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Some m-fold symmetric bi-univalent function classes and their associated Taylor-Maclaurin coefficient bounds

Hari Mohan Srivastava[1,2,3], Pishtiwan Othman Sabir[4], Sevtap Sümer Eker[5], Abbas Kareem Wanas[6], Pshtiwan Othman Mohammed[7,8*], Nejmeddine Chorfi[9] and Dumitru Baleanu[10,11*]

*Correspondence:
pshtiwansangawi@gmail.com;
dumitru.baleanu@lau.edu.lb
[7]Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Iraq
[10]Department of Computer Science and Mathematics, Lebanese American University, Beirut 11022801, Lebanon
Full list of author information is available at the end of the article

Abstract

The Ruscheweyh derivative operator is used in this paper to introduce and investigate interesting general subclasses of the function class Σ_m of m-fold symmetric bi-univalent analytic functions. Estimates of the initial Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are obtained for functions of the subclasses introduced in this study, and the consequences of the results are discussed. Additionally, the Fekete-Szegö inequalities for these classes are investigated. The results presented could generalize and improve some recent and earlier works. In some cases, our estimates are better than the existing coefficient bounds. Furthermore, within the engineering domain, the utilization of the Ruscheweyh derivative operator can encompass a broad spectrum of engineering applications, including the robotic manipulation control, optimizing optical systems, antenna array signal processing, image compression, and control system filter design. It emphasizes the potential for innovative solutions that can significantly enhance the reliability and effectiveness of engineering applications.

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1 Introduction

Let $\mathcal{A}$ denote the class of the functions f that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$ of the Taylor-Maclaurin series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Assume that $\mathcal{S}$ is the subclass of $\mathcal{A}$ that contains all univalent functions in $\mathbb{U}$ of the form (1.1), and $\mathcal{P}$ is the subclass of all functions $h(z)$ of the form

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots, \tag{1.2}$$

which is analytic in the open unit disk $\mathbb{U}$ and $\operatorname{Re}(h(z)) > 0$, $z \in \mathbb{U}$.

For a function $f \in \mathcal{A}$ defined by (1.1), the Ruscheweyh derivative operator [1] is defined by

$$\mathcal{R}^{\delta} f(z) = z + \sum_{k=2}^{\infty} \Omega(\delta, k) a_k z^k,$$

where $\delta \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $z \in \mathbb{U}$, and

$$\Omega(\delta, k) = \frac{\Gamma(\delta + k)}{\Gamma(k)\Gamma(\delta + 1)}.$$

The Koebe 1/4-theorem [2] asserts that every univalent function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U}) \quad \text{and} \quad f\big(f^{-1}(w)\big) = w \quad \left(|w| < r_0(f), r_0(f) \geqq \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ has the form

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \big(2a_2^2 - a_3\big) w^3 - \big(5a_2^3 - 5a_2 a_3 + a_4\big) w^4 + \cdots. \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent. The class of bi-univalent functions in $\mathbb{U}$ is denoted by Σ. The following are some examples of functions in the class Σ:

$$\frac{z}{1 - z}, \qquad -\log(1 - z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1 + z}{1 - z}\right),$$

with the corresponding inverse functions:

$$\frac{w}{1 + w}, \qquad \frac{e^w - 1}{e^w} \quad \text{and} \quad \frac{e^{2w} - 1}{e^{2w} + 1},$$

respectively.

Estimates on the bounds of the Taylor-Maclaurin coefficients $|a_n|$ are an important concern problem in geometric function theory because they provides information about the geometric properties of these functions. Lewin [3] studied the class Σ of bi-univalent functions and discovered that $|a_2| < 1.51$ for the functions belonging to the class Σ. Later on, Brannan and Clunie [4] conjectured that $|a_2| \leqq \sqrt{2}$. Subsequently, Netanyahu [5] showed that $\max |a_2| = 4/3$ for $f \in \Sigma$. Recently, many works have appeared devoted to studying the bi-univalent functions class Σ and obtaining non-sharp bounds on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In fact, in their pioneering work, Srivastava et al. [6] have revived and significantly improved the study of the analytic and bi-univalent function class Σ in recent years. They also discovered bounds on $|a_2|$ and $|a_3|$ and were followed by such authors (see, for example, [7–14] and references therein). The coefficient estimates on the bounds of $|a_n|$ $(n \in \{4,5,6,\dots\})$ for a function $f \in \Sigma$ defined by (1.1) remains an unsolved problem. In fact, for coefficients greater than three, there is no natural way to obtain an upper bound. There are a few articles where the Faber polynomial techniques were used to find upper bounds for higher-order coefficients (see, for example, [15–18]).

For each function $f \in \mathcal{S}$, the function

$$h(z) = \left(f\left(z^m\right)\right)^{\frac{1}{m}}, \quad (z \in \mathbb{U}, m \in \mathbb{N}) \tag{1.4}$$

is univalent and maps the unit disk into a region with m-fold symmetry. A function f is said to be m-fold symmetric (see [19]) and is denoted by $\mathcal{A}_m$ if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in \mathbb{U}, m \in \mathbb{N}). \tag{1.5}$$

Assume that $\mathcal{S}_m$ denotes the class of m-fold symmetric univalent functions in $\mathbb{U}$ that are normalized by the series expansion (1.5). In fact, the functions in class $\mathcal{S}$ are 1-fold symmetric. According to Koepf [19], the m-fold symmetric function $h \in \mathcal{P}$ has the form

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots. \tag{1.6}$$

Analogous to the concept of m-fold symmetric univalent functions, Srivastava et al. [20] defined the concept of m-fold symmetric bi-univalent function in a direct way. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. The normalized form of f is given as (1.5), and the extension $g = f^{-1}$ is given by as follows:

$$\begin{aligned} g(w) &= w - a_{m+1} w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots. \end{aligned} \tag{1.7}$$

We denote the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$ by Σ_m. For $m = 1$, the series (1.7) coincides with the series expansion (1.3) of the class Σ. Following are some examples of m-fold symmetric bi-univalent functions:

$$\left[\frac{z^m}{1-z^m}\right]^{\frac{1}{m}}, \qquad \left[-\log\left(1-z^m\right)\right]^{\frac{1}{m}} \quad \text{and} \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}},$$

with the corresponding inverse functions:

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \qquad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}} \quad \text{and} \quad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}},$$

respectively.

Recently, authors have expressed an interest in studying the m-fold symmetric bi-univalent functions class Σ_m (see, for example, [21–24]) and obtaining non-sharp bounds estimates on the first two Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$.

For a function $f \in \mathcal{A}_m$ defined by (1.5), one can think of the m-fold Ruscheweyh derivative operator $\mathcal{R}^\delta : \mathcal{A}_m \to \mathcal{A}_m$, which is analogous to the Ruscheweyh derivative $\mathcal{R}^\delta : \mathcal{A} \to \mathcal{A}$ and can define as follows:

$$\mathcal{R}^\delta f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(\delta + k + 1)}{\Gamma(k+1)\Gamma(\delta+1)} a_{mk+1} z^{mk+1}, \quad (\delta \in \mathbb{N}_0, m \in \mathbb{N}, z \in \mathbb{U}).$$

In engineering, optimizing optical systems and designing effective control systems pose enormous challenges. Describing complex wavefronts necessitates the use of analytic and univalent functions tailored to specific optical constraints, while in signal processing for antenna arrays, employing m-fold symmetric univalent functions is crucial for beamforming amidst electromagnetic wave complexities, demanding innovation and precision. Control systems engineering utilizes univalent functions for filter design, where achieving the desired frequency response must align with system stability and minimal phase distortion, posing a continual challenge. Additionally, modeling complex mechanical systems requires leveraging the Ruscheweyh derivative operator to analyze functions representing system dynamics, facilitating critical parameter identification for system performance optimization. In robotics, univalent functions aid in controlling manipulators while navigating constraints related to joint angles and velocities. Moreover, in image compression and transmission for communication systems, the use of m-fold symmetric bi-univalent functions offers the potential for optimizing compression ratios while preserving image quality, representing an ongoing engineering challenge (see, for example, [25, 26]).

This paper aims to introduce new general subclasses of m-fold symmetric bi-univalent functions in $\mathbb{U}$ applying the m-fold Ruscheweyh derivative operator, obtain estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in subclasses $\mathcal{Q}_{\Sigma_m}(\tau, \lambda, \gamma, \delta; \alpha)$ and $\Theta_{\Sigma_m}(\tau, \lambda, \gamma, \delta; \beta)$, and improve many recent works. Moreover, we have derived the Fekete-Szegö inequalities for these classes. To derive our main results, we need to use the following lemmas that will be useful in proving the basic theorems in Sects. 2 and 3.

Lemma 1 [2] *If $h \in \mathcal{P}$ with $h(z)$ given by* (1.2), *then*

$$|h_k| \leq 2, \quad k \in \mathbb{N}.$$

Lemma 2 [27] *If $h \in \mathcal{P}$ with $h(z)$ given by* (1.2) *and μ is a complex number, then*

$$\left|h_2 - \mu h_1^2\right| \leq 2 \max\left\{1, |2\mu - 1|\right\}.$$

2 Coefficient bounds for the function class $\mathcal{Q}_{\Sigma_m}(\delta, \lambda, \gamma, n; \alpha)$

In this section, we assume that

$$\lambda \geqq 0, \qquad 0 \leqq \gamma \leqq 1, \qquad 0 < \alpha \leqq 1, \qquad \tau \in \mathbb{C} \setminus \{0\}, \qquad \delta \in \mathbb{N}_0 \quad \text{and} \quad m \in \mathbb{N}.$$

For a function $h \in \mathcal{P}$ given by (1.2). If $\mathcal{K}(z)$ is any complex-valued function such that $\mathcal{K}(z) = [h(z)]^\alpha$, then

$$\left|\arg\left(\mathcal{K}(z)\right)\right| = \alpha \left|\arg\left(h(z)\right)\right| < \frac{\alpha\pi}{2}.$$

Definition 1 A function $f \in \Sigma_m$ given by (1.5) is called in the class $Q_{\Sigma_m}(\tau, \lambda, \gamma, \delta; \alpha)$ if it satisfies the following conditions:

$$\left| \arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^\delta f(z)}{z} + \big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^\delta f(z)\big)' + \lambda\gamma\Big(z\big(\mathcal{R}^\delta f(z)\big)'' - 2\Big) - 1 \right]\right) \right| < \frac{\alpha\pi}{2}, \tag{2.1}$$

and

$$\left| \arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^\delta g(w)}{z} + \big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^\delta g(w)\big)' + \lambda\gamma\Big(w\big(\mathcal{R}^\delta g(w)\big)'' - 2\Big) - 1 \right]\right) \right| < \frac{\alpha\pi}{2}, \tag{2.2}$$

where $z, w \in \mathbb{U}$ and the function $g = f^{-1}$ is given by (1.7).

Theorem 1 *Let $f \in Q_{\Sigma_m}(\tau, \lambda, \gamma, \delta; \alpha)$ be given by (1.5). Then,*

$$|a_{m+1}| \leqq \frac{2\sqrt{2}|\tau|\alpha}{\sqrt{(\delta+1)|\tau\alpha(\delta+2)(m+1)\Phi_1(\lambda,\gamma,m) + 2(1-\alpha)(\delta+1)\Phi_2(\lambda,\gamma,m)|}}, \tag{2.3}$$

and

$$|a_{2m+1}| \leqq \frac{2|\tau|\alpha}{(\delta+1)(\delta+2)\Phi_1(\lambda,\gamma,m)} + \frac{2|\tau|^2\alpha^2(m+1)}{(\delta+1)^2\Phi_2(\lambda,\gamma,m)}, \tag{2.4}$$

where

$$\Phi_1(\lambda,\gamma,m) = 1 + 2(\lambda+\gamma)m + \lambda\gamma\big((2m+1)^2+1\big), \tag{2.5}$$

and

$$\Phi_2(\lambda,\gamma,m) = \big(1 + (\lambda+\gamma)m + \lambda\gamma\big((m+1)^2+1\big)\big)^2. \tag{2.6}$$

Proof It follows from (2.1) and (2.2) that

$$1 + \frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^\delta f(z)}{z} + \big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^\delta f(z)\big)' + \lambda\gamma\Big(z\big(\mathcal{R}^\delta f(z)\big)'' - 2\Big) - 1 \right] = \big[p(z)\big]^\alpha, \tag{2.7}$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^\delta g(w)}{z} + \big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^\delta g(w)\big)' + \lambda\gamma\Big(w\big(\mathcal{R}^\delta g(w)\big)'' - 2\Big) - 1 \right] = \big[q(w)\big]^\alpha, \tag{2.8}$$

where $p, q \in \mathcal{P}$ have the following representations

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots, \tag{2.9}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{2.10}$$

Clearly, we have

$$\begin{aligned}\left[p(z)\right]^{\alpha} &= 1 + \alpha p_m z^m + \left(\frac{1}{2}\alpha(\alpha-1)p_m^2 + \alpha p_{2m}\right)z^{2m} \\ &\quad + \left(\frac{1}{6}\alpha(\alpha-1)(\alpha-2)p_m^3 + \alpha(1-\alpha)p_m p_{2m} + \alpha p_{3m}\right)z^{3m} + \cdots,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\left[q(w)\right]^{\alpha} &= 1 + \alpha q_m w^m + \left(\frac{1}{2}\alpha(\alpha-1)q_m^2 + \alpha q_{2m}\right)w^{2m} \\ &\quad + \left(\frac{1}{6}\alpha(\alpha-1)(\alpha-2)q_m^{3m} + \alpha(1-\alpha)q_m q_{2m} + \alpha q_{3m}\right)w^{3m} + \cdots.\end{aligned} \tag{2.12}$$

We also find that

$$\begin{aligned}&1 + \frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^{\delta} f(z)}{z} + \big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^{\delta} f(z)\big)' + \lambda\gamma\left(z\big(\mathcal{R}^{\delta} f(z)\big)'' - 2\right) - 1\right] \\ &= 1 + \frac{(1 + m(\lambda+\gamma) + \lambda\gamma((m+1)^2+1))(\delta+1)}{\tau} a_{m+1} z^m \\ &\quad + \frac{(1 + 2m(\lambda+\gamma) + \lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)}{2\tau} a_{2m+1} z^{2m} + \cdots,\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&1 + \frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^{\delta} g(w)}{z} + \big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^{\delta} g(w)\big)' + \lambda\gamma\left(w\big(\mathcal{R}^{\delta} g(w)\big)'' - 2\right) - 1\right] \\ &= 1 - \frac{(1 + m(\lambda+\gamma) + \lambda\gamma((m+1)^2+1))(\delta+1)}{\tau} a_{m+1} w^m \\ &\quad + \frac{(1 + 2m(\lambda+\gamma) + \lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)}{2\tau} \\ &\quad \times \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m} + \cdots.\end{aligned} \tag{2.14}$$

Comparing the corresponding coefficients of (2.13) and (2.14) yields

$$\frac{(1 + m(\lambda+\gamma) + \lambda\gamma((m+1)^2+1))(\delta+1)}{\tau} a_{m+1} = \alpha p_m, \tag{2.15}$$

$$\frac{(1 + 2m(\lambda+\gamma) + \lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)}{2\tau} a_{2m+1} = \frac{\alpha(\alpha-1)}{2} p_m^2 + \alpha p_{2m}, \tag{2.16}$$

$$-\frac{(1 + m(\lambda+\gamma) + \lambda\gamma((m+1)^2+1))(\delta+1)}{\tau} a_{m+1} = \alpha q_m, \tag{2.17}$$

and

$$\frac{(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)}{2\tau}\left[(m+1)a_{m+1}^2-a_{2m+1}\right]$$
$$=\frac{\alpha(\alpha-1)}{2}q_m^2+\alpha q_{2m}. \tag{2.18}$$

In view of (2.15) and (2.17), we find that

$$p_m=-q_m, \tag{2.19}$$

and

$$\frac{2(1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1))^2(\delta+1)^2}{\tau^2}a_{m+1}^2=\alpha^2\left(p_m^2+q_m^2\right). \tag{2.20}$$

Adding (2.16) to (2.18) and substituting the value of $p_m^2+q_m^2$ form (2.20), we obtain

$$\frac{(\delta+1)(\delta+2)(m+1)\Phi_1(\lambda,\gamma,m)}{2\tau}a_{m+1}^2$$
$$=\frac{(\alpha-1)(\delta+1)^2\Phi_2(\lambda,\gamma,m)}{\tau^2\alpha}a_{m+1}^2+\alpha(p_{2m}+q_{2m}), \tag{2.21}$$

where $\Phi_1(\lambda,\gamma,m)$ and $\Phi_2(\lambda,\gamma,m)$ are given by (2.5) and (2.6), respectively.

Further computations using (2.21) yield

$$a_{m+1}^2=\frac{2\tau^2\alpha^2(p_{2m}+q_{2m})}{(\delta+1)[\tau\alpha(\delta+2)(m+1)\Phi_1(\lambda,\gamma,m)+2(1-\alpha)(\delta+1)\Phi_2(\lambda,\gamma,m)]}. \tag{2.22}$$

Taking the absolute value of (2.22) and applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we deduce that

$$|a_{m+1}|\leqq\frac{2\sqrt{2}|\tau|\alpha}{\sqrt{(\delta+1)|\tau\alpha(\delta+2)(m+1)\Phi_1(\lambda,\gamma,m)+2(1-\alpha)(\delta+1)\Phi_2(\lambda,\gamma,m)|}}.$$

Next, to determine the bound on $|a_{2m+1}|$, by subtracting (2.18) from (2.16), we obtain

$$\frac{(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)}{\tau}a_{2m+1}$$
$$-\frac{(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)(m+1)}{2\tau}a_{m+1}^2$$
$$=\frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right)+\alpha(p_{2m}-q_{2m}). \tag{2.23}$$

Now, substituting the value of a_{m+1}^2 from (2.20) into (2.23) and using (2.19), we conclude that

$$a_{2m+1}=\frac{\tau\alpha(p_{2m}-q_{2m})}{2(\delta+1)(\delta+2)\Phi_1(\lambda,\gamma,m)}+\frac{\tau^2\alpha^2(m+1)(p_m^2+q_m^2)}{4(\delta+1)^2\Phi_2(\lambda,\gamma,m)}. \tag{2.24}$$

Finally, taking the absolute value of (2.24) and applying Lemma 1 once again for the coefficients p_m, p_{2m}, q_m, and q_{2m}, we deduce that

$$|a_{2m+1}| \leqq \frac{2|\tau|\alpha}{(\delta+1)(\delta+2)\Phi_1(\lambda,\gamma,m)} + \frac{2|\tau|^2\alpha^2(m+1)}{(\delta+1)^2\Phi_2(\lambda,\gamma,m)}.$$

This completes the proof. □

Theorem 2 *Let $f \in Q_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\alpha)$ be given by (1.5). Then,*

$$\begin{aligned} \left|a_{2m+1} - \mu a_{m+1}^2\right| \leq{}& \frac{\alpha|\tau||4\mu\tau\alpha\sigma_2 - \sigma_1|}{\sigma_1\sigma_2} \max\left\{1, \left|\frac{\tau\alpha\sigma_1\sigma_2(m+1)}{(4\mu\tau\alpha\sigma_2-\sigma_1)\sigma_3} - 1\right|\right\} \\ &+ \frac{\alpha|\tau||4\mu\tau\alpha\sigma_2 + \sigma_1|}{\sigma_1\sigma_2} \max\left\{1, \left|\frac{\tau\alpha\sigma_1\sigma_2(m+1)}{(4\mu\tau\alpha\sigma_2+\sigma_1)\sigma_3} - 1\right|\right\} \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} \sigma_1 ={}& (\delta+1)\big[\tau\alpha(\delta+2)(m+1)\big(1+2(\lambda+\gamma)m+\lambda\gamma\big((2m+1)^2+1\big)\big) \\ &+ 2(1-\alpha)(\delta+1)\big(1+(\lambda+\gamma)m+\lambda\gamma\big((m+1)^2+1\big)\big)^2\big], \end{aligned} \tag{2.26}$$

$$\sigma_2 = (\delta+1)(\delta+2)\big(1+2(\lambda+\gamma)m+\lambda\gamma\big((2m+1)^2+1\big)\big), \tag{2.27}$$

and

$$\sigma_3 = (\delta+1)^2\big(1+(\lambda+\gamma)m+\lambda\gamma\big((m+1)^2+1\big)\big)^2. \tag{2.28}$$

Proof For $\mu \in \mathbb{C}$, using equations (2.22) and (2.24) and arranging, we find

$$\begin{aligned} a_{2m+1} - \mu a_{m+1}^2 ={}& \left(\frac{\alpha\tau}{2\sigma_2} - \frac{2\mu\tau^2\alpha^2}{\sigma_1}\right)p_{2m} + \frac{\tau^2\alpha^2(m+1)}{4\sigma_3}p_m^2 \\ &- \left(\frac{\alpha\tau}{2\sigma_2} + \frac{2\mu\tau^2\alpha^2}{\sigma_1}\right)q_{2m} + \frac{\tau^2\alpha^2(m+1)}{4\sigma_3}q_m^2 \end{aligned} \tag{2.29}$$

where σ_1, σ_2, and σ_3 are given by (2.26), (2.27), and (2.28), respectively.

Further computations using (2.29) yield

$$\begin{aligned} a_{2m+1} - \mu a_{m+1}^2 ={}& \frac{\alpha\tau(4\mu\tau\alpha\sigma_2-\sigma_1)}{2\sigma_1\sigma_2}\left[p_{2m} - \frac{\tau\alpha\sigma_1\sigma_2(m+1)}{2(4\mu\tau\alpha\sigma_2-\sigma_1)\sigma_3}p_m^2\right] \\ &- \frac{\alpha\tau(4\mu\tau\alpha\sigma_2+\sigma_1)}{2\sigma_1\sigma_2}\left[q_{2m} - \frac{\tau\alpha\sigma_1\sigma_2(m+1)}{2(4\mu\tau\alpha\sigma_2+\sigma_1)\sigma_3}q_m^2\right]. \end{aligned} \tag{2.30}$$

If we take

$$\mu_1 = \frac{\tau\alpha\sigma_1\sigma_2(m+1)}{2(4\mu\tau\alpha\sigma_2-\sigma_1)\sigma_3} \tag{2.31}$$

and

$$\mu_2 = \frac{\tau\alpha\sigma_1\sigma_2(m+1)}{2(4\mu\tau\alpha\sigma_2+\sigma_1)\sigma_3}, \tag{2.32}$$

then from (2.30), we get

$$\left|a_{2m+1}-\mu a_{m+1}^2\right| \leq \frac{\alpha|\tau||4\mu\tau\alpha\sigma_2-\sigma_1|}{2\sigma_1\sigma_2}\left|p_{2m}-\mu_1 p_m^2\right| + \frac{\alpha|\tau||4\mu\tau\alpha\sigma_2+\sigma_1|}{2\sigma_1\sigma_2}\left|q_{2m}-\mu_2 q_m^2\right|. \tag{2.33}$$

Hence, applying Lemmas 2 and (2.33) yields the Fekete-Szegö inequality for the class $Q_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\alpha)$, as given by (2.25). □

3 Coefficient bounds for the function class $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$

In this section, we assume that

$$\lambda \geqq 0, \quad 0\leqq\gamma\leqq 1, \quad 0\leqq\beta<1, \quad \tau\in\mathbb{C}\backslash\{0\}, \quad \delta\in\mathbb{N}_0 \quad \text{and} \quad m\in\mathbb{N}.$$

If $\mathcal{L}(z)$ is any complex-valued function such that $\mathcal{L}(z)=\beta+(1-\beta)h(z)$, then

$$\operatorname{Re}\bigl(\mathcal{L}(z)\bigr)=\beta+(1-\beta)\operatorname{Re}\bigl(h(z)\bigr)>\beta.$$

Definition 2 A function $f\in\Sigma_m$ given by (1.5) is called in the class $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ if it satisfies the following conditions:

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^\delta f(z)}{z}+\bigl(\lambda(\gamma+1)+\gamma\bigr)\bigl(\mathcal{R}^\delta f(z)\bigr)' + \lambda\gamma\Bigl(z\bigl(\mathcal{R}^\delta f(z)\bigr)''-2\Bigr)-1\right]\right)>\beta, \tag{3.1}$$

and

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^\delta g(w)}{z}+\bigl(\lambda(\gamma+1)+\gamma\bigr)\bigl(\mathcal{R}^\delta g(w)\bigr)' + \lambda\gamma\Bigl(w\bigl(\mathcal{R}^\delta g(w)\bigr)''-2\Bigr)-1\right]\right)>\beta, \tag{3.2}$$

where $z,w\in\mathbb{U}$ and the function $g=f^{-1}$ is given by (1.7).

Theorem 3 *Let $f\in\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ be given by (1.5). Then,*

$$|a_{m+1}|\leqq\min\left\{\frac{2|\tau|(1-\beta)}{(\delta+1)(1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1))}, 2\sqrt{\frac{2|\tau|(1-\beta)}{(\delta+1)(\delta+2)(m+1)\Phi_1(\lambda,\gamma,m)}}\right\}, \tag{3.3}$$

and

$$|a_{2m+1}|\leqq\frac{4|\tau|(1-\beta)}{(\delta+1)(\delta+2)(1+2(\lambda+\gamma)m+\lambda\gamma((2m+1)^2+1))} \tag{3.4}$$

where $\Phi_1(\lambda,\gamma,m)$ is defined by (2.5).

Proof It follows from (3.1) and (3.2) that

$$1+\frac{1}{\tau}\bigg[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^{\delta}f(z)}{z}+\big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^{\delta}f(z)\big)' + \lambda\gamma\big(z\big(\mathcal{R}^{\delta}f(z)\big)''-2\big)-1\bigg]=\beta+(1-\beta)p(z), \tag{3.5}$$

and

$$1+\frac{1}{\tau}\bigg[(1-\lambda)(1-\gamma)\frac{\mathcal{R}^{\delta}g(w)}{z}+\big(\lambda(\gamma+1)+\gamma\big)\big(\mathcal{R}^{\delta}g(w)\big)' + \lambda\gamma\big(w\big(\mathcal{R}^{\delta}g(w)\big)''-2\big)-1\bigg]=\beta+(1-\beta)q(z), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.9) and (2.10), respectively.

Clearly, we have

$$\beta+(1-\beta)p(z)=1+(1-\beta)p_m z^m+(1-\beta)p_{2m}z^{2m}+(1-\beta)p_{3m}z^{3m}+\cdots \tag{3.7}$$

and

$$\beta+(1-\beta)q(w)=1+(1-\beta)q_m w^m+(1-\beta)q_{2m}w^{2m}+(1-\beta)q_{3m}w^{3m}+\cdots. \tag{3.8}$$

Equating the corresponding coefficients of (3.5) and (3.6) yields

$$\frac{(1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1))(\delta+1)}{\tau}a_{m+1}=(1-\beta)p_m, \tag{3.9}$$

$$\frac{(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)}{2\tau}a_{2m+1}=(1-\beta)p_{2m}, \tag{3.10}$$

$$-\frac{(1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1))(\delta+1)}{\tau}a_{m+1}=(1-\beta)q_m, \tag{3.11}$$

and

$$\frac{(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))(\delta+1)(\delta+2)}{2\tau}\big[(m+1)a_{m+1}^2-a_{2m+1}\big] = (1-\beta)q_{2m}. \tag{3.12}$$

In view of (3.9) and (3.11), we find that

$$p_m=-q_m, \tag{3.13}$$

and

$$\frac{2(1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1))^2(\delta+1)^2}{\tau^2}a_{m+1}^2=(1-\beta)^2\big(p_m^2+q_m^2\big). \tag{3.14}$$

Adding (3.10) to (3.12), we obtain

$$\frac{(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))(m+1)(\delta+1)(\delta+2)}{2\tau}a_{m+1}^2$$
$$=(1-\beta)(p_{2m}+q_{2m}). \tag{3.15}$$

Hence, we find from (3.14) and (3.15) that

$$a_{m+1}^2=\frac{\tau^2(1-\beta)^2(p_m^2+q_m^2)}{2(\delta+1)^2(1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1))^2}, \tag{3.16}$$

and

$$a_{m+1}^2=\frac{2\tau(1-\beta)(p_{2m}+q_{2m})}{(\delta+1)(\delta+2)(m+1)(1+2(\lambda+\gamma)m+\lambda\gamma((2m+1)^2+1))}, \tag{3.17}$$

respectively. By taking the absolute value of (3.16) and (3.17) and applying Lemma 1 for the coefficients p_m, p_{2m}, q_m, and q_{2m}, we deduce that

$$|a_{m+1}|\leqq\frac{2|\tau|(1-\beta)}{(\delta+1)(1+(\lambda+\gamma)m+\lambda\gamma((m+1)^2+1))},$$

and

$$|a_{m+1}|\leqq 2\sqrt{\frac{2|\tau|(1-\beta)}{(\delta+1)(\delta+2)(m+1)(1+2(\lambda+\gamma)m+\lambda\gamma((2m+1)^2+1))^2}},$$

respectively. To determine the bound on $|a_{2m+1}|$, by subtracting (3.12) from (3.10), we get

$$\frac{(\delta+1)(\delta+2)(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))}{\tau}a_{2m+1}$$
$$-\frac{(\delta+1)(\delta+2)(m+1)(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))}{2\tau}a_{m+1}^2$$
$$=(1-\beta)(p_{2m}-q_{2m}). \tag{3.18}$$

Upon substituting the value of a_{m+1}^2 from (3.16) and (3.17) into (3.18), we conclude that

$$a_{2m+1}=\frac{\tau^2(1-\beta)^2(m+1)(p_m^2+q_m^2)}{4(\delta+1)^2(1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1))^2}+\frac{\tau(1-\beta)(p_{2m}-q_{2m})}{(\delta+1)(\delta+2)\Phi_1(\lambda,\gamma,m)} \tag{3.19}$$

and

$$a_{2m+1}=\frac{2\tau(1-\beta)p_{2m}}{(\delta+1)(\delta+2)(1+2(\lambda+\gamma)m+\lambda\gamma((2m+1)^2+1))}. \tag{3.20}$$

Now, taking the absolute value of (3.19) and (3.20) and applying Lemma 1 once again for the coefficients p_m, p_{2m}, q_m, and q_{2m}, we deduce that

$$|a_{2m+1}|\leqq\frac{2|\tau|^2(1-\beta)^2(m+1)}{(\delta+1)^2(1+(\lambda+\gamma)m+\lambda\gamma((m+1)^2+1))^2}+\frac{4|\tau|(1-\beta)}{(\delta+1)(\delta+2)\Phi_1(\lambda,\gamma,m)},$$

and

$$|a_{2m+1}| \leqq \frac{4|\tau|(1-\beta)}{(\delta+1)(\delta+2)(1+2(\lambda+\gamma)m+\lambda\gamma((2m+1)^2+1))},$$

respectively. This completes the proof. □

Next, we derive the Fekete-Szegö inequality for the class $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$.

Theorem 4 *Let $f \in \Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ be given by (1.5). Then,*

$$\begin{aligned}\left|a_{2m+1}-\mu a_{m+1}^2\right| \leq \frac{2|\tau|(1-\beta)}{\rho_1}\Bigg[&\max\left\{1,\left|\frac{\tau(1-\beta)(2\mu-m-1)}{2\rho_2}-1\right|\right\}\\ &+\max\left\{1,\left|\frac{\tau(1-\beta)(1+m-2\mu)}{2\rho_2}-1\right|\right\}\Bigg]\end{aligned} \tag{3.21}$$

where

$$\rho_1=(\delta+1)(\delta+2)\big(1+2(\lambda+\gamma)m+\lambda\gamma\big((2m+1)^2+1\big)\big) \tag{3.22}$$

and

$$\rho_2=(\delta+1)^2\big(1+m(\lambda+\gamma)+\lambda\gamma\big((m+1)^2+1\big)\big)^2. \tag{3.23}$$

Proof For $\nu \in \mathbb{C}$, using equations (3.16) and (3.19) and arranging, we have

$$\begin{aligned}a_{2m+1}-\nu a_{m+1}^2 = \frac{\tau(1-\beta)}{\rho_1}&\left[p_{2m}-\frac{\tau(1-\beta)(2\nu-m-1)\rho_1}{4\rho_2}p_m^2\right]\\ &-\frac{\tau(1-\beta)}{\rho_1}\left[q_{2m}-\frac{\tau(1-\beta)(1+m-2\nu)\rho_1}{4\rho_2}q_m^2\right]\end{aligned} \tag{3.24}$$

where ρ_1 and ρ_2 are given by (2.29) and (3.23), respectively.

If we take

$$\nu_1=\frac{\tau(1-\beta)(2\nu-m-1)\rho_1}{4\rho_2}$$

and

$$\nu_2=\frac{\tau(1-\beta)(1+m-2\nu)\rho_1}{4\rho_2},$$

then from (3.24), we get

$$\left|a_{2m+1}-\nu a_{m+1}^2\right| \leq \frac{|\tau|(1-\beta)}{\rho_1}\left|p_{2m}-\nu_1 p_m^2\right|+\frac{|\tau|(1-\beta)}{\rho_1}\left|q_{2m}-\nu_2 q_m^2\right|. \tag{3.25}$$

Hence, our result follows from (3.25) by applying Lemma 2. □

4 Corollaries and consequences

This section is devoted to demonstrating of some special cases of the definitions and theorems. These results are given in the form of remarks and corollaries.

Remark 1 It should be noted that the classes $Q_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\alpha)$ and $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ are generalizations of well-known classes considered earlier. These classes are:

1. For $\delta=\gamma=0$ and $\tau=\lambda=1$, the classes $\mathcal{Q}_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\alpha)$ and $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{H}^{\alpha}_{\Sigma,m}$ and $\mathcal{H}_{\Sigma,m}(\beta)$, respectively, which were given by Srivastava et al. [20].
2. For $\delta=\gamma=0$ and $\tau=1$, the classes $\mathcal{Q}_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\alpha)$ and $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{A}^{\alpha,\lambda}_{\Sigma,m}$ and $\mathcal{A}^{\lambda}_{\Sigma,m}(\beta)$, respectively, which were recently investigated by Eker [21].
3. For $\gamma=0$ and $\tau=1$, the class $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ reduces to the class $\Xi_{\Sigma_m}(\lambda,\delta;\beta)$, which was studied by Sabir et al. [28].
4. For $\delta=0$, the classes $Q_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\alpha)$ and $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{WS}_{\Sigma_m}(\lambda,\gamma,\tau;\alpha)$ and $\mathcal{WS}^{*}_{\Sigma_m}(\lambda,\gamma,\tau;\beta)$, respectively, which were considered recently by Srivastava and Wanas [29].
5. For $\delta=\gamma=0$, the classes $\mathcal{Q}_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\alpha)$ and $\Theta_{\Sigma_m}(\tau,\lambda,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{B}_{\Sigma_m}(\tau,\lambda;\alpha)$ and $\mathcal{B}^{*}_{\Sigma_m}(\tau,\lambda;\beta)$, respectively, which were recently introduced by Srivastava et al. [30].

Remark 2 In Theorem 1, if we choose

1. $\delta=0$, then we obtain the results, which were proven by Srivastava and Wanas [29, Theorem 2.1].
2. $\delta=0$ and $\gamma=0$, then we obtain the results, which were given by Srivastava et al. [30, Theorem 2.1].
3. $\delta=0$, $\gamma=0$ and $\tau=1$, then we obtain the results, which were obtained by Eker [21, Theorem 1].
4. $\delta=0$, $\gamma=0$, $\lambda=1$ and $\tau=1$, then we obtain the results, which were proven by Srivastava et al. [20, Theorem 2].

By taking $\delta=0$ in Theorem 3, we conclude the following result.

Corollary 1 *Let $f\in\Theta_{\Sigma_m}(\tau,\lambda,\gamma;\beta)$ be given by* (1.5). *Then,*

$$|a_{m+1}|\leqq\min\left\{\frac{2|\tau|(1-\beta)}{1+m(\lambda+\gamma)+\lambda\gamma((m+1)^2+1)},\right.$$
$$\left.2\sqrt{\frac{|\tau|(1-\beta)}{(m+1)(1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1))}}\right\},$$

and

$$|a_{2m+1}|\leqq\frac{2|\tau|(1-\beta)}{1+2m(\lambda+\gamma)+\lambda\gamma((2m+1)^2+1)}.$$

Remark 3 The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 1 are better than those given in [29, Theorem 3.1].

By taking $\gamma = 0$ in Corollary 1, we conclude the following result.

Corollary 2 *Let $f \in \Theta_{\Sigma_m}(\tau, \lambda; \beta)$ be given by* (1.5). *Then,*

$$|a_{m+1}| \leqq \min\left\{\frac{2|\tau|(1-\beta)}{1+m\lambda}, 2\sqrt{\frac{|\tau|(1-\beta)}{(m+1)(1+2m\lambda)}}\right\},$$

and

$$|a_{2m+1}| \leqq \frac{2|\tau|(1-\beta)}{1+2m\lambda}.$$

Remark 4 The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 2 are better than those given in [30, Theorem 3.1].

By setting $\gamma = 0$ and $\tau = 1$ in Corollary 1, we conclude the following result.

Corollary 3 *Let $f \in \Theta_{\Sigma_m}(\lambda; \beta)$ be given by* (1.5). *Then,*

$$|a_{m+1}| \leqq \min\left\{\frac{2(1-\beta)}{1+m\lambda}, 2\sqrt{\frac{(1-\beta)}{(m+1)(1+2m\lambda)}}\right\},$$

and

$$|a_{2m+1}| \leqq \frac{2(1-\beta)}{1+2m\lambda}.$$

Remark 5 The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3 are better than those given in [21, Theorem 2].

By setting $\gamma = 0$ and $\lambda = \tau = 1$ in Corollary 1, we conclude the following result.

Corollary 4 *Let $f \in \Theta_{\Sigma_m}(\beta)$ be given by* (1.5). *Then,*

$$|a_{m+1}| \leqq \min\left\{\frac{2(1-\beta)}{1+m}, 2\sqrt{\frac{(1-\beta)}{(m+1)(1+2m)}}\right\},$$

and

$$|a_{2m+1}| \leqq \frac{2(1-\beta)}{1+2m}.$$

Remark 6 The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4 are better than those given in [20, Theorem 3].

Remark 7 For 1-fold symmetric bi-univalent functions, the classes $\mathcal{Q}_{\Sigma_1}(\tau, \lambda, \gamma, \delta; \alpha) \equiv \mathcal{Q}_{\Sigma}(\tau, \lambda, \gamma, \delta; \alpha)$ and $\Theta_{\Sigma_1}(\tau, \lambda, \gamma, \delta; \beta) \equiv \Theta_{\Sigma}(\tau, \lambda, \gamma, \delta; \beta)$ are special cases of these classes illustrated below:

1. For $\delta = 0$, the classes $\mathcal{Q}_{\Sigma}(\tau, \lambda, \gamma, \delta; \alpha)$ and $\Theta_{\Sigma}(\tau, \lambda, \gamma, \delta; \beta)$ reduce to the classes $\mathcal{WS}_{\Sigma}(\lambda, \gamma, \tau; \alpha)$ and $\mathcal{WS}^*_{\Sigma}(\lambda, \gamma, \tau; \beta)$, respectively, which were recently introduced by Srivastava and Wanas [29].

2. For $\delta = \gamma = 0$ and $\tau = 1$, the classes $Q_\Sigma(\tau,\lambda,\gamma,\delta;\alpha)$ and $\Theta_\Sigma(\tau,\lambda,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{B}_\Sigma(\alpha,\lambda)$ and $\mathcal{B}_\Sigma(\beta,\lambda)$, respectively, which were recently investigated by Frasin and Aouf [8].
3. For $\delta = \gamma = 0$ and $\tau = \lambda = 1$, the classes $Q_\Sigma(\tau,\lambda,\gamma,\delta;\alpha)$ and $\Theta_\Sigma(\tau,\lambda,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{H}_\Sigma(\alpha)$ and $\mathcal{H}_\Sigma(\beta)$, respectively, which were given by Srivastava et al. [6].

For 1-fold symmetric bi-univalent functions, Theorem 1 reduces to the following corollary:

Corollary 5 *Let*

$$f \in \mathcal{Q}_\Sigma(\tau,\lambda,\gamma,\delta;\alpha) \quad \big(\lambda \geqq 0, 0 \leqq \gamma \leqq 1, 0 < \alpha \leqq 1, \tau \in \mathbb{C}\backslash\{0\}, \delta \in \mathbb{N}_0\big)$$

be given by (1.1). *Then,*

$$|a_2| \leqq \frac{2|\tau|\alpha}{\sqrt{(\delta+1)|\tau\alpha(\delta+2)(1+2(\lambda+\gamma+5\lambda\gamma)) + (1-\alpha)(\delta+1)(1+\lambda+\gamma+5\lambda\gamma)^2|}},$$

and

$$|a_3| \leqq \frac{2|\tau|\alpha}{(\delta+1)(\delta+2)(1+2(\lambda+\gamma+5\lambda\gamma))} + \frac{4|\tau|^2\alpha^2}{(\delta+1)^2(1+\lambda+\gamma+5\lambda\gamma)^2}.$$

Remark 8 In Corollary 5, if we choose

1. $\delta = 0$, then we obtain the results, which were given by Srivastava and Wanas [29, Corollary 2.1].
2. $\delta = 0$, $\gamma = 0$ and $\tau = 1$, then we obtain the results, which were proven by Frasin and Aouf [8, Theorem 2.2].
3. $\delta = 0$, $\gamma = 0$, $\lambda = 1$ and $\tau = 1$, then we obtain the results, which were obtained by Srivastava et al. [6, Theorem 1].

For 1-fold symmetric bi-univalent functions, Theorem 3 reduces to the following corollary:

Corollary 6 *Let*

$$f \in \Theta_\Sigma(\tau,\lambda,\gamma,\delta;\beta) \quad \big(\lambda \geqq 0, 0 \leqq \gamma \leqq 1, 0 \leqq \beta < 1, \tau \in \mathbb{C}\backslash\{0\}, \delta \in \mathbb{N}_0\big)$$

be given by (1.1). *Then,*

$$|a_2| \leqq \min\left\{\frac{2|\tau|(1-\beta)}{(\delta+1)(1+\lambda+\gamma+5\lambda\gamma)}, 2\sqrt{\frac{|\tau|(1-\beta)}{(\delta+1)(\delta+2)(1+2(\lambda+\gamma+5\lambda\gamma))}}\right\},$$

and

$$|a_3| \leqq \frac{4|\tau|(1-\beta)}{(\delta+1)(\delta+2)(1+2(\lambda+\gamma+5\lambda\gamma))}.$$

By taking $\delta = 0$ in Corollary 6, we have the following result.

Corollary 7 *Let*

$$f \in \Theta_\Sigma(\tau, \lambda, \gamma; \beta) \quad \big(\lambda \geqq 0, 0 \leqq \gamma \leqq 1, 0 \leqq \beta < 1, \tau \in \mathbb{C}\backslash\{0\}\big)$$

be given by (1.1). *Then,*

$$|a_2| \leqq \min\left\{\frac{2|\tau|(1-\beta)}{1+\lambda+\gamma+5\lambda\gamma}, \sqrt{\frac{2|\tau|(1-\beta)}{1+2(\lambda+\gamma+5\lambda\gamma)}}\right\},$$

and

$$|a_3| \leqq \frac{2|\tau|(1-\beta)}{1+2(\lambda+\gamma+5\lambda\gamma)}.$$

Remark 9 The bounds on $|a_2|$ and $|a_3|$ given in Corollary 7 are better than those given in [29, Corollary 3.1].

By setting $\delta = \gamma = 0$ and $\tau = 1$ in Corollary 6, we conclude the following result.

Corollary 8 *Let*

$$f \in \Theta_\Sigma(\lambda; \beta) \quad (\lambda \geqq 0, 0 \leqq \beta < 1)$$

be given by (1.1). *Then,*

$$|a_2| \leqq \min\left\{\frac{2(1-\beta)}{1+\lambda}, \sqrt{\frac{2(1-\beta)}{1+2\lambda}}\right\},$$

and

$$|a_3| \leqq \frac{2(1-\beta)}{1+2\lambda}.$$

Remark 10 The bounds on $|a_2|$ and $|a_3|$ given in Corollary 8 are better than those given in [8, Theorem 3.2].

By setting $\delta = \gamma = 0$ and $\lambda = \tau = 1$ in Corollary 6, we conclude the following result.

Corollary 9 *Let* $f \in \Theta_\Sigma(\beta)$ $(0 \leqq \beta < 1)$ *be given by* (1.1). *Then,*

$$|a_2| \leqq \min\left\{1-\beta, \sqrt{\frac{2(1-\beta)}{3}}\right\},$$

and

$$|a_3| \leqq \frac{2(1-\beta)}{3}.$$

Remark 11 The bounds on $|a_2|$ and $|a_3|$ given in Corollary 9 are better than those given in [6, Theorem 2].

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Author contributions
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Author details
[1]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada. [2]Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy. [3]Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea. [4]Department of Mathematics, College of Science, University of Sulaimani, Sulaymaniyah 46001, Iraq. [5]Department of Mathematics, Faculty of Science, Dicle University, TR-21280, Diyarbakır, Turkey. [6]Department of Mathematics, College of Science, University of Al-Qadisiyah, Al-Diwaniyah 58001, Iraq. [7]Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Iraq. [8]Research and Development Center, University of Sulaimani, Sulaymaniyah 46001, Iraq. [9]Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia. [10]Department of Computer Science and Mathematics, Lebanese American University, Beirut 11022801, Lebanon. [11]Institute of Space Sciences, R76900 Magurele-Bucharest, Romania.

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