g(\varphi\mathcal{Y}_1,\mathcal{Y}_3)g(\varphi\mathcal{Y}_2,\mathcal{Y}_4) \\ & + 2g(\mathcal{Y}_1,\varphi\mathcal{Y}_2)g(\varphi\mathcal{Y}_3,\mathcal{Y}_4)\},\end{aligned} \tag{2.21}$$

where $\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\mathcal{Y}_4 \in \mathfrak{X}(T\widetilde{\mathcal{F}}^{2m+1})$.

The hyperbolic space $\mathbb{H}^{2n+1} = \{x_1 \cdots x_{2n+1} \in \mathbb{R}^{2n+1} | x_1 > 0\}$ equipped with the almost contact structure $((\varphi,\xi,\eta,g)$ constructed by Chinea and Gonzales [15] is a Kenmotsu manifold. Many geometers have paid considerable attention to minimal Legendrian submanifolds.

We recall the following important algebraic lemma

Lemma 2.1 *Let $t_1, t_2 \cdots t_n$, s be $(n+1)(n \geq 2)$ real number such that*

$$\sum_{i=1}^{n}(t_i)^2 = (d_1)\left(\sum_{i=1}^{n} t_1^2 + s\right). \tag{2.22}$$

Then $2t_1t_2 \geq s$, with equality holds if and only if $t_1 + t_2 = t_3 = \cdots t_n$.

Theorem 2.1 *Let $\phi : \mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ be an isometric immersion of a warped product Legendrian submanifold $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ into a Kenmotsu space form $\widetilde{\mathcal{F}}^{2n+1}(\epsilon)$. Then, for each point $x \in \mathbb{F}^n$ and each plane section $\pi_i \subset T_x\mathbb{F}_i^{n_i}$, for $i = 1, 2$, we get*

(1) *Let $\pi_1 \subset T_x\mathbb{F}_1^{d_1}$, then*

$$\begin{aligned}\delta_{\widetilde{\mathbb{F}}^{d_1}}(x) \leq {}& \frac{n^2}{2}\|\mathbb{H}\|^2 + d_2\|\nabla(\ln f)\|^2 - d_2\Delta(\ln f) \\ & + \left\{\frac{d_1}{2}(d_1 + 2d_2 - 1) - 1\right\}\left(\frac{\epsilon-3}{4}\right).\end{aligned} \tag{2.23}$$

Equality of the above inequality holds at $x \in \mathbb{F}^n$ if and only if there exists an orthonormal basis $\{e_1 \cdots e_n\}$ of $T_x\mathbb{F}^n$ and orthonormal basis $\{e_n + 1 \cdots e_{2n+1}\}$ of $T_x^{\perp}$ such that $(a), \pi = \operatorname{Span}\{e_1, e_2\}$, and (b) shape operators take the following form

$$(i) \quad A_{e_{n+1}} = \left[\begin{array}{ccccc|ccc} \mu_1 & \mathcal{B}_{12}^{n+1} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\ \mathcal{B}_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{d_11} & 0 & 0 & \cdots & \mu & 0_{d_1d_1+1} & \cdots & 0_{d_1n} \\ \hline 0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}_{d_1+1d_1+1}^{n+1} & \cdots & \mathcal{B}_{d_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \mathcal{B}_{nd_1+1}^{n+1} & \cdots & \mathcal{B}_{nn}^{n+1} \end{array}\right],$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \dots m\}$, then we have the matrix

$$(ii) \quad A_{e_r} = \left[\begin{array}{ccccc|ccc} \mathcal{B}_{11}^{r} & \mathcal{B}_{12}^{r} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\ \mathcal{B}_{21}^{r} & -\mathcal{B}_{11}^{r} & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{d_11} & 0 & 0 & \cdots & 0_{d_1d_1} & 0_{d_1d_1+1} & \cdots & 0_{d_1n} \\ \hline 0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}_{d_1+1d_1+1}^{n+1} & \cdots & \mathcal{B}_{d_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \mathcal{B}_{nd_1+1}^{n+1} & \cdots & \mathcal{B}_{nn}^{n+1} \end{array}\right],$$

(2) *If $\pi_2 \subset T_x\mathbb{F}_2^{d_2}$, then*

$$\delta_{\widetilde{\mathbb{F}}^{d_2}}(x) \leq \frac{n^2}{2}\|\mathbb{H}\|^2 + d_2\left\|\nabla(\ln f)\right\|^2 - d_2\Delta(\ln f) + \left\{\frac{d_2}{2}(d_2 + 2d_1 - 1) - 1\right\}\left(\frac{\epsilon - 3}{4}\right). \tag{2.24}$$

Equalities of the above equation hold if and only if

$$(iii) \quad A_{e_{n+1}} = \left[\begin{array}{cccc|ccccc} \mathcal{B}_{11}^{n+1} & \cdots & \cdots & \mathcal{B}_{1d_1}^{n+1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{B}_{d_11}^{n+1} & \cdots & \cdots & \mathcal{B}_{d_1d_1}^{n+1} & 0_{d_1d_1+1} & \cdots & \cdots & \cdots & 0_{d_1n} \\ \hline 0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mu_1 & \mathcal{B}_{d_1+1d_1+2}^{n+1} & 0 & \cdots & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \mathcal{B}_{d_1+2d_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & \mu \end{array}\right],$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2,\dots,m\}$, thus we have

$$(iv)\quad A_{e_r} = \left[\begin{array}{cccc|ccccc} \mathcal{B}^r_{11} & \cdots\cdots & & \mathcal{B}^r_{1d_1} & 0_{1d_1+1} & \cdots & & \cdots\cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & & \ddots\ \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & & \ddots\ \ddots & \vdots \\ \mathcal{B}^r_{d_111} & \cdots\cdots & & \mathcal{B}^r_{d_1d_1} & 0_{d_1d_1+1} & \cdots & & \cdots\cdots & 0_{d_1n} \\ \hline 0_{d_1+11} & \cdots\cdots & & 0_{d_1+1d_1} & \mathcal{B}^r_{d_1+1d_1+1} & \mathcal{B}^{n+1}_{d_1+1d_1+2} & 0 & \cdots & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \mathcal{B}^{n+1}_{d_1+2d_1+1} & -\mathcal{B}^r_{d_1+1d_1+1} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots\cdots & & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & 0 \end{array}\right].$$

(v) If the equality holds in (1) *or* (2)*, then $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ is mixed totally geodesic in $\mathbb{F}_\epsilon^{2n+1}$. Moreover, $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ is both $\mathbb{F}_1^{d_1}$-minimal and $\mathbb{F}_2^{d_2}$-minimal. Thus, $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ is a minimal warped product submanifold in $\mathbb{F}_\epsilon^{2n+1}$.*

Proof Let $\pi_1 \subset T_x\mathbb{F}_1$ be a 2-plane for $x \in \mathbb{F}^n$, then we consider the orthonormal basis $\{e_1 \cdots e_{d_1}, e_{d_1+1}, \dots e_n\}$ of $T_x\mathbb{F}^n$ such that $\{e_1, \dots e_{d_1}\}$ is an orthonomal basis for $T_x\mathbb{F}_1$ and $\{e_{d_1+1}, \dots e_n\}$ is for $T_x\mathbb{F}_2$. Similarly, $\{e_{n+1}, \dots e_{2n+1}\}$ is an orthonormal basis for $T_x^\perp\mathbb{F}^n$. Assuming that $\pi = \operatorname{Span}\{e_1, e_2\}$ such that the normal vector e_{n+1} is in the direction of mean curvature vector $\mathbb{H}$ and using (2.21) and the Gauss equation (2.6), we get

$$n^2\|\mathbb{H}\|^2 = 2\tau\left(T_x\mathbb{F}^n\right) + \|\mathcal{B}\|^2 - n(n-1)\left(\frac{\epsilon-3}{4}\right), \tag{2.25}$$

which implies that

$$\left(\sum_{i=1}^{d_1}\mathcal{B}^{n+1}_{ii}\right)^2 = 2\tau\left(T_x\mathbb{F}^n\right) + \|\mathcal{B}\|^2 - n(n-1)\left(\frac{\epsilon-3}{4}\right) - \left(\sum_{j=d_1+1}^{n}\mathcal{B}^{n+1}_{jj}\right)^2 - 2\sum_{A=d_1+1}^{d_1}\sum_{B=d_1+1}^{n}\mathcal{B}^{n+1}_{AA}\mathcal{B}^{n+1}_{BB}. \tag{2.26}$$

Let us consider the following

$$\Omega = 2\tau\left(T_x\mathbb{F}^n\right) - n(n-1)\left(\frac{\epsilon-3}{4}\right) - \frac{(d_1-2)}{(d_1-1)}\left(\sum_{i=1}^{d_1}\mathcal{B}^{n+1}_{ii}\right)^2 - \left(\sum_{j=d_1+1}^{n}\mathcal{B}^{n+1}_{jj}\right)^2 - 2\sum_{A=d_1+1}^{d_1}\sum_{B=d_1+1}^{n}\mathcal{B}^{n+1}_{AA}\mathcal{B}^{n+1}_{BB}. \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$\left(\sum_{i=1}^{d_1}\mathcal{B}_{ii}^{n+1}\right)^2=(d_1-1)\left(\Omega+\|\mathcal{B}\|^2\right). \tag{2.28}$$

The above equation can be expressed as:

$$\begin{aligned}\left(\sum_{i=1}^{d_1}\mathcal{B}_{ii}^{n+1}\right)^2=(d_1-1)\Bigg\{\Omega+\sum_{i=1}^{d_1}\left(\mathcal{B}_{ii}^{n+1}\right)^2+\sum_{j=d_1+1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2\\+\sum_{i\neq j=1}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2\Bigg\}.\end{aligned} \tag{2.29}$$

Therefore, we apply Lemma 2.1 on the above equation, i.e.,

$$t_\alpha=\mathcal{B}_{\alpha\alpha}^{n+1},\quad \forall t_\alpha\in\{1\cdots,d_1\}\quad\text{and}$$

$$s=\Omega+\sum_{j=d_1+1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2+\sum_{i\neq j=1}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2.$$

Thus, we obtain that

$$\mathcal{B}_{11}^{n+1}\mathcal{B}_{22}^{n+1}\geq\frac{1}{2}\left\{\Omega+\sum_{j=d_1+1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2+\sum_{i\neq j=1}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2\right\}. \tag{2.30}$$

Then from (2.21) and (2.14), we derive

$$K(\pi_1)=\left(\frac{\epsilon-3}{4}\right)+\sum_{r=n+1}^{2n+1}\left(\mathcal{B}_{11}^{r}\mathcal{B}_{22}^{r}-\left(\mathcal{B}_{12}^{r}\right)^2\right). \tag{2.31}$$

If we combine equations (2.30) and (2.31), we get

$$\begin{aligned}K(\pi_1)\geq\left(\frac{\epsilon-3}{4}\right)+\frac{1}{2}\Omega+\frac{1}{2}\sum_{j=d_1+1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2\\+\sum_{r=n+1}^{2n+1}\left(\mathcal{B}_{11}^{r}\mathcal{B}_{22}^{r}-\left(\mathcal{B}_{12}^{r}\right)^2\right)+\frac{1}{2}\sum_{i\neq j=1}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\frac{1}{2}\sum_{r=n+2}^{2n+1}\sum_{i,j=1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2.\end{aligned} \tag{2.32}$$

We choose the last two terms of the above equation and derive

$$\begin{aligned}\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2=\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+2\sum_{j=3}^{n}\left(\mathcal{B}_{1j}^{n+1}\right)^2\\+2\left(\mathcal{B}_{12}^{n+1}\right)^2+2\sum_{j=3}^{n}\left(\mathcal{B}_{2j}^{n+1}\right)^2.\end{aligned} \tag{2.33}$$

Moreover, for the last term, we obtain

$$\begin{aligned}\sum_{r=n+2}^{2n+1}\sum_{i,j=1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2 &= \sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2 + 2\sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left(\mathcal{B}_{1j}^{n+1}\right)^2 \\ &\quad + 2\sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left(\mathcal{B}_{2j}^{n+1}\right)^2 + 2\left(\mathcal{B}_{12}^{n+1}\right)^2 \\ &\quad + \sum_{r=n+2}^{2n+1}\left((\mathcal{B}_{11})^2 + (\mathcal{B}_{22})^2\right).\end{aligned}$$

Furthermore, we have

$$\sum_{r=n+2}^{2n+1}\mathcal{B}_{11}^{r}\mathcal{B}_{22}^{2} + \frac{1}{2}\sum_{r=n+2}^{2n+1}\left(\left(\mathcal{B}_{11}^{r}\right)^2 + \left(\mathcal{B}_{22}^{r}\right)^2\right) = \frac{1}{2}\sum_{r=n+2}^{2n+1}\left(\mathcal{B}_{11}^{r} + \mathcal{B}_{22}^{r}\right)^2, \tag{2.34}$$

$$\begin{aligned}&\sum_{j=3}^{n}\left(\left(\mathcal{B}_{1j}^{n+1}\right)^2 + \left(\mathcal{B}_{2j}^{n+1}\right)^2\right) + \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left(\mathcal{B}_{1j}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left(\mathcal{B}_{2j}^{n+1}\right)^2 \\ &\quad = \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left\{\left(\mathcal{B}_{2j}^{n+1}\right)^2 + \left(\mathcal{B}_{2j}^{n+1}\right)^2\right\}.\end{aligned} \tag{2.35}$$

After adding (2.33) and (2.60), using (2.34) and (2.35), and taking into account that

$$\left(\mathcal{B}_{12}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1}\left(\mathcal{B}_{12}^{n+1}\right)^2 = \sum_{r=n+1}^{2n+1}\left(\mathcal{B}_{12}^{n+1}\right)^2,$$

we get

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1}\sum_{i,j=1}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2 &= 2\sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left\{\left(\mathcal{B}_{2j}^{n+1}\right)^2 + \left(\mathcal{B}_{2j}^{n+1}\right)^2\right\} \\ &\quad + \sum_{\substack{i,j=3\\ i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2 \\ &\quad - 2\sum_{r=n+2}^{2n+1}\left\{\mathcal{B}_{11}^{r}\mathcal{B}_{22}^{r} - \left(\mathcal{B}_{12}^{r}\right)^2\right\} \\ &\quad + \sum_{r=n+2}^{2n+1}\left(\mathcal{B}_{11}^{r} + \mathcal{B}_{22}^{2}\right)^2.\end{aligned} \tag{2.36}$$

It follows from (2.32) and (2.36) that

$$\begin{aligned}K(\pi_1) &\geq \left(\frac{\epsilon - 3}{4}\right) + \frac{1}{2}\Omega + \frac{1}{2}\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\beta\beta}^{n+1}\right)^2 \\ &\quad + \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left\{\left(\mathcal{B}_{2j}^{n+1}\right)^2 + \left(\mathcal{B}_{2j}^{n+1}\right)^2\right\}\end{aligned}$$

$$+\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2\right\}+\frac{1}{2}\sum_{r=n+2}^{2n+1}\left(\mathcal{B}_{11}^{r}+\mathcal{B}_{22}^{2}\right)^2,$$

which implies that

$$K(\pi_1)\geq\left(\frac{\epsilon-3}{4}\right)+\frac{1}{2}\left\{\Omega+\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2+\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\beta\beta}^{n+1}\right)^2\right\}.$$

From (2.27), we arrive at

$$\begin{aligned}K(\pi_1)\geq{}&\left(\frac{\epsilon-3}{4}\right)+\tau\left(T_x\mathbb{F}^n\right)+\frac{1}{2(d_1-1)}\left(\sum_{\alpha=d_1+1}^{n}\mathcal{B}_{\alpha\alpha}^{n+1}\right)^2\\&-\frac{n^2}{2}\|\mathbb{H}\|^2-\frac{n(n-1)}{2}\left(\frac{\epsilon-3}{4}\right)\\&+\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{jj}^{n+1}\right)^2+\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\beta\beta}^{n+1}\right)^2\right\}.\end{aligned}\tag{2.37}$$

Using together (2.11) and (2.19) in (2.37), we obtain

$$\begin{aligned}K(\pi_1)\geq{}&\tau\left(T_x\mathbb{F}_1^{d_1}\right)+\tau\left(T_x\mathbb{F}_2^{d_2}\right)+\frac{d_2\nabla f}{f}-\frac{n^2}{2}\|\mathbb{H}\|^2\\&+\left(1-\frac{n(n-1)}{2}\right)\left(\frac{\epsilon-3}{4}\right)\\&+\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\beta\beta}^{n+1}\right)^2\right\}.\end{aligned}$$

This implies that

$$\begin{aligned}\tau\left(T_x\mathbb{F}_1^{d_1}\right)-K(\pi_1)\leq{}&\frac{n^2}{2}\|\mathbb{H}\|^2-\tau\left(T_x\mathbb{F}_2^{d_2}\right)-\frac{d_2\nabla f}{f}\\&+\left(\frac{n(n-1)}{2}-1\right)\left(\frac{\epsilon-3}{4}\right)\\&-\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\beta\beta}^{n+1}\right)^2\right\}.\end{aligned}\tag{2.38}$$

The Gauss equation (2.6)(i) for $\tau(T_x\mathbb{F}_2^{d_2})$ gives us

$$\begin{aligned}\tau\left(T_x\mathbb{F}_2^{d_2}\right)={}&\frac{d_2(d_2-1)}{2}\left(\frac{\epsilon-3}{4}\right)\\&-\frac{1}{2}\sum_{r=n+1}^{2n+1}\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{n+1}\right)^2-\frac{1}{2}\sum_{r=n+1}^{2n+1}\left(\mathcal{B}_{d_1+1d_1+1}+\cdots+\mathcal{B}_{nn}^{r}\right).\end{aligned}\tag{2.39}$$

In view of the equations (2.38) and (2.39), we find that

$$\begin{aligned}\tau\left(T_x\mathbb{F}_1^{d_1}\right)-K(\pi_1)\le{}&\frac{n^2}{2}\|\mathbb{H}\|^2-\frac{d_2(d_2-1)}{2}\left(\frac{\epsilon-3}{4}\right)\\&-\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\mathcal{B}_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\mathcal{B}_{ij}^{r}\right)^2\right.\\&\left.+\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\beta\beta}^{n+1}\right)^2-\sum_{r=n+1}^{2n+1}\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{n+1}\right)^2\right\}\\&+\left(\frac{n(n-1)}{2}-1\right)\left(\frac{\epsilon-3}{4}\right)-\frac{d_2\nabla f}{f}.\end{aligned}\tag{2.40}$$

Then the last relation turns into

$$\begin{aligned}\tau\left(T_x\mathbb{F}_1^{d_1}\right)-K(\pi_1)\le{}&\frac{n^2}{2}\|\mathbb{H}\|^2-\frac{d_2(d_2-1)}{2}\left(\frac{\epsilon-3}{4}\right)+\left(\frac{n(n-1)}{2}-1\right)\left(\frac{\epsilon-3}{4}\right)\\&-\frac{1}{2}\left\{\sum_{\substack{k,l=3\\k\neq l}}^{d_1}\left(\mathcal{B}_{kl}^{n+1}\right)^2+2\sum_{k=3}^{2n+1}\sum_{l=d_1+1}^{n}\left(\mathcal{B}_{kl}^{n+1}\right)^2+\sum_{\substack{A,B=d_1+1\\A\neq B}}^{d_1}\left(\mathcal{B}_{kl}^{n+1}\right)^2\right.\\&+\sum_{r=n+2}^{2n+1}\sum_{k,l=3}^{d_1}\left(\mathcal{B}_{kl}^{n+1}\right)^2+2\sum_{r=n+2}^{2n+1}\sum_{k=3}^{d_1}\sum_{A=d_1+1}^{n}\left(\mathcal{B}_{kl}^{r}\right)^2\\&+\sum_{r=n+2}^{2n+1}\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{r}\right)^2+\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\beta\beta}^{n+1}\right)^2\\&\left.-\sum_{r=n+1}^{2n+1}\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{r}\right)^2\right\}-\frac{d_2\nabla f}{f}.\end{aligned}\tag{2.41}$$

Using the above equation and the following two relations

$$\sum_{A=d_1+1}^{n}\left(\mathcal{B}_{AA}^{2}\right)^2+\sum_{\substack{A,B=3\\A\neq B}}^{n}\left(\mathcal{B}_{AB}^{n+1}\right)^2=\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{n+1}\right)^2\quad\text{and}$$

$$\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{r}\right)^2=\sum_{r=n+1}^{2n+1}\sum_{A,B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{r}\right)^2,$$

the assertion (2.41) follows as:

$$\begin{aligned}&\tau\left(T_x\mathbb{F}_1^{d_1}\right)-K(\pi_1)\\&\quad\le\left\{\frac{d_1}{2}(d_1+2d_2-1)-1\right\}\left(\frac{\epsilon-3}{4}\right)\\&\quad-\frac{1}{2}\left\{\sum_{\substack{k,l=3\\k\neq l}}^{d_1}\left(\mathcal{B}_{kl}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{k,l=3}^{d_1}\left(\mathcal{B}_{kl}^{n+1}\right)^2\right.\end{aligned}\tag{2.42}$$

$$+2\sum_{\alpha=3}^{d_1}\sum_{\beta=d_1+1}^{n}\left(\mathcal{B}_{\alpha\beta}^{n+1}\right)^2+2\sum_{r=n+2}^{2n+1}\sum_{A=3}^{d_1}\sum_{B=d_1+1}^{n}\left(\mathcal{B}_{AB}^{r}\right)^2\Bigg\}+\frac{n^2}{2}\|\mathbb{H}\|^2-\frac{d_2\nabla f}{f}. \tag{2.43}$$

The first inequality of Theorem 2.1 holds from the above equation and (2.15). For the second case, if $\pi\subset T_x\mathbb{F}_2^{d_2}$, we consider $\pi_2=\operatorname{Span}\{e_{d_1+1},e_{d_1+1}\}$, following the same methodology as in the first case:

$$\left(\sum_{\alpha=d_1+1}^{n}\mathcal{B}_{\alpha\alpha}^{n+1}\right)^2=2\tau\left(T_x\mathbb{F}^n\right)+\|\mathcal{B}\|^2-n(n-1)\left(\frac{\epsilon-3}{4}\right)-\left(\sum_{\beta=d_1+1}^{n}\mathcal{B}_{\beta\beta}^{d_1}\right)^2 -2\sum_{\alpha=1}^{d_1}\sum_{\beta=d_1+1}^{n}\mathcal{B}_{\alpha\alpha}^{n+1}\mathcal{B}_{\beta\beta}^{n+1}.$$

Consider the following

$$\Psi=2\tau\left(T_x\mathbb{F}^n\right)-n(n-1)\left(\frac{\epsilon-3}{4}\right) -\frac{(d_1-2)}{(d_1-1)}\left(\sum_{\alpha=d_1+1}^{d_1}\mathcal{B}_{\alpha\alpha}^{n+1}\right)^2-\left(\sum_{\beta=d_1+1}^{n}\mathcal{B}_{\beta\beta}^{d_1}\right)^2-2\sum_{\alpha=1}^{d_1}\sum_{\beta=d_1+1}^{n}\mathcal{B}_{\alpha\alpha}^{n+1}\mathcal{B}_{\beta\beta}^{n+1}.$$

The last two equations imply that

$$\left(\sum_{\alpha=d_1+1}^{n}\mathcal{B}_{\alpha\alpha}^{n+1}\right)^2=(d_2-1)\left(\Psi+\|\mathcal{B}\|^2\right),$$

which implies that

$$\left(\sum_{\alpha=d_1+1}^{n}\mathcal{B}_{\alpha\alpha}^{n+1}\right)^2=(d_2-1)\Bigg\{\Psi+\left(\sum_{\alpha=1}^{d_1}\mathcal{B}_{\alpha\alpha}^{n+1}\right)^2+\left(\sum_{\beta=d_1+1}^{n}\mathcal{B}_{\beta\beta}^{n+1}\right)^2 +\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n}\left(\mathcal{B}_{\alpha\beta}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{\alpha=,\beta=1}^{n}\left(\mathcal{B}_{\alpha\beta}^{r}\right)^2\Bigg\}. \tag{2.44}$$

Similarly, applying Lemma 2.1 to the above equation, we get

$$\mathcal{B}_{d_1+1d_1+1}^{n+1}\mathcal{B}_{d_1+2d_1+2}^{n+1}\geq\frac{1}{2}\Bigg\{\Psi+\left(\sum_{\alpha=1}^{d_1}\mathcal{B}_{\alpha\alpha}^{n+1}\right)^2+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n}\left(\mathcal{B}_{\alpha\beta}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{\alpha=,\beta=1}^{n}\left(\mathcal{B}_{\alpha\beta}^{r}\right)^2\Bigg\}. \tag{2.45}$$

From (2.21) and (2.14), we find that

$$K(\pi_2)=\left(\frac{\epsilon-3}{4}\right)+\sum_{r=n+1}^{2n+1}\left(\mathcal{B}_{d_1+1d_1+1}^{r}\mathcal{B}_{d_1+2d_1+2}^{r}-\left(\mathcal{B}_{d_1+1d_1+2}^{r}\right)^2\right) \tag{2.46}$$

The equations (2.45) and (2.46) are implied that

$$K(\pi_2)\geq\left(\frac{\epsilon-3}{4}\right)+\sum_{r=n+1}^{2n+1}\left(\mathcal{B}_{d_1+1d_1+1}^{r}\mathcal{B}_{d_1+2d_1+2}^{r}-\left(\mathcal{B}_{d_1+1d_1+2}^{r}\right)^2\right)$$

$$\times \frac{1}{2}\left\{\Psi + \left(\sum_{\alpha=1}^{d_1} \mathcal{B}^{n+1}_{\alpha\alpha}\right)^2 + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} \left(\mathcal{B}^{n+1}_{\alpha\beta}\right)^2 + \sum_{r=n+2}^{2n+1} \sum_{\alpha=,\beta=1}^{n} \left(\mathcal{B}^{r}_{\alpha\beta}\right)^2\right\}. \tag{2.47}$$

Following the method from (2.27) and (2.42), we get the second inequality of Theorem 2.1. On the other hand, for the equality condition, we define two different cases whether the 2-plane π_i is tangent to the first factor or to the second factor. In the first case, we consider $\pi_1 \subset T_x\mathbb{F}_1^{d_1}$, then the equality holds if and only if equalities hold in (2.30), (2.32), (2.38), (2.39), and (2.42), then we get following condition

$$\mathcal{B}^{n+1}_{11} + \mathcal{B}^{n+1}_{22} = \mathcal{B}^{n+1}_{33} = \cdots = \mathcal{B}^{n+1}_{d_1d_1} \tag{2.48}$$

$$\sum_{r=n+2}^{2n+1} \sum_{j=3}^{n} \left(\left(\mathcal{B}^{n+1}_{2j}\right)^2 + \left(\mathcal{B}^{n+1}_{2j}\right)^2\right) + \sum_{r=n+2}^{2n+1} \left(\mathcal{B}^{r}_{11} + \mathcal{B}^{2}_{22}\right)^2 = 0, \tag{2.49}$$

$$\sum_{r=n+1}^{2n+1} \left(\mathcal{B}_{d_1+1d_1+1} + \cdots + \mathcal{B}^{r}_{nn}\right) = \left(\sum_{\alpha=1}^{d_1} \mathcal{B}^{n+1}_{\alpha\alpha}\right)^2 = 0, \tag{2.50}$$

$$\sum_{\substack{k,l=3\\ k\neq l}}^{d_1} \left(\mathcal{B}^{n+1}_{kl}\right)^2 + \sum_{r=n+2}^{2n+1} \sum_{k,l=3}^{d_1} \left(\mathcal{B}^{n+1}_{kl}\right)^2 + \sum_{\alpha=3}^{d_1} \sum_{\beta=d_1+1}^{n} \left(\mathcal{B}^{n+1}_{\alpha\beta}\right)^2 + \sum_{r=n+2}^{2n+1} \sum_{A=3}^{d_1} \sum_{B=d_1+1}^{n} \left(\mathcal{B}^{r}_{AB}\right)^2 = 0. \tag{2.51}$$

Equation (2.50) clearly indicates that the warped product $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ is both $\mathbb{F}_1^{d_1}$-minimal and $\mathbb{F}_2^{d_2}$-minimal warped product Legendrian submanifold in $\mathbb{F}_\epsilon^{2n+1}$. It is concluded that the warped product Legendrian submanifold $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ is minimal in $\mathbb{F}_\epsilon^{2n+1}$. Moreover, we divide the other case into two methods, depending on the vector fields r. Assuming that $r = n + 1$, we define the following

$$\mathcal{B}^{n+1}_{11} + \mathcal{B}^{n+1}_{22} = \mathcal{B}^{n+1}_{33} = \cdots = \mathcal{B}^{n+1}_{d_1d_1} \quad \text{and}$$
$$\sum_{j=3}^{n} \mathcal{B}^{n+1}_{1j} = \sum_{j=3}^{n} \mathcal{B}^{n+1}_{2j} = \sum_{\substack{k,l=3\\ k\neq l}}^{d_1} \left(\mathcal{B}^{n+1}_{kl}\right)^2 = \sum_{\alpha=3}^{d_1} \sum_{\beta=d_1+1}^{n} \left(\mathcal{B}^{n+1}_{\alpha\beta}\right)^2 = 0.$$

Thus, the above condition is equivalent to the following matrices.

$$\text{(i)} \quad A_{e_{n+1}} = \left[\begin{array}{ccccc|ccc}
\mu_1 & \mathcal{B}^{n+1}_{12} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\
\mathcal{B}^{n+1}_{12} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0_{d_11} & 0 & 0 & \cdots & \mu & 0_{d_1d_1+1} & \cdots & 0_{d_1n} \\
\hline
0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}^{n+1}_{d_1+1d_1+1} & \cdots & \mathcal{B}^{n+1}_{d_1+1n} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \mathcal{B}^{n+1}_{nd_1+1} & \cdots & \mathcal{B}^{n+1}_{nn}
\end{array}\right],$$

where $\mu = \mu_1 + \mu_2$ gives the (i) theorem. Similarly, if $r \in \{n+2,\dots,m\}$, then the above condition implies that

$$\begin{aligned}
\mathcal{B}_{11}^{n+1} + \mathcal{B}_{22}^{n+1} &= \sum_{j=3}^{n} \mathcal{B}_{1j}^{n+1} = \sum_{j=3}^{n} \mathcal{B}_{2j}^{n+1} \\
&= \sum_{\substack{k,l=3 \\ k\neq l}}^{d_1} \left(\mathcal{B}_{kl}^{n+1}\right)^2 \\
&= \sum_{\alpha=3}^{d_1} \sum_{\beta=d_1+1}^{n} \left(\mathcal{B}_{\alpha\beta}^{n+1}\right)^2 = 0.
\end{aligned}$$

It is equivalent to the second metric:

$$\text{(ii)} \quad A_{e_r} = \left[\begin{array}{ccccc|ccc}
\mathcal{B}_{11}^r & \mathcal{B}_{12}^r & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\
\mathcal{B}_{21}^r & -\mathcal{B}_{11}^r & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots\vdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0_{d_1 1} & 0 & 0 & \cdots & 0_{d_1 d_1} & 0_{d_1 d_1+1} & \cdots & 0_{d_1 n} \\
\hline
0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}_{d_1+1d_1+1}^{n+1} & \cdots & \mathcal{B}_{d_1+1n}^{n+1} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \mathcal{B}_{nd_1+1}^{n+1} & \cdots & \mathcal{B}_{nn}^{n+1}
\end{array}\right].$$

It is clear that two above conditions show that $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ is mixed totally geodesic Legendrian subamnifold in $\mathbb{F}_\epsilon^{2n+1}$.

Furthermore, the equality sign in (ii) holds if and only if the following two matrices are satisfied:

$$\text{(iii)} \quad A_{e_{n+1}} = \left[\begin{array}{cccc|ccccc}
\mathcal{B}_{11}^{n+1} & \cdots & \cdots & \mathcal{B}_{1d_1}^{n+1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\mathcal{B}_{d_1 11}^{n+1} & \cdots & \cdots & \mathcal{B}_{d_1 d_1}^{n+1} & 0_{d_1 d_1+1} & \cdots & \cdots & \cdots & 0_{d_1 n} \\
\hline
0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mu_1 & \mathcal{B}_{d_1+1d_1+2}^{n+1} & 0 & \cdots & 0_{d_1+1n} \\
\vdots & \ddots & \ddots & \vdots & \mathcal{B}_{d_1+2d_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & \mu
\end{array}\right],$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \dots, 2n+1\}$ thus we have

$$(iv) \quad A_{e_r} = \left[\begin{array}{cccc|ccccc} \mathcal{B}^r_{11} & \cdots & \cdots & \mathcal{B}^r_{1d_1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{B}^r_{d_1 1} & \cdots & \cdots & \mathcal{B}^r_{d_1 d_1} & 0_{d_1 d_1+1} & \cdots & \cdots & \cdots & 0_{d_1 n} \\ \hline 0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}^r_{d_1+1d_1+1} & \mathcal{B}^{n+1}_{d_1+1d_1+2} & 0 & \cdots & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \mathcal{B}^{n+1}_{d_1+2d_1+1} & -\mathcal{B}^r_{d_1+1d_1+1} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & 0 \end{array}\right].$$

From the above, it is also clear that $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ is both $\mathbb{F}_1^{d_1}$-minimal and $\mathbb{F}_2^{d_2}$-minimal warped product Legendrian submanifold in $\mathbb{F}_\epsilon^{2n+1} \times \mathbb{R}$, which implies the minimality of warped product Legendrian submanifold $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_1}$ in $\mathbb{F}_\epsilon^{2n+1}$. □

Warped product manifolds have studied themselves to be a profitable ambient space to obtain a wide range of distinct geometrical properties for immersion. We now find the inequalities for the Riemannian manifold that has constant sectional curvature $\epsilon \in \{1, -3\}$ and can be expressed as a product manifold of $\mathbb{F}_\epsilon^{2n+1}$. We find the following results as follows:

An application for warped product Legendrian submanifold in $\mathbb{H}^{2n+1}$ with $\epsilon = -1$

Theorem 2.2 *Assume that $\phi : \mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ is an isometric immersion of a warped product submanifold $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ into a hyperbolic spaces $\mathbb{H}^{2n+1}$. Then, for each point $x \in \mathbb{F}^n$ and each plane section $\pi_i \subset T_x\mathbb{F}_i^{n_i}$, for $i = 1, 2$, we get the following for*

(a) $\pi_1 \subset T_x\mathbb{F}_1^{d_1}$

$$\delta_{\widetilde{\mathbb{F}}^{d_1}}(x) \leq \frac{n^2}{2}\|\mathbb{H}\|^2 + d_2\big\|\nabla(\ln f)\big\|^2 - d_2\Delta(\ln f) - \left\{\frac{d_1}{2}(d_1 + 2d_2 - 1) - 1\right\}.$$

Equality of the above inequality holds at $x \in \mathbb{F}^n$ if and only if there exists an orthonormal basis $\{e_1 \cdots e_n\}$ of $T_x\mathbb{F}^n$ and orthonormal basis $\{e_{n+1} \cdots e_{2n+1}\}$ of $T_x^\perp$ such that $(a), \pi = \mathrm{Span}\{e_1, e_2\}$, and (b) shape operators take the following form

$$(i) \quad A_{e_{n+1}} = \left[\begin{array}{ccccc|ccc} \mu_1 & \mathcal{B}^{n+1}_{12} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\ \mathcal{B}^{n+1}_{12} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{d_1 1} & 0 & 0 & \cdots & \mu & 0_{d_1 d_1+1} & \cdots & 0_{d_1 n} \\ \hline 0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}^{n+1}_{d_1+1d_1+1} & \cdots & \mathcal{B}^{n+1}_{d_1+1n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \mathcal{B}^{n+1}_{nd_1+1} & \cdots & \mathcal{B}^{n+1}_{nn} \end{array}\right],$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \dots m\}$, then we have the matrix

$$(ii) \quad A_{e_r} = \left[\begin{array}{ccccc|ccc} \mathcal{B}^r_{11} & \mathcal{B}^r_{12} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\ \mathcal{B}^r_{21} & -\mathcal{B}^r_{11} & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots \vdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{d_1 1} & 0 & 0 & \cdots & 0_{d_1 d_1} & 0_{d_1 d_1+1} & \cdots & 0_{d_1 n} \\ \hline 0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}^{n+1}_{d_1+1d_1+1} & \cdots & \mathcal{B}^{n+1}_{d_1+1n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \mathcal{B}^{n+1}_{nd_1+1} & \cdots & \mathcal{B}^{n+1}_{nn} \end{array}\right],$$

(b) *for $\pi_2 \subset T_x\mathbb{F}_2^{d_2}$*

$$\delta_{\widetilde{\mathbb{F}}^{d_2}}(x) \leq \frac{n^2}{2}\|\mathbb{H}\|^2 + d_2\big\|\nabla(\ln f)\big\|^2 - d_2\Delta(\ln f) - \left\{\frac{d_2}{2}(d_2 + 2d_1 - 1) - 1\right\}.$$

The equality of the above equation holds if and only if

$$(iii) \quad A_{e_{n+1}} = \left[\begin{array}{cccc|ccccc} \mathcal{B}^{n+1}_{11} & \cdots & \cdots & \mathcal{B}^{n+1}_{1d_1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{B}^{n+1}_{d_1 11} & \cdots & \cdots & \mathcal{B}^{n+1}_{d_1 d_1} & 0_{d_1 d_1+1} & \cdots & \cdots & \cdots & 0_{d_1 n} \\ \hline 0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mu_1 & \mathcal{B}^{n+1}_{d_1+1d_1+2} & 0 & \cdots & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \mathcal{B}^{n+1}_{d_1+2d_1+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & \mu \end{array}\right],$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \dots, 2n+1\}$, thus we have

$$(iv) \quad A_{e_r} = \left[\begin{array}{cccc|ccccc} \mathcal{B}^r_{11} & \cdots & \cdots & \mathcal{B}^r_{1d_1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{B}^r_{d_1 11} & \cdots & \cdots & \mathcal{B}^r_{d_1 d_1} & 0_{d_1 d_1+1} & \cdots & \cdots & \cdots & 0_{d_1 n} \\ \hline 0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mathcal{B}^r_{d_1+1d_1+1} & \mathcal{B}^{n+1}_{d_1+1d_1+2} & 0 & \cdots & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \mathcal{B}^{n+1}_{d_1+2d_1+1} & -\mathcal{B}^r_{d_1+1d_1+1} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & 0 \end{array}\right].$$

(*v*) *If the equality holds in* (1) *or* (2), *then* $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ *is mixed totally geodesic in space form* $\mathbb{F}_\epsilon^{2n+1}$. *Moreover,* $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ *is both* $\mathbb{F}_1^{d_1}$*-minimal and* $\mathbb{F}_2^{d_2}$*-minimal. Thus,* $\mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ *is a minimal warped product submanifold in Kenmotsu space form* $\mathbb{F}_\epsilon^{2n+1}$.

Proof Now we consider the constant sectional curvature $\epsilon = -1$ and $\mathbb{F}_\epsilon^{2n+1} = \mathbb{H}^{2n+1}$ for the product manifold $\mathbb{H}^{2n+1}$. Then, by inserting the proceeding value in (2.23) and (2.24), we get the required result. □

Some applications to obtain Dirichlet eigenvalue inequalities A crucial part of Riemannian geometry is determining the bound of the eigenvalue of the Laplacian on a particular manifold. The study of eigenvalues that show up as solutions to the Dirichlet boundary value problems for curvature functions is a key goal of this purpose. Due to the diversity of boundary conditions on a manifold, and from the perspective of the Dirichlet boundary condition, one can consider determining the upper bound of the eigenvalue as a method of locating the proper bound of the Laplacian on the particular manifold. Now, if the first eigenvalue of the Dirichlet boundary condition is denoted by $\upsilon_1(\Sigma) > 0$ on a complete noncompact Riemannian manifold $\mathbb{F}^n$ with the compact domain Σ in $\mathbb{F}^n$, then we have

$$\Delta\sigma + \upsilon\sigma = 0, \quad \text{on } \Sigma \quad \text{and} \quad \sigma = 0 \quad \text{on } \partial\Sigma, \tag{2.52}$$

where Δ is the Laplacian on $\mathbb{F}^n$, and σ is a nonzero function defined on $\mathbb{F}^n$. Then, $\upsilon_1(\mathbb{F}^n)$ is expressed as $\inf_\Sigma \upsilon_1(\Sigma)$.

The Dirichlet eigenvalues are the eigenvalues of the Laplace operator on a domain with Dirichlet boundary conditions. They have a number of important consequences in various areas of mathematics, including differential geometry, number theory, and mathematical physics. For example, the Dirichlet eigenvalues determine the geometry of a domain. For example, the first Dirichlet eigenvalue of a domain is related to the diameter of the domain. The higher eigenvalues are related to the curvature of the domain and the way it is embedded in the Euclidean space. Consequently, the Dirichlet eigenvalues appear in the solution of the heat equation on a domain. The eigenvalues and the corresponding eigenfunctions determine the rate of decay of the solution. Assume that f is the nonconstant warping function on compact warped product submanifold $\mathbb{F}^n$, then the minimum principle on υ_1 leads to (see, for instance, [3, 10])

$$\int_{\mathbb{F}^n} \|\nabla\sigma\|^2\, dV \geq \upsilon_1 \int_{\mathbb{F}^n} (\sigma)^2\, dV, \tag{2.53}$$

and the equality is satisfied if and only if

$$\Delta\sigma = \upsilon_1\sigma. \tag{2.54}$$

Implementing the integration along the base manifold $\mathbb{F}^{d_1}$ in Equations (2.23) and (2.24), we get the following result.

Theorem 2.3 *Assume that* $\phi : \mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ *is a compact warped product Legendrian submanifold* $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ *into a Kenmotsu space form* $\widetilde{\mathcal{F}}^{2n+1}(\epsilon)$. *Then we have*

$$\int_{\mathbb{F}_1 \times d_2} \delta_{\mathbb{F}^{d_1}}(x)\, dV \leq \frac{n^2}{2} \int_{\mathbb{F}_1 \times d_2} \|\mathbb{H}\|^2\, dV + d_2 \int_{\mathbb{F}_1 \times d_2} (\ln f)^2\, dV$$

$$+\int_{\mathbb{F}_1\times d_2}\left\{\left(\frac{d_1}{2}(d_1+2d_2-1)-1\right)\left(\frac{\epsilon-3}{4}\right)\right\}dV, \tag{2.55}$$

for $\pi_1\subset T\mathbb{F}_1$. *Moreover, we have*

$$\begin{aligned}\int_{\mathbb{F}_1\times d_2}\delta_{\mathbb{F}^{d_2}}(x)\,dV &\le \frac{n^2}{2}\int_{\mathbb{F}_1\times d_2}\|\mathbb{H}\|^2\,dV + d_2\int_{\mathbb{F}_1\times d_2}(\ln f)^2\,dV\\ &\quad+\int_{\mathbb{F}_1\times d_2}\left\{\left(\frac{d_2}{2}(d_2+2d_1-1)-1\right)\left(\frac{\epsilon-3}{4}\right)\right\}dV,\end{aligned} \tag{2.56}$$

for $\pi_2\subset T\mathbb{F}_2$.

Proof As we know from the Stokes theorem, $\int\Delta\sigma\,dV=0$ for a compact support. Then we use the proceeding condition in (2.23) and (2.24) by replacing $\sigma=\ln f$ and get easily the result. □

An applications for Brochler formulas

Theorem 2.4 *Assume that* $\phi:\mathbb{F}^n=\mathbb{F}_1^{d_1}\times_f\mathbb{F}_2^{d_2}$ *is a compact warped product Legendrian submanifold* $\mathbb{F}^n=\mathbb{F}_1^{d_1}\times_f\mathbb{F}_2^{d_2}$ *into a Kenmotsu space form* $\widetilde{\mathcal{F}}^{2n+1}(\epsilon)$. *Then we have*

$$\begin{aligned}\int_{\mathbb{F}_1\times d_2}\mathrm{Ric}(\nabla\ln f,\nabla\ln f)\,dV &\ge \frac{\upsilon_1}{d_2}\int_{\mathbb{F}_1\times d_2}\delta_{\widetilde{\mathbb{F}}^{d_1}}(x)\,dV-\frac{n^2\upsilon_1}{2d_2}\int_{\mathbb{F}_1\times d_2}\|\mathbb{H}\|^2\,dV\\ &\quad+\frac{\upsilon_1}{d_2}\int_{\mathbb{F}_1\times d_2}\left\{1-\left(\frac{d_1}{2}(d_1+2d_2-1)\right)\right\}\left(\frac{\epsilon-3}{4}\right)dV\\ &\quad-\int_{\mathbb{F}_1\times d_2}\left\|\nabla^2\ln f\right\|^2 dV,\end{aligned} \tag{2.57}$$

for $\pi_1\subset T\mathbb{F}_1$. *Moreover, we have*

$$\begin{aligned}\int_{\mathbb{F}_1\times d_2}\mathrm{Ric}(\nabla\ln f,\nabla\ln f)\,dV &\ge \frac{\upsilon_1}{d_2}\int_{\mathbb{F}_1\times d_2}\delta_{\mathbb{F}^{d_2}}(x)\,dV-\frac{n^2\upsilon_1}{2d_2}\int_{\mathbb{F}_1\times d_2}\|\mathbb{H}\|^2\,dV\\ &\quad+\frac{\upsilon_1}{d_2}\int_{\mathbb{F}_1\times d_2}\left\{1-\left(\frac{d_2}{2}(d_2+2d_1-1)\right)\right\}\left(\frac{\epsilon-3}{4}\right)dV\\ &\quad-\int_{\mathbb{F}_1\times d_2}\left\|\nabla^2\ln f\right\|^2 dV,\end{aligned} \tag{2.58}$$

for $\pi_2\subset T\mathbb{F}_2$.

Proof If σ is the first eigenfunction of the Laplacian $\Delta\sigma=\mathrm{div}(\nabla\sigma)$ for $\mathbb{F}^n$ connected to the first nonzero eigenvalue υ_1, such that, $\Delta\sigma=-\upsilon_1\sigma$, then recalling the Bochner formula [2] that gives the following relation of the differentiable function σ denoted at the Riemannian manifold as:

$$\frac{1}{2}\Delta\|\nabla\sigma\|^2=\left\|\nabla^2\sigma\right\|^2+\mathrm{Ric}(\nabla\sigma,\nabla\sigma)+g\big(\nabla\sigma,\nabla(\Delta\sigma)\big).$$

By the integration of the previous equation using the Stokes theorem, we have

$$\int_{\mathbb{F}_1\times d_2}\left\|\nabla^2\sigma\right\|^2 dV+\int_{\mathbb{F}_1\times d_2}\mathrm{Ric}(\nabla\sigma,\nabla\sigma)\,dV+\int_{\mathbb{F}_1\times d_2}g\big(\nabla\sigma,\nabla(\Delta\sigma)\big)\,dV=0. \tag{2.59}$$

Now, using $\Delta\sigma = \upsilon_1\sigma$ and making some rearrangement in Equation (2.59), we derive

$$\int_{\mathbb{F}_1\times d_2} \|\nabla\sigma\|^2\, dV = \frac{1}{\upsilon_1}\left(\int_{\mathbb{F}_1\times d_2} \left\|\nabla^2\sigma\right\|^2 dV + \int_{\mathbb{F}_1\times d_2} \mathrm{Ric}(\nabla\sigma,\nabla\sigma)\, dV\right). \tag{2.60}$$

Taking integration in (2.23) and (2.24) and inserting the above equation, we get the desired results. □

3 Chen's problem: finding the conditions under which warped products must be minimal

In this section, we provide the partial answer to the Chen problem, that is, the necessary condition for the warped product Legendrian submanifold to be a minimal in Kenmotsu space form $\widetilde{\mathcal{F}}^{2n+1}(\epsilon)$.

Corollary 3.1 *Let $\phi : \mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ be an isometric immersion of a warped product Legendrian submanifold $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ into a Kenmotsu space form $\widetilde{\mathcal{F}}^{2n+1}(\epsilon)$. Then, for each point $x \in \mathbb{F}^n$ and each $\pi_1 \subset T_x\mathbb{F}_1^{n_1}$, we have*

$$\delta_{\widetilde{\mathbb{F}}^{d_1}}(x) + d_2\Delta(\ln f) \leq \left\{\frac{d_1}{2}(d_1 + 2d_2 - 1) - 1\right\}\left(\frac{\epsilon - 3}{4}\right) + d_2\left\|\nabla(\ln f)\right\|^2, \tag{3.1}$$

and if the equality satisfies, then ϕ is minimal.

The second result is:

Corollary 3.2 *Let $\phi : \mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ be an isometric immersion of a warped product Legendrian submanifold $\mathbb{F}^n = \mathbb{F}_1^{d_1} \times_f \mathbb{F}_2^{d_2}$ into a Kenmotsu space form $\widetilde{\mathcal{F}}^{2n+1}(\epsilon)$. Then, for each point $x \in \mathbb{F}^n$ and each $\pi_2 \subset T_x\mathbb{F}_2^{n_2}$, we have*

$$\delta_{\widetilde{\mathbb{F}}^{d_2}}(x) + d_2\Delta(\ln f) \leq \left\{\frac{d_2}{2}(d_2 + 2d_1 - 1) - 1\right\}\left(\frac{\epsilon - 3}{4}\right) + d_2\left\|\nabla(\ln f)\right\|^2. \tag{3.2}$$

and if the equality satisfies, then ϕ is minimal.

4 Conclusion remarks

The Chen delta invariant is a numerical invariant in algebraic topology that measures the extent to which a loop in space fails to be a boundary of a surface. More precisely, if a loop is the boundary of a surface, then the Chen delta invariant is zero. Otherwise, it measures how "far" the loop is from being a boundary. Applications of the delta invariant can be found in various areas of mathematics, including topology, geometry, and algebraic geometry. For example, it has been used to study the topology of moduli spaces of algebraic curves, the geometry of the Kähler-Einstein metric on a complex manifold, and the topology of configuration spaces of particles in a Euclidean space. It has also found applications in physics, particularly in the study of topological field theories [7, 11].

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There is no data used for this manuscript.

Declarations

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The authors declare no competing interests.

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