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Approximation by modified (p, q)-gamma-type operators

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Abstract

The main object of this paper is to construct a new class of modified (p, q)-Gamma-type operators. For this new class of operators, in section one, the general moments are find; in section two, the Korovkin-type theorem and some direct results are proved by considering the modulus of continuity and modulus of smoothness and their behavior in Lipschitz-type spaces. In section three, some results in the weighted spaces are given, and in the end, some shape-preserving properties are proven.

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1 Introduction

One of the central theorems in the approximation theory is a Korovkin-type theorem. It is studied in various function spaces and in the various forms of convergence, starting from standard convergence [1, 12, 18, 27, 29], statistical convergence [3, 9, 10, 16, 23], power summability form of it [4–8, 24], and many other forms. In this paper, we will study the kind of the modified (p, q)-Gamma-type operators, and for these operators, we will prove the Korovkin-type theorem and some direct results by considering the modulus of continuity and modulus of smoothness and their behavior in Lipschitz-type spaces. In Sect. 3, some results in the weighted spaces are given, and in the end, some shape-preserving properties are proven. In [25], the following Gamma-type operators were introduced:

$$G_n(f, x) = \int_0^\infty K_n(x, u) f\left(\frac{n}{u}\right) du, \tag{1.1}$$

where

$$K_n(x, u) = \frac{x^{n+1}}{\Gamma(n+1)} e^{-xu} u^n, \quad x \in (0, \infty).$$

Later one, in [29], the above operators have been modified to the following form:

$$\mathcal{G}_n(f,x) = \int_0^\infty K_n(x,u) f(nu)\,du, \tag{1.2}$$

where

$$K_n(x,u) = \frac{x^{n+3}}{\Gamma(n+3)} e^{-\frac{x}{u}} u^{-n-4}, \quad x \in (0,\infty).$$

Recently, in [21], the above operators have been modified as follows:

$$\mathcal{G}_{n,q}(f,x) = \int_0^{\frac{\infty}{A}} K_{n,q}(x,u) f\bigl([n]_q u\bigr)\,d_q u, \tag{1.3}$$

where

$$\mathcal{K}_{n,q}(x,u) = \frac{q x^{n+1}}{\Gamma_q(n+1)} E(-qx/u) u^{-n-4}, \quad x \in (0,\infty).$$

For any function f, the (p,q)-derivative is given by (for example, see [11, 19])

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and in case where f is differentiable at 0, then $D_{p,q} f(0) = f'(0)$. We know that

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \qquad [n]_{p,q}! = \prod_{j=1}^{n} [j]_{p,q}, \qquad [0]_{p,q}! = 1, \qquad \binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

for all $0 \le k \le n$. In [13], it is proved that (Theorem 1)

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{pq} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq}. \tag{1.4}$$

Based on this relation, we have

Lemma 1.1 *The* (p,q)*-factorial satisfies the following relation*:

$$[n+1]_{pq} = p^2 [n-1]_{pq} + [2]_{pq} \cdot q^{n-1}.$$

Proof From relation (1.4) and definition of the (p,q)– factorial, for $k = 1$, we get

$$\begin{aligned} &\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{pq} = p^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_{pq} + q^{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix}_{pq} \\ &\Rightarrow \quad \frac{[n+1]_{pq}}{[n-1]_{pq}[2]_{pq}} = \frac{p^2}{[2]_{pq}} + \frac{q^{n-1}}{[n-1]_{pq}}, \end{aligned}$$

and we obtain the desired result. □

Some relation related to the p, q-exponential function and p, q-integral are given by the following relations:

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{[n]_{p,q}!},$$

$e_{p,q}(x)E_{p,q}(-x) = 1.$

$$\int f(x)\, d_{p,q}x = (p-q)x \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}x\right)\frac{q^k}{p^{k+1}}.$$

Further, the p, q-Gamma function is given by

$$\Gamma_{p,q}(n) = \int_0^{\infty} u^{n-1} E_{p,q}(-qu)\, d_{p,q}u.$$

It is known that the following relation is valid (Proposition 3.3, [26]):

$$\Gamma_{p,q}(x+1) = [x]_{p,q}\Gamma_{p,q}(x), \tag{1.5}$$

for every x.

In this paper, we introduce modified (p,q)-Gamma-type operators:

$$G^{(1)}_{n;p,q}(f,x) = \int_0^{\infty} K_{n;p,q}(x,u) f\bigl([n]_{p,q}u\bigr)\, d_{p,q}u, \tag{1.6}$$

with

$$K_{n;p,q}(x,u) = \frac{pqx^{n+3}}{\Gamma_{p,q}(n+3)} E_{p,q}\left(-\frac{qx}{u}\right)u^{-n-4}. \tag{1.7}$$

Remark 1.2 Our operators are a generalization of the operators given in [29]; for $p \to 1$, we obtain their class of operators. For $p \in (0,1)$ and $q = 0$, we obtain operators defined in [21].

Now, we give some basic results.

Lemma 1.3 *For $p, q \in (0,1)$ and $x \in (0,\infty)$, the operators $G^{(1)}_{n;p,q}$ satisfy*

$$G^{(1)}_{n;p,q}\bigl(u^k, x\bigr) = \frac{[n]^k_{p,q} x^k \Gamma_{p,q}(n+3-k)}{\Gamma_{p,q}(n+3)} = \frac{[n]^k_{p,q} x^k}{\prod_{j=0}^{k-1}[n+2-j]_{p,q}}.$$

Proof By setting $t = x/u$, we have

$$\begin{aligned} G^{(1)}_{n;p,q}\bigl(u^k, x\bigr) &= \int_0^{\infty} \frac{pqx^{n+3}}{\Gamma_{p,q}(n+3)} E_{p,q}\left(-\frac{qx}{u}\right)u^{-n-4}\bigl([n]_{p,q}u\bigr)^k\, d_{p,q}u \\ &= \frac{pq[n]^k_{p,q} x^{n+3}}{\Gamma_{p,q}(n+3)} \int_0^{\infty} u^{k-n-4} E_{p,q}(-qx/u)\, d_{p,q}u \end{aligned}$$

$$= \frac{[n]_{p,q}^k x^k}{\Gamma_{p,q}(n+3)} \int_0^\infty t^{n+2-k} E_{p,q}(-qt)\, d_{p,q}t$$
$$= \frac{[n]_{p,q}^k x^k \Gamma_{p,q}(n+3-k)}{\Gamma_{p,q}(n+3)},$$

as required. □

As an application of the above Lemma, we have

Corollary 1.4 *For $p,q \in (0,1)$ and $x \in (0,\infty)$, the operators $G^{(1)}_{n;p,q}$ fulfill*

(1) $G^{(1)}_{n;p,q}(1,x) = 1$,
(2) $G^{(1)}_{n;p,q}(u,x) = \frac{[n]_{pq}x}{[n+2]_{pq}}$,
(3) $G^{(1)}_{n;p,q}(u^2,x) = \frac{[n]^2_{pq}x^2}{[n+1]_{pq}[n+2]_{pq}}$,
(4) $G^{(1)}_{n;p,q}(u^3,x) = \frac{[n]^2_{pq}x^3}{[n+1]_{pq}[n+2]_{pq}}$,
(5) $G^{(1)}_{n;p,q}(u^4,x) = \frac{[n]^3_{pq}x^4}{[n-1]_{pq}[n+1]_{pq}[n+2]_{pq}}$.

Proof The first one is obvious. For the second, we have:

$$G^{(1)}_{n;p,q}(u,x) = \frac{[n]_{pq}x\Gamma_{pq}(n+2)}{\Gamma_{pq}(n+3)}.$$

From relation (1.5), we obtain

$$G^{(1)}_{n;p,q}(u,x) = \frac{[n]_{pq}x\Gamma_{pq}(n+2)}{\Gamma_{pq}(n+3)} = \frac{[n]_{pq}x\Gamma_{pq}(n+2)}{[n+2]_{pq}\Gamma_{pq}(n+2)} = \frac{[n]_{pq}x}{[n+2]_{pq}}.$$

Similarly, we obtain

$$G^{(1)}_{n;p,q}\left(u^2,x\right) = \frac{[n]^2_{pq}x^2}{[n+1]_{pq}[n+2]_{pq}},$$
$$G^{(1)}_{n;p,q}\left(u^3,x\right) = \frac{[n]^2_{pq}x^3}{[n+1]_{pq}[n+2]_{pq}},$$
$$G^{(1)}_{n;p,q}\left(u^4,x\right) = \frac{[n]^3_{pq}x^4}{[n-1]_{pq}[n+1]_{pq}[n+2]_{pq}}.$$
□

As a result of Lemma 1.3 and the linearity of the operator $G^{(1)}_{n;p,q}$, we obtain the following:

Lemma 1.5 *For $p,q \in (0,1)$ and $x \in (0,\infty)$, the operators $G^{(1)}_{n;p,q}$ satisfy*

$$G^{(1)}_{n;p,q}\left((u-x)^k,x\right) = x^k \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{[n]^j_{p,q}}{\prod_{i=0}^{j-1}[n+2-i]_{p,q}}.$$

Lemma 1.6 *For $p,q \in (0,1)$ and $x \in (0,\infty)$, the operators $G^{(1)}_{n;p,q}$ satisfy*

(1) $G^{(1)}_{n;p,q}((u-x),x) = \frac{[n]_{pq}(1-p^2)-[2]_{pq}q^n}{[n+2]_{pq}}x$,
(2) $G^{(1)}_{n;p,q}((u-x)^2,x) = \frac{[n]_{pq}([n]_{pq}+(p^2-2)[n+1]_{pq})+[2]_{pq}[n+1]_{pq}q^n}{[n+1]_{pq}[n+2]_{pq}}x^2$,
(3) $G^{(1)}_{n;p,q}((u-x)^3,x) = \frac{[n]_{pq}(-2[n]_{pq}+(3-p^2))[n+1]_{pq})-[2]_{pq}[n+1]_{pq}q^n}{[n+1]_{pq}[n+2]_{pq}}x^3$,

(4) $G^{(1)}_{n;p,q}((u-x)^4,x) = \frac{[n]_{pq}([n]^2_{pq}+2[n]_{pq}[n-1]_{pq}+(p^2-4)[n-1]_{pq}[n+1]_{pq})+[2]_{pq}[n-1]_{pq}[n+1]_{pq}q^n}{[n-1]_{pq}[n+1]_{pq}[n+2]_{pq}}x^4$.

Proof Applying Lemma 1.1 and Lemma 1.5 will give:

(1) $G^{(1)}_{n;p,q}((u-x),x) = G^{(1)}_{n;p,q}(u,x) - x = \frac{[n]_{pq}x\Gamma_{pq}(n+2)}{\Gamma_{pq}(n+3)} - x = \frac{[n]_{pq}-[n+2]_{pq}}{[n+2]_{pq}}x = \frac{[n]_{pq}-(p^2[n]_{pq}+[2]_{pq}q^n)}{[n+2]_{pq}}x = \frac{[n]_{pq}(1-p^2)-[2]_{pq}q^n}{[n+2]_{pq}}x$.

(2) Similarly, we obtain: $G^{(1)}_{n;p,q}((u-x)^2,x) = \frac{[n]^2_{pq}x^2-2x^2[n]_{pq}[n+1]_{pq}+x^2[n+1]_{pq}[n+2]_{pq}}{[n+1]_{pq}[n+2]_{pq}} = \frac{[n]_{pq}([n]_{pq}+(p^2-2)[n+1]_{pq})+[2]_{pq}[n+1]_{pq}q^n}{[n+1]_{pq}[n+2]_{pq}}x^2$.

(3) $G^{(1)}_{n;p,q}((u-x)^4,x) = \frac{[n]_{pq}([n]^2_{pq}+2[n]_{pq}[n-1]_{pq}+(p^2-4)[n-1]_{pq}[n+1]_{pq})+[2]_{pq}[n-1]_{pq}[n+1]_{pq}q^n}{[n-1]_{pq}[n+1]_{pq}[n+2]_{pq}}x^4$. □

Remark 1.7 Throughout this paper, we assume that $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are two sequences such that $0 < p_n, q_n < 1$, $p_n \neq q_n$, satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} p_n^n = \alpha$ and $\lim_{n\to\infty} q_n^n = \beta$, where $0 \leq \alpha, \beta < 1$. Then, from Lemma 1.6, we have

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}G^{(1)}_{n;p_n,q_n}\big((u-x),x\big) \\
&\quad= \lim_{n\to\infty}[n]_{p_n,q_n}\frac{[n]_{p_n,q_n}(1-p_n^2)-[2]_{p_n,q_n}q_n^n}{[n+2]_{p_n,q_n}}x = (2\alpha-4\beta)x, \\
&\lim_{n\to\infty}[n]_{p_n,q_n}G^{(1)}_{n;p_n,q_n}\big((u-x)^2,x\big) \\
&\quad= \lim_{n\to\infty}[n]_{p_n,q_n}\frac{[n]_{p_n,q_n}([n]_{p_n,q_n}+(p_n^2-2)[n+1]_{p_n,q_n})+[2]_{p_n,q_n}[n+1]_{p_n,q_n}q_n^n}{[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}}x^2 \\
&\quad= 2\alpha x^2, \\
&\lim_{n\to\infty}[n]_{pq}G^{(1)}_{n;p,q}\big((u-x)^3,x\big) \\
&\quad= \lim_{n\to\infty}[n]_{pq}\frac{[n]_{pq}(-2[n]_{pq}+(3-p^2))[n+1]_{pq})-[2]_{pq}[n+1]_{pq}q^n}{[n+1]_{pq}[n+2]_{pq}}x^3 = (2\alpha-4\beta)x^3, \\
&\lim_{n\to\infty}[n]_{p_n,q_n}G^{(1)}_{n;p_n,q_n}\big((u-x)^4,x\big) \\
&\quad= \lim_{n\to\infty}[n]_{p_n,q_n} \\
&\qquad\times\frac{[n]_{pq}([n]^2_{pq}+2[n]_{pq}[n-1]_{pq}+(p^2-4)[n-1]_{pq}[n+1]_{pq})+[2]_{pq}[n-1]_{pq}[n+1]_{pq}q^n}{[n-1]_{pq}[n+1]_{pq}[n+2]_{pq}}x^4 \\
&\quad= 2\alpha x^4.
\end{aligned}$$

Next results prove the Korovkin-type theorem for the $G^{(1)}_{n;p,q}$. The Korovkin-type theorem and its versions are widely studied; see, for example, [2–9, 17, 20, 23].

Theorem 1.8 *Let $G^{(1)}_{n;p,q}$ be a sequence of positive linear operators defined on $C[0,\infty)$, such that for every $i \in \{0,1,2\}$,*

$$\lim_{n\to\infty}\big\|G^{(1)}_{n;p_n,q_n}(e_i;x)-e_i\big\| = 0, \tag{1.8}$$

where $e_i = x^i$. Then, for every $f \in C[0,\infty)$,

$$\lim_{n\to\infty}\big\|G^{(1)}_{n;p_n,q_n}(f;x)-f\big\| = 0, \tag{1.9}$$

uniformly for every $x \in [a,b] \subset [0,\infty)$.

Proof From Corollary 1.4, we have

$$\left\| G^{(1)}_{n;p,q}(e_0;x) - e_0 \right\| = 1 - 1 = 0,$$

$$\left\| G^{(1)}_{n;p_n,q_n}(e_1;x) - e_1 \right\| = \left\| \frac{[n]_{p_nq_n}x}{[n+2]_{p_nq_n}} - x \right\| = 0$$

and

$$\begin{aligned}&\left\| G^{(1)}_{n;p_n,q_n}(e_2;x) - e_2 \right\| \\ &\quad = \left\| \frac{[n]^2_{p_nq_n}x^2}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}} - x^2 \right\| = 0.\end{aligned}$$

The proof of theorem follows from the Korovkin theorem [1]. □

2 Some direct results

With $B[0,\infty)$, $C[0,\infty)$ and $C_B([0,\infty))$, we will denote the space of all bounded functions, continuous functions, and continuous, bounded functions defined in the interval $[0,\infty)$. Let be given $\eta > 0$, then the Petree K-functional [28] is defined as follows:

$$K(t,\eta) = \inf_{r\in C^2_B([0,\infty))} \left\{ \|t-r\| + \eta \left\| r'' \right\| \right\},$$

and $C^2_B([0,\infty)) = \{r/r', r'' \in C_B([0,\infty))\}$, with the norm

$$\|t\|_{C^2_B} = \|t\|_\infty + \left\| t' \right\|_\infty + \left\| t'' \right\|_\infty.$$

It is proven in [14] and [15] that exists a constant $C > 0$ such that

$$K(t,\eta) \le C \cdot \omega_2(t,\sqrt{\eta}), \tag{2.1}$$

where

$$\omega_2(t,\eta) = \sup_{0<|h|\le\eta} \sup_{u,u+\eta\in[0,\infty)} \left| t(u+2h) - 2t(u+h) + t(u) \right|.$$

Theorem 2.1 *If $t \in C_B[0,\infty)$, then*

$$\begin{aligned}&\left\| G^{(1)}_{n;p_n,q_n} t - t \right\| \\ &\quad \leqq \omega(t;\sqrt{n}) \\ &\qquad \times \left(1 + \frac{1}{\sqrt{n}} \left[\frac{[n]_{p_nq_n}([n]_{p_nq_n} + (p_n^2 - 2)[n+1]_{p_nq_n}) + [2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}} y^2 \right]^{\frac{1}{2}} \right).\end{aligned}$$

Proof From properties of the modulus of continuity and fact that operators $G^{(1)}_{n;p_n,q_n}$ are positive and linear, for any $t \in C_B[0,\infty)$, we obtain

$$\begin{aligned}\left|G^{(1)}_{n;p_n,q_n}(t;y)-t(y)\right| &\leqq \int_0^\infty K_{n;p_n,q_n}(y,u)\left|t\big([n]_{p_n,q_n}u\big)-t(y)\right| d_{p_n,q_n}u \\ &\leqq \omega(t;\eta)\left(1+\int_0^\infty K_{n;p_n,q_n}(y,u)\frac{|[n]_{p_n,q_n}u-y|}{\eta}\, d_{p_n,q_n}u\right).\end{aligned} \tag{2.2}$$

Let us set

$$B := \frac{1}{\eta}\int_0^\infty K_{n;p_n,q_n}(y,u)\left|[n]_{p_n,q_n}u-y\right| d_{p_n,q_n}u.$$

Then, using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} B &\leqq \left[\int_0^\infty K_{n;p_n,q_n}(y,u)\, d_{p_n,q_n}u\right]^{\frac{1}{2}}\cdot\left[\int_0^\infty K_{n;p_n,q_n}(y,u)\left|[n]_{p_n,q_n}u-y\right|^2 d_{p_n,q_n}u\right]^{\frac{1}{2}} \qquad (2.3)\\ &= \left[G^{(1)}_{n,p_n,q_n}\big((s-y)^2,y\big)\right]^{\frac{1}{2}} \\ &= \left[\frac{[n]_{p_nq_n}([n]_{p_nq_n}+(p_n^2-2)[n+1]_{p_nq_n})+[2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^2\right]^{\frac{1}{2}}.\end{aligned}$$

Putting $\eta = \sqrt{n}$, we get the result. □

Next result gives an upper bound for $G^{(1)}_{n,p_n,q_n}$-Gamma operators.

Theorem 2.2 *For any $g \in C_B[0,\infty)$,*

$$\left|G^{(1)}_{n,p_n,q_n}(g;y)\right| \leqq \|g\|_C.$$

Proof From the definition of the modified (p,q)-Gamma-type operators in (1.6), we have

$$\left|G^{(1)}_{n,p_n,q_n}(g;y)\right| \leqq \sup_{s\in\mathbb{R}^+}\left|g(s)\right|\cdot\int_0^\infty\left|K_{n;p_n,q_n}(y,u)\right| d_{p_n,q_n}u = \|g\|_C.$$ □

Theorem 2.3 *For $y \in (0,\infty)$, $g \in C_B[0,\infty)$, there exists a $M \in \mathbb{R}^+$, such that*

$$\left|G^{(1)}_{n,p_n,q_n}(g,y)-g(y)\right| \le M\omega_2\Big(g,\sqrt{\left|J(y)\right|+I^2(y)}\Big)+\omega\big(g,\left|I(y)\right|\big),$$

where $I(y) = \frac{[n]_{p_nq_n}(1-p_n^2)-[2]_{p_nq_n}q_n^n}{[n+2]_{p_nq_n}}y$ and

$$J(y) = \frac{[n]_{p_nq_n}([n]_{p_nq_n}+(p_n^2-2)[n+1]_{p_nq_n})+[2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^2.$$

Proof For any $y \in (0,\infty)$, we denote by

$$G^{(2)}(n,p_n,q_n)(g,y) = G^{(1)}_{n,p_n,q_n}(g,y)+g(y)-g\big(I(y)+y\big).$$

Then, from Lemma (1.5), we obtain

$$G^{(2)}_{n,p_n,q_n}\big((s-y),y\big) = G_{n,p_n,q_n}\big((s-y),y\big) + (s-y) - \big(I(y)+y-y\big) = I(y) - I(y) = 0.$$

Let $y, s \in (0,\infty)$ and $r(y) \in C^2_B([0,\infty))$. Using the Taylor formula, we get:

$$r(s) = r(y) + r'(y)(s-y) + \int_y^s \big(r''(v)(s-v)\big)\,dv,$$

and it yields

$$\begin{aligned}
&\big|G^{(2)}_{n,p_n,q_n}(r,y) - r(y)\big| \\
&\quad= \left|r'(y)G^{(2)}_{n,p_n,q_n}\big((s-y),y\big) + G^{(2)}_{n,p_n,q_n}\left(\int_y^s \big(r''(v)(s-v)\big)\,dv, y\right)\right| \\
&\quad= \left|G^{(2)}_{n,p_n,q_n}\left(\int_y^s \big(r''(v)(s-v)\big)\,dv, y\right)\right| \\
&\quad= \left|G^{(2)}_{n,p_n,q_n}\left(\int_y^s \big(r''(v)(s-v)\big)\,dv, y\right) - \int_y^{I(y)+y} r''(v)\big(I(y)+y-v\big)\,dv\right| \\
&\quad\le G^{(1)}_{n,p_n,q_n}\left(\int_y^s \big|r''(v)\big|(s-v)\,dv, y\right) + \int_y^{I(y)+y} \big|r''(v)\big|\big|\big(I(y)+y-v\big)\big|\,dv \\
&\quad\le \big(\big|J(y)\big| + I^2(y)\big)\big\|r''\big\|.
\end{aligned}$$

From Theorem 2.2, we have that $|G^{(1)}_{n,p_n,q_n}(g,y)| \le \|f\|$, then

$$\begin{aligned}
&\big|G^{(1)}_{n,p_n,q_n}(g,y) - g(y)\big| \\
&\quad= \big|G^{(2)}(n,p_n,q_n)(g,y) + g\big(I(y)+y\big) - 2g(y)\big| \\
&\quad\le \big|G^{(2)}(n,p_n,q_n)(g-r,y) - (g-r)y\big| \\
&\qquad+ \big|G^{(2)}(n,p_n,q_n)(r,y) - r(y)\big| + \big|g\big(I(y)+y\big) - g(y)\big| \\
&\quad\le 4\|g-r\| + \big(\big|J(y)\big| + I^2(y)\big)\big\|r''\big\| + \omega\big(g,\big|I(y)\big|\big).
\end{aligned}$$

Taking infimum for all $r \in C^2_B([0,\infty))$ and relation (2.1), we obtain our result. □

In [15], the following modulus are given:

$$\omega_\gamma(g;\eta) := \sup_{0<|h|\leqq\eta}\ \sup_{y,y+h\gamma(y)\in[0,\infty)} \big\{\big|g\big(y+h\gamma(y)\big) - g(y)\big|\big\}$$

and

$$\omega_2^\rho(g;\eta) := \sup_{0<|h|\leqq\eta}\ \sup_{y,y\pm h\rho(y)\in[0,\infty)} \big\{\big|g\big(y+h\rho(y)\big) - 2g(y) + g\big(y-h\rho(y)\big)\big|\big\},$$

$\rho(y) = \sqrt{(y-a)(b-y)}$, and K-functional:

$$K_{2,\rho(y)}(g,\eta) = \inf_{r\in W^2(\rho)} \big\{\|g-r\|_{C[0,\infty)} + \eta\big\|\rho^2 r''\big\|_{C[0,\infty)}\big\},$$

where $\eta > 0$.

$$W^2(\rho) = \{r \in C_B[0,\infty) : r' \in AC[0,\infty),\ \rho^2 r'' \in C_B[0,\infty)\} \quad \text{and} \quad r' \in AC[0,\infty).$$

Theorem 2.4 *Let $\rho = \sqrt{y(1-y)}$, $g \in C_B[0,1]$ and $y \in [0,1]$, $n \in \mathbb{N}$. Then,*

$$\begin{aligned}\left\| G^{(1)}_{n,p_n,q_n}(g;y) - g(y) \right\| \leqq 4K_{2,\rho(y)}\left(g, \frac{G^{(1)}_{n,p_n,q_n}((s-y)^2;y) + \alpha_1(n,p_n,q_n)}{4\rho^2(y)}\right) \\ + \omega_\gamma\left(g; \frac{\alpha_1(n,p_n,q_n)}{\gamma(y)}\right),\end{aligned}$$

where $\alpha_1(n,p_n,q_n) = \frac{[n]_{p_nq_n}}{[n+2]_{p_nq_n}}$.

Proof Let

$$G^{(3)}_{n,p_n,q_n}(g;y) = G^{(1)}_{n,p_n,q_n}(g;y) + g(y) - g\big(y + \beta_1(n,p_n,q_n,y)\big),$$

where

$$\beta_1(n,p_n,q_n,y) = \frac{[n]_{p_nq_n}(1-p_n^2) - [2]_{p_nq_n}q_n^n}{[n+2]_{p_nq_n}}y.$$

Then,

$$G^{(3)}_{n,p_n,q_n}(1;y) = 1 \quad \text{and} \quad G^{(3)}_{n,p_n,q_n}\big((s-y);y\big) = 0.$$

Let $r \in W^2(\rho)$. Using the Taylor formula, we obtain

$$r(s) = r(y) + r'(y)(s-y) + \int_y^s (s-\nu)r''(\nu)\,\mathrm{d}\nu \quad \big(s \in [0,\infty)\big),$$

and

$$\begin{aligned}G^{(3)}_{n,p_n,q_n}(r;y) - r(y) = G^{(1)}_{n,p_n,q_n}\left(\int_y^s (s-\nu)r''(\nu)\,\mathrm{d}\nu; y\right) \\ - \int_y^{y+\beta_1(n,p_n,q_n,y)} \big[y + \beta_1(n,p_n,q_n,y) - \nu\big] r''(\nu)\,\mathrm{d}\nu.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|G^{(3)}_{n,p_n,q_n}(r;y) - r(y)\right| \\ &\quad \leqq G^{(1)}_{n,p_n,q_n}\left(\left|\int_y^s (s-\nu)r''(\nu)\,\mathrm{d}\nu\right|; y\right) \\ &\qquad + \int_y^{y+\beta_1(n,p_n,q_n,y)} \left|y + \beta_1(n,p_n,q_n,y) - \nu\right| \cdot \left|r''(\nu)\right|\,\mathrm{d}\nu\end{aligned}$$

$$\leqq \left\| \rho^2 r''(y)\, G^{(1)}_{n,p_n,q_n}\left(\left|\int_y^s \frac{|s-\nu|}{\rho^2(\nu)}\,\mathrm{d}\nu\right|;y\right) + \left\|\rho^2 r''(y)\right\|\right\| \cdot \left|\int_y^{y+\beta_1(n,p_n,q_n,y)} \frac{|y+\beta_1(n,p_n,q_n,y)-\nu|}{\rho^2(\nu)}\,\mathrm{d}\nu\right|.$$

For $\nu = \nu y + (1-\nu)s$ ($\nu \in [0,1]$). Since ρ^2 is concave on $[0,\infty)$, it follows that $\rho^2(\nu) \geq \nu\rho^2(y) + (1-\nu)\rho^2(s)$ and hence

$$\frac{|s-\nu|}{\rho^2(\nu)} = \frac{\nu|y-s|}{\rho^2(\nu)} \leqq \frac{\nu|y-s|}{\nu\rho^2(y)+(1-\nu)\rho^2(s)} \leqq \frac{|y-s|}{\rho^2(y)}.$$

Thus, we have

$$\left\|G^{(3)}_{n,p_n,q_n}(r) - r\right\| \leqq \frac{\|\rho^2 r''\|_{C[0,\infty)}}{\rho^2(y)}\left\{\left[G^{(1)}_{n,p_n,q_n}\left((s-y)^2;y\right)\right] + y\beta_1(n,p_n,q_n,y)\right\}.$$

From the above relations, we obtain

$$\begin{aligned}
&\left\|G^{(3)}_{n,p_n,q_n}(g,y) - g(y)\right\| \\
&\quad\leqq \left\|G^{(3)}_{n,p_n,q_n}(g-r)\right\| + \left\|G^{(3)}_{n,p_n,q_n}(r) - r\right\| + \|g-r\| + \left\|g\left(y+\beta_1(n,p_n,q_n,y)\right) - g(y)\right\| \\
&\quad\leqq 4\|g-r\| + \frac{\|\rho^2 r''\|}{\rho^2(y)}\left[G^{(1)}_{n,p_n,q_n}\left((s-y)^2;y\right) + y\beta_1(n,p_n,q_n,y)\right] \\
&\qquad + \left\|g\left(y+\beta_1(n,p_n,q_n,y)\right) - g(y)\right\|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\|g\left(y+\beta_1(n,p_n,q_n,y)\right) - g(y)\right\| &\leqq \left\|g\left(y+\gamma(y)\frac{G^{(1)}_{n,p_n,q_n}((s-y);y)}{\gamma(y)}\right) - g(y)\right\| \\
&\leqq \omega_\gamma\left(g;\frac{\beta_1(n,p_n,q_n,y)}{\gamma(y)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|G^{(1)}_{n,p_n,q_n}(g,y) - g(y)\right\| &\leqq 4K_{2,\rho(y)}\left(g,\frac{G^{(1)}_{n,p_n,q_n}((s-y)^2;y) + y\beta_1(n,p_n,q_n,y)}{4\rho^2(y)}\right) \\
&\quad + \omega_\gamma\left(g;\frac{\beta_1(n,p_n,q_n,y)}{\gamma(y)}\right).
\end{aligned} \tag{2.4}$$

From inequality

(1)

$$\frac{[n]_{p_nq_n}(1-p_n^2) - [2]_{p_nq_n}q_n^n}{[n+2]_{p_nq_n}}y \leqq \frac{[n]_{p_nq_n}}{[n+2]_{p_nq_n}}.$$

It follows from Theorem 2.4

$$K_{2,\rho(y)}\left(g,\frac{G^{(1)}_{n,p_n,q_n}((s-y)^2;y)+y\beta_1(n,p_n,q_n,y)}{4\rho^2(y)}\right) \leqq K_{2,\rho(y)}\left(g,\frac{G^{(1)}_{n,p_n,q_n}((s-y)^2;y)+\alpha_1(n,p_n,q_n)}{4\rho^2(y)}\right),$$

(2)

$$\omega_\gamma\left(g;\frac{\beta_1(n,p_n,q_n,y)}{\gamma(y)}\right)\leqq \omega_\gamma\left(g;\frac{\alpha_1(n,p_n,q_n)}{\gamma(y)}\right)$$

$\forall y\in[0,1]$. Finally, we have

$$\left\|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right\|\leqq 4K_{2,\rho(y)}\left(g,\frac{G^{(1)}_{n,p_n,q_n}((s-y)^2;y)+\alpha_1(n,p_n,q_n)}{4\rho^2(y)}\right) + \omega_\gamma\left(g;\frac{\alpha_1(n,p_n,q_n)}{\gamma(y)}\right),$$

as asserted by the theorem. □

Theorem 2.5 *Let $g\in C[0,N]$, N is a finite number. Then,*

$$\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right|\leqq \frac{2}{N}\|g\|c^2+\frac{3}{4}\left(N+c^2+2\right)\omega_2(g;c),$$

where

$$c=\sqrt[4]{G^{(1)}_{n,p_n,q_n}\left((s-y)^2;y\right)}.$$

Proof Let g_S be the Steklov function of the second order for $g(y)$. We know that

$$G^{(1)}_{n,p_n,q_n}(e_0;y)=1,$$

which follows from Corollary (1.4), and

$$\begin{aligned}\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right| &\leqq \left|G^{(1)}_{n,p_n,q_n}(g-g_S;y)\right|+\left|G^{(1)}_{n,p_n,q_n}(g_S;y)-y_S(y)\right|+\left|g_S(y)-g(y)\right|\\ &\leqq 2\|g_S-g\|+\left|G^{(1)}_{n,p_n,q_n}(g_S;y)-g_S(y)\right|.\end{aligned}\tag{2.5}$$

It follows from Lemmas in [30]

$$\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right|\leqq \frac{3}{2}\omega_2(g;c)+\left|G^{(1)}_{n,p_n,q_n}(g_S;y)-g_S(y)\right|.\tag{2.6}$$

As $g_S\in C^2[0,N]$, and Lemmas in [17], we get

$$\left|G^{(1)}_{n,p_n,q_n}(g_S;y)-g_S(y)\right|\leqq \|g_S'\|\sqrt{G^{(1)}_{n,p_n,q_n}\left((s-y)^2;y\right)}+\frac{1}{2}\|g_S''\|G^{(1)}_{n,p_n,q_n}\left((s-y)^2;y\right).$$

The following inequality is valid [30]:

$$\|g_S''\| \leqq \frac{3}{2c^2}\omega_2(g;c). \tag{2.7}$$

In the light of (2.6) and (2.7), we obtain:

$$\left|G^{(1)}_{n,p_n,q_n}(g_S;y) - g_S(y)\right| \leqq \|g_S'\| \sqrt{G^{(1)}_{n,p_n,q_n}\big((s-y)^2;y\big)} + \frac{3}{4c^2}\omega_2(g;c)G^{(1)}_{n,p_n,q_n}\big((s-y)^2;y\big).$$

From relation (2.7) and the Landau inequality [22], we get

$$\|g_S'\| \leqq \frac{2}{N}\|g\| + \frac{3N}{4c^2}\omega_2(g;c). \tag{2.8}$$

Using relations (2.7) and (2.8) and upon setting

$$c = \sqrt[4]{G^{(1)}_{n,p_n,q_n}\big((s-y)^2;y\big)},$$

we obtain

$$\left|G^{(1)}_{n,p_n,q_n}(g_S;y) - g_S(y)\right| \leqq \frac{2}{N}\|g\|c^2 + \frac{3}{4}\big(N + c^2\big)\omega_2(g;c).$$

The proof of the theorem follows from relation (2.6). □

Theorem 2.6 *Let $g \in C_B[0,\infty)$. Then,*

$$\left|G^{(1)}_{n,p_n,q_n}(g;y) - g(y)\right| \leqq D(n,p_n,q_n,y)\|g\|_{C_B^2},$$

for $y \geqq 0$, where

$$D(n,p_n,q_n,y) = \left[\frac{[n]_{p_nq_n}(1-p_n^2) - [2]_{p_nq_n}q_n^n}{[n+2]_{p_nq_n}}y\right] + \left[\frac{[n]_{p_nq_n}([n]_{p_nq_n} + (p_n^2-2)[n+1]_{p_nq_n}) + [2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^2\right].$$

Proof From the Taylor formula, it follows

$$G^{(1)}_{n,p_n,q_n}(g;y) - g(y) = G^{(1)}_{n,p_n,q_n}\big((s-y);y\big)g'(y) + \frac{1}{2}G^{(1)}_{n,p_n,q_n}\big((s-y)^2;y\big)g''(\iota),$$

where $\iota \in (y,s)$. From the above relation, we have

$$\begin{aligned}
&\left|G^{(1)}_{n,p_n,q_n}(g;y) - g(y)\right| \\
&= \|g'\| \cdot \left[\frac{[n]_{p_nq_n}(1-p_n^2) - [2]_{p_nq_n}q_n^n}{[n+2]_{p_nq_n}}y\right] \\
&\quad + \frac{\|g''\|}{2}\left[\frac{[n]_{p_nq_n}([n]_{p_nq_n} + (p_n^2-2)[n+1]_{p_nq_n}) + [2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^2\right] \\
&\leqq D(n,p_n,q_n,y)\|g\|_{C_B^2}.
\end{aligned}$$

□

Theorem 2.7 *Let $g \in C[0,\infty)$. Then,*

$$\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right| \leqq 2\mathcal{M}\left[\omega_2\left(g;\sqrt{\frac{1}{2}D(n,p_n,q_n,y)}\right) + \min\left\{1,\frac{1}{2}D(n,p_n,q_n,y)\right\}\|g\|_\infty\right],$$

where $\mathcal{M}>0$ is a constant, and $D(n,p_n,q_n,y)$ is as in Theorem 2.6.

Proof Let

$$g(t)-g(y)=g(t)-r(t)+r(t)-r(y)+r(y)-g(y),$$

then

$$\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right| \leqq \left|G^{(1)}_{n,p_n,q_n}(g-r;y)\right| + \left|G^{(1)}_{n,p_n,q_n}(r;y)-r(y)\right| + \left|g(y)-r(y)\right|.$$

Considering that $g \in C_B^2$ and Theorems 2.2 and 2.6, we get

$$\begin{aligned}\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right| &\leqq 2\|g-r\| + D(n,p_n,q_n,y)\|r\|_{C_B^2}\\ &= 2K\left(g;\frac{1}{2}D(n,p_n,q_n,y)\right).\end{aligned}$$

The following relation is valid [15]

$$K(g;\eta)\leqq L\left[\omega_2(g;\sqrt{\eta})+\min\{1,\eta\}\|g\|_\infty\right],$$

for $\forall\eta>0$, and $L>0$ is a positive constant. The proof of the theorem follows from the last two relations. □

The next result gives an estimation of $G^{(1)}_{n,p_n,q_n}$-operators in Lipschitz space $\mathrm{Lip}_L\gamma$ [27] given by the relation:

$$\mathrm{Lip}_L(\gamma) := \left\{g\in C_B[0,\infty) : \left|g(s)-g(y)\right| \leqq L\frac{|s-y|^\gamma}{(y+s)^{\frac{\gamma}{2}}},\ y\in(0,\infty)\ s\in(0,\infty)\right\},$$

$L>0$ is a constant, $\gamma\in(0,1]$.

Theorem 2.8 *Let $g\in \mathrm{Lip}_L(\gamma)$. Then, $\forall y,t\in(0,\infty)$, $n\in\mathbb{N}$ and $\gamma\in(0,1]$,*

$$\begin{aligned}&\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right|\\ &\quad\leqq \frac{T}{(y+t)^{\frac{\gamma}{2}}}\left(\frac{L}{(y+t)^{\frac{\gamma}{2}}}\right.\\ &\qquad\left.\times\left\{\frac{[n]_{p_nq_n}([n]_{p_nq_n}+(p_n^2-2)[n+1]_{p_nq_n})+[2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^2\right\}^{\frac{\gamma}{2}}\right)^{\frac{\gamma}{2}},\end{aligned}$$

$T>0$ is a constant.

Proof Let $g \in \mathrm{Lip}_L^*(\gamma)$ and $\gamma \in (0,1]$. Then,

I. For $\gamma = 1$, we have

$$\begin{aligned}
&\left|G^{(1)}_{n,p_n,q_n}\big(g(s);y\big)-g(y)\right| \\
&\quad\leqq \left|G^{(1)}_{n,p_n,q_n}\big(\left|g(s)-g(y)\right|;y\big)\right| \\
&\quad\leqq T\cdot G^{(1)}_{n,p_n,q_n}\left(\frac{|s-y|}{(y+s)^{\frac{1}{2}}};y\right) \\
&\quad\leqq \frac{T}{(y+s)^{\frac{1}{2}}}G^{(1)}_{n,p_n,q_n}\big(|s-y|;y\big)
\end{aligned}$$

for $T>0$ constant.

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\left|G^{(1)}_{n,p_n,q_n}\big(g(s);y\big)-g(y)\right| \\
&\quad\leqq \frac{T}{(y+s)^{\frac{1}{2}}}\,G^{(1)}_{n,p_n,q_n}\big(|s-y|;y\big) \\
&\quad\leqq \frac{T}{(y+s)^{\frac{1}{2}}}\,\sqrt{G^{(1)}_{n,p_n,q_n}\big((s-y)^2;y\big)} \\
&\quad= \frac{T}{(y+s)^{\frac{1}{2}}}\left(\frac{[n]_{p_nq_n}([n]_{p_nq_n}+(p_n^2-2)[n+1]_{p_nq_n})+[2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^2\right)^{\frac{1}{2}}.
\end{aligned}$$

II. For $\gamma \in (0,1)$, we have

$$\begin{aligned}
&\left|G^{(1)}_{n,p_n,q_n}\big(g(s);y\big)-g(y)\right| \\
&\quad\leqq \left|G^{(1)}_{n,p_n,q_n}\big(\left|g(s)-g(y)\right|;y\big)\right| \\
&\quad\leqq T\cdot G^{(1)}_{n,p_n,q_n}\left(\frac{|s-y|^{\gamma}}{(y+s)^{\frac{\gamma}{2}}};y\right) \\
&\quad\leqq \frac{T}{(y+s)^{\frac{\gamma}{2}}}\,G^{(1)}_{n,p_n,q_n}\big(|s-y|^{\gamma};y\big).
\end{aligned}$$

From the Hölder inequality under the following conditions

$$\frac{1}{\gamma},\frac{1}{1-\gamma},$$

it follows

$$\left|G^{(1)}_{n,p_n,q_n}\big(g(s);y\big)-g(y)\right| \leqq \frac{T}{(y+s)^{\frac{\gamma}{2}}}\left[G^{(1)}_{n,p_n,q_n}\big(|s-y|;y\big)\right]^{\gamma}$$

for $T>0$ constant. Applying the Cauchy–Schwarz inequality, we have:

$$\begin{aligned}
&\left|G^{(1)}_{n,p_n,q_n}\big(g(s);y\big)-g(y)\right| \\
&\quad\leqq \frac{T}{(y+s)^{\frac{\gamma}{2}}}\left[\sqrt{G^{(1)}_{n,p_n,q_n}\big((s-y)^2;y\big)}\right]^{\gamma}
\end{aligned}$$

$$= \frac{T}{(y+s)^{\frac{\gamma}{2}}} \left\{ \frac{[n]_{p_nq_n}([n]_{p_nq_n} + (p_n^2-2)[n+1]_{p_nq_n}) + [2]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}} y^2 \right\}^{\frac{\gamma}{2}}. \qquad \square$$

3 Weighted approximation

Let $\zeta(y) = y^2 + 1$ be the weight function. We denote by $B_\zeta[0,\infty)$, $C_\zeta[0,\infty)$ and $C_\zeta^*[0,\infty)$ the space of functions g defined on $[0,\infty)$ and satisfying, respectively: $|g(y)| \leqq T_g\zeta(y)$, where T_g is a constant, space of all continuous functions and subspace of $C_\zeta[0,\infty)$ for which $\frac{g(y)}{\zeta(y)}$ is convergent as $y \to \infty$.

The space $B_\zeta[0,\infty)$ is a normed linear space defined by the norm as follows:

$$\|g\|_\zeta = \sup_{y \geqq 0} \frac{|g(y)|}{\zeta(y)}.$$

Next we will consider the weighted modulus of continuity $\Omega(g;\kappa)$ defined on $[0,\infty)$ as

$$\Omega(g;\kappa) = \sup_{y\geqq 0;\, 0<|j|\leqq\kappa} \frac{|g(y+j)-g(y)|}{(1+j^2)\zeta(y)} \quad \bigl(\forall\, g \in C_\zeta^*[0,\infty)\bigr).$$

It is know that for any $\mu \in [0,\infty)$, the following inequality:

$$\Omega(g;\mu\kappa) \leqq 2(1+\mu)\bigl(1+\kappa^2\bigr)\Omega(g;\kappa)$$

holds true $\forall g \in C_\zeta^*[0,\infty)$, and

$$\bigl|g(s)-g(y)\bigr| \leqq 2\left(\frac{|s-y|}{\kappa}+1\right)\bigl(1+\kappa^2\bigr)\Omega(g;\kappa)\bigl(1+y^2\bigr)\bigl(1+(s-y)^2\bigr).$$

Theorem 3.1 *For $g \in C_\zeta^*[0,\infty)$,*

$$\lim_{n\to\infty} \bigl\| G_{n,p_n,q_n}^{(1)}(g;y) - g(y) \bigr\|_\rho = 0.$$

Proof We will achieve our result from the Korovkin-type theorem and relations

$$\lim_n \bigl\| G_{n,p,q}^{(1)} e_i - e_i \bigr\|_\zeta = 0 \quad (i = 0),$$

which follows from Corollary 1.4.

In what follows, we will prove it for $i = 1$ and $i = 2$. Letting $g \in C_\zeta^*[0,\infty)$, we get

$$\begin{aligned} \bigl\| G_{n,p_n,q_n}^{(1)} e_1 - e_1 \bigr\|_\zeta &= \sup_{y\geqq 0} \left\{ \frac{|G_{n,p_n,q_n}^{(1)} e_1 - e_1|}{\zeta(y)} \right\} \\ &\leqq \sup_{y\geqq 0} \frac{\left| \frac{[n]_{p_nq_n}(1-p_n^2) - [2]_{p_nq_n}q_n^n}{[n+2]_{p_nq_n}} y \right|}{\zeta(y)} \\ &\leq \sup_{y\geqq 0} \frac{\left| \frac{[n]_{p_nq_n}}{[n+2]_{p_nq_n}} \right|}{\zeta(y)} = 0. \end{aligned}$$

Using a similar consideration, we have

$$\begin{aligned}\left\| G_{n,p_n,q_n}^{(1)} e_2 - e_2 \right\|_{\zeta} &= \sup_{y \geqq 0} \left\{ \frac{|G_{n,p_n,q_n}^{(1)} e_2 - e_2|}{\zeta(y)} \right\} \\ &\leqq \sup_{y \geqq 0} \left\{ \frac{\left| \frac{[n]_{p_n q_n}([n]_{p_n q_n} + (p_n^2 - 2)[n+1]_{p_n q_n}) + [2]_{p_n q_n}[n+1]_{p_n q_n} q_n^n}{[n+1]_{p_n q_n}[n+2]_{p_n q_n}} y^2 \right|}{\zeta(y)} \right\} \\ &= \frac{\frac{|[n]_{p_n q_n}^2 - 2[n]_{p_n,q_n}[n+1]_{p_n,q_n} + [n+1]_{p_n,q_n}[n+2]_{p_n,q_n}|}{[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}}}{\zeta(y)} = 0.\end{aligned}$$

We thus conclude that

$$\lim_{n \to \infty} \left\| G_{n,p_n,q_n}^{(1)} e_i - e_i \right\|_{\zeta} = 0 \quad (i = 0, 1, 2).$$ □

Theorem 3.2 *Let $g \in C_{\zeta}^{*}[0, \infty)$. Then,*

$$\sup_{y \in [0,\infty)} \frac{|G_{n,p_n,q_n}^{(1)}(g; y) - g(y)|}{(1 + y^2)(1 + Fy^4)} \leqq S\Omega\left(g; n^{-\frac{1}{4}}\right)$$

for large n, where S is a constant, and $F > 0$ is constants dependent only on n, p, q.

Proof For $y \in [0, \infty)$, we have

$$G_{n,p_n,q_n}^{(1)}(g; y) - g(y) = \int_0^{\infty} K_{n;p_n,q_n}(y, \nu)\left[g\left([n]_{p_n,q_n}\nu\right) - g(y)\right] d_{p_n,q_n}\nu.$$

Then,

$$\begin{aligned}&\left|G_{n,p_n,q_n}^{(1)}(g; y) - g(y)\right| \\ &\quad\leqq \int_0^{\infty} K_{n,p_n,q_n}(y, \nu) 2\left(1 + \kappa_n^2\right)\Omega(g; \kappa_n)\left(1 + y^2\right) \\ &\qquad\times \cdot \left(\frac{|[n]_{p_n,q_n}\nu - y|}{\kappa_n} + 1\right)\left(1 + \left([n]_{p_n,q_n}\nu - y\right)^2\right) d_{p_n,q_n}\nu.\end{aligned}$$

Let us define

$$S(\nu, p_n, q_n, y) = \left(\frac{|[n]_{p_n,q_n}\nu - y|}{\kappa_n} + 1\right)\left(1 + \left([n]_{p_n,q_n}\nu - y\right)^2\right).$$

Then,

$$S(\nu, p_n, q_n, y) \leqq \begin{cases} 2(1 + \kappa_n^2) & (|1 + ([n]_{p_n,q_n}\nu - y)| \leqq \kappa_n), \\ 2(1 + \kappa_n^2)\frac{([n]_{p_n,q_n}\nu - y)^4}{\kappa_n^4} & (|[n]_{p_n,q_n}\nu - y| \geqq \kappa_n), \end{cases}$$

and

$$S(\nu, p_n, q_n, y) \leqq 2\left(1 + \kappa_n^2\right)\left(1 + \frac{([n]_{p_n,q_n}\nu - y)^4}{\kappa_n^4}\right).$$

So, clearly, we get

$$\begin{aligned}&\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right|\\&\quad\leqq 4\left(1+\kappa_n^2\right)^2\Omega(g;\kappa_n)\left(1+y^2\right)\int_0^\infty K_{n,p_n,q_n}(y,\nu)\cdot\left(1+\frac{([n]_{p_n,q_n}\nu-y)^4}{\kappa_n^4}\right)d_{p_n,q_n}\nu.\end{aligned}$$

From Lemma 1.6, it yields

$$\begin{aligned}&\int_0^\infty K_{n,p_n,q_n}(y,\nu)\left(1+\frac{([n]_{p_n,q_n}\nu-y)^4}{\kappa_n^4}\right)d_{p_n,q_n}\nu\\&\quad=\int_0^\infty K_{n,p_n,q_n}(y,\nu)\,d_{p_n,q_n}\nu+\int_0^\infty K_{n,p_n,q_n}(y,\nu)\frac{([n]_{p_n,q_n}\nu-y)^4}{\kappa_n^4}\,d_{p_n,q_n}\nu\\&\quad=1\\&\qquad+\frac{1}{\kappa_n^4}\left(\frac{[n]_{p_nq_n}([n]^2_{p_nq_n}+2[n]_{p_nq_n}[n-1]_{p_nq_n}+(p_n^2-4)[n-1]_{p_nq_n}[n+1]_{p_nq_n})+[2]_{p_nq_n}[n-1]_{p_nq_n}[n+1]_{p_nq_n}q_n^n}{[n-1]_{p_nq_n}[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^4\right).\end{aligned}$$

For $\kappa_n=n^{-\frac{1}{4}}$, we get

$$\left|G^{(1)}_{n,p_n,q_n}(g;y)-g(y)\right|\leqq S\Omega\left(g;n^{-\frac{1}{4}}\right)\left(1+y^2\right)\left(1+Fy^4\right).\qquad\square$$

4 Shape-preserving properties

Next we will prove that modified (p,q)-Gamma-type operators preserve the monotonicity and convexity under certain conditions. We start with

Theorem 4.1 *Let $g\in C[0,\infty)$. If $g'(x)>0$ and g convex on $[0,\infty)$, then modified (p_n,q_n)-Gamma-type operators are increasing.*

Proof We will prove our result in two steps.

Step one. In this case, we will prove the monotonicity of modified (p_n,q_n)-Gamma-type operators for the Lagrange interpolation polynomial of function $g(y)$. Let us suppose that y_0,y_1 are distinct numbers in the interval $[t,z]$, where $t<y_0<y_1<z$. Then, the Lagrangian interpolation polynomial through points $(y_0,g(y_0))$ and $(y_1,g(y_1))$ is:

$$P(y)=\frac{y-y_1}{y_0-y_1}g(y_0)+\frac{y-y_0}{y_1-y_0}g(y_1).$$

Based on Corollary 1.4, we have:

$$G^{(1)}_{n,p_n,q_n}(P,t)-G^{(1)}_{n,p_n,q_n}(P,z)=(t-z)\frac{g(y_0)-g(y_1)}{y_0-y_1}\frac{[n]_{p_nq_n}}{[n+2]_{p_nq_n}}<0,$$

which proves that $G^{(1)}_{n,p_n,q_n}(P(s),y)$ is also increasing.

Step two. From the above condition, it follows

$$g(y)=P(y)+\frac{g''(\xi_y)}{2!}(y-y_0)(y-y_1),$$

for number $\xi_y \in (\min\{y_0, y_1\}, \max\{y_0, y_1\})$. For $t < y_0 < y_1 < z$ and Corollary 1.4, we have

$$\begin{aligned}
&G^{(1)}_{n,p_n,q_n}(g,t) - G^{(1)}_{n,p_n,q_n}(g,z) \\
&\quad = \left[G^{(1)}_{n,p_n,q_n}(P,t) - G^{(1)}_{n,p_n,q_n}(P,z)\right] \\
&\qquad + \frac{g''(\xi_s)}{2!}\left[G^{(1)}_{n,p_n,q_n}\left(s^2,y\right) - (y_0+y_1)G^{(1)}_{n,p_n,q_n}(s,y) + y_0y_1\right] \\
&\qquad - \frac{g''(\xi_s)}{2!}\left[G^{(1)}_{n,p_n,q_n}\left(s^2,z\right) - (y_0+y_1)G^{(1)}_{n,p_n,q_n}(s,z) + y_0y_1\right] \\
&\quad = \left[G^{(1)}_{n,p_n,q_n}(P,t) - G^{(1)}_{n,p_n,q_n}(P,z)\right] \\
&\qquad + (t-z)\frac{[n]_{p_nq_n}}{[n+1]_{p_nq_n}}\frac{g''(\xi_s)}{2!}\left[(t+z)\frac{[n]_{p_nq_n}}{[n+1]_{p_nq_n}} - (y_0+y_1)\right] < 0.
\end{aligned}$$

Therefore, it proves the theorem. □

Question Prove that the above theorem is valid just only if $f'(x) > 0$, on $[0,\infty)$.

Thus, the next results show that modified (p,q)-Gamma-type operators preserve the convexity.

Theorem 4.2 *Let $g \in C[0,\infty)$. If $g(y)$ is convex on $[0,\infty)$, then (p_n,q_n)-Gamma-type operators are also convex.*

Proof Let us consider that $g''(y) > 0$. Then,

$$\left[G^{(1)}_{n,p_n,q_n}\left(P(s),y\right)\right]''_y = \left[\frac{g(y_0)-g(y_1)}{y_0-y_1}\frac{[n]_{p_nq_n}}{[n+1]_{p_nq_n}}y - \frac{y_1g(y_0)}{y_0-y_1} - \frac{y_0g(y_1)}{y_1-y_0}\right]'' = 0.$$

On the other hand,

$$\begin{aligned}
&G^{(1)}_{n,p_n,q_n}\left(g(s),y\right) \\
&\quad = G^{(1)}_{n,p_n,q_n}\left(P(s),y\right) + \frac{g''(\xi_s)}{2!}\left[G^{(1)}_{n,p_n,q_n}\left(s^2,y\right) - (y_0+y_1)G^{(1)}_{n,p_n,q_n}(s,y) + y_0y_1\right] \\
&\quad = G^{(1)}_{n,p_n,q_n}(P,y) + \frac{g''(\xi_s)}{2!}\left[\frac{[n]^2_{p_nq_n}}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}}y^2 - (y_0+y_1)\frac{[n]_{p_nq_n}}{[n+2]_{p_nq_n}}y + y_0y_1\right].
\end{aligned}$$

From the last relation, it follows

$$\left[G^{(1)}_{n,p_n,q_n}\left(g(s),y\right)\right]''_y = g''(\xi_s)\cdot\frac{[n]^2_{p_nq_n}}{[n+1]_{p_nq_n}[n+2]_{p_nq_n}} > 0.$$

Hence, it proves the theorem. □

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