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Generalized integral Jensen inequality

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Abstract
In this paper we introduce necessary and sufficient conditions for a real-valued function to be preinvex. Some properties of preinvex functions and new versions of Jensen's integral type inequality in this setting are given. Several examples are also involved.

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1 Introduction and preliminary

The important role played by Jensen's inequality as an application of convex functions in mathematics, statistics, economics, probability theory, etc. is well known, see [14, 20]. Many other inequalities can be obtained from it. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y)$$

for every $x, y \in I$ and $t \in [0,1]$. The classical integral form of Jensen's inequality states that

$$f\left(\frac{1}{d-c}\int_c^d g(x)\,dx\right) \le \frac{1}{d-c}\int_c^d f\big(g(x)\big)\,dx, \tag{1}$$

where g is an integrable function on $[c,d]$ with $a \le g(x) \le b$ and f is a convex function on $[a,b]$. In recent years, many papers dealing with refinements of Jensen's inequality for important generalized convex functions have appeared in the literature, see [6–8, 11, 16, 18, 22, 23] and the references therein.

The analogue of the arithmetic mean in the context of finite measure spaces (X, Σ, μ) is the integral arithmetic mean, which, for a μ-integrable function $f : X \to \mathbb{R}$, is the number

$$M_1(f) := \frac{1}{\mu(X)}\int_X f\,d\mu.$$

In probability theory, $M_1(f)$ represents the mathematical expectation of the random variable f. There are many results on the integral arithmetic mean. A basic one is the integral form of Jensen's inequality:

Theorem 1.1 *Let* (X, Σ, μ) *be a finite measure space and* $g : X \to \mathbb{R}$ *be a* μ*-integrable function. If* f *is a convex function given on an interval* $I \subseteq \mathbb{R}$ *that includes the image of* g*, then* $M_1(g) \in I$ *and*

$$f\big(M_1(g)\big) \leq M_1(fog),$$

provided that fog *is* μ*-integrable.*

A significant generalization of convex functions is that of invex functions introduced by Hanson in [5]. Recall some notions in the invexity analysis that will be used throughout the paper. A set $S \subseteq \mathbb{R}$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}$ if

$$y + t\eta(x,y) \in S \tag{2}$$

for every $x, y \in S$ and $t \in [0,1]$. It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x - y$, but there exist invex sets that are not convex. Recall that for $x, y \in S$ the η-path P_{xy} is a subset of S defined by

$$P_{xy} := \big\{x + t\eta(y,x) | 0 \leq t \leq 1\big\}.$$

An important generalization of convex functions is the class of preinvex functions introduced in [24, 25] by Weir and Mond and Weir and Jeyakumar and then applied to the establishment of the sufficient optimality conditions and duality in nonlinear programming. There have been some works in the literature that are investigated by preinvex functions (e.g. see [1, 2, 10, 12, 13, 21, 25–27] and the references therein).

Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Then the function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η if

$$f\big(y + t\eta(x,y)\big) \leq tf(x) + (1-t)f(y) \tag{3}$$

for every $x, y \in S$ and $t \in [0,1]$. Every convex function is preinvex with respect to the map $\eta(x,y) = x - y$, but the converse does not hold. Recall that the mapping $\eta : S \times S \to \mathbb{R}$ is said to satisfy the conditions C if

$$\eta\big(y, y + t\eta(x,y)\big) = -t\eta(x,y),$$
$$\eta\big(x, y + t\eta(x,y)\big) = (1-t)\eta(x,y)$$

for every $x, y \in S$ and $t \in [0,1]$. From conditions C we have

$$\eta\big(y + t_2\eta(x,y), y + t_1\eta(x,y)\big) = (t_2 - t_1)\eta(x,y) \tag{4}$$

for every $x, y \in S$ and every $t_1, t_2 \in [0,1]$. The Hermite–Hadamard inequality for preinvex functions is introduced in [19] as follows:

$$f\left(a + \frac{1}{2}\eta(b,a)\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{5}$$

where $a, b \in S$. Since then, numerous articles have been published in this category (see, for example, [9, 12] and the references therein). It would be worthwhile to give the exact (precise conditions) Jensen's inequality for preinvex functions. We also recall the following theorem from [14, p. 25].

Theorem 1.2 *Let $f : I \to \mathbb{R}$ be a convex function on the interval $I \subseteq \mathbb{R}$. Then f is continuous on the $int(I)$ and has finite one-sided derivatives $f'_-(x)$ and $f'_+(x)$ at every point $x \in int(I)$. Moreover,*

$$f(y) \geq f(x) + (y - x)f'_+(x)$$

for every $y \in I$.

The main purpose of this paper is to introduce some generalized versions of integral Jensen's inequality for preinvex functions defined on the invex subsets of a real line.

2 Main results

In this section we establish some versions of integral Jensen-type inequality for preinvex functions. At first, to reach our goal, in the following result some necessary and sufficient conditions for a real-valued function to be preinvex are introduced.

Proposition 2.1 *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Suppose that f is a real-valued function on S. Then*:

(*i*) *If $f : S \to \mathbb{R}$ is a preinvex function and η satisfies conditions C, then the restriction of f to any η-path in S is a convex function.*

(*ii*) *If for every $x, y \in S, f(x + \eta(y, x)) \leq f(y)$ and the restriction of f to any η-path in S is a convex function, then f is a preinvex function on S.*

Proof (i) Suppose that f is preinvex on S and $x, y \in S$. Assume that $z, w \in P_{xy}$ with $z = x + t_1\eta(y, x)$ and $w = x + t_2\eta(y, x)$ for some $t_1, t_2 \in [0, 1]$. By using (4) for every $\lambda \in [0, 1]$, we obtain

$$\begin{aligned}(1-\lambda)z + \lambda w &= z + \lambda(w - z) \\ &= z + \lambda(t_2 - t_1)\eta(y, x) \\ &= z + \lambda\eta\bigl(x + t_2\eta(y, x), x + t_1\eta(y, x)\bigr) = z + \lambda\eta(w, z).\end{aligned}$$

From this and the preinvexity of f we deduce that

$$f\bigl((1-\lambda)z + \lambda w\bigr) = f\bigl(z + \lambda\eta(w, z)\bigr) \leq (1-\lambda)f(z) + \lambda f(w),$$

which shows that$f$ is convex on P_{xy}.

(ii) Let $x, y \in S$ and $t \in [0, 1]$. By the convexity of f on the P_{xy}, we have

$$\begin{aligned}f\bigl(x + t\eta(y, x)\bigr) &= f\bigl((1-t)x + t\bigl(x + \eta(y, x)\bigr)\bigr) \\ &\leq (1-t)f(x) + tf\bigl(x + \eta(y, x)\bigr) \leq (1-t)f(x) + tf(y),\end{aligned}$$

which shows the preinvexity of f on S, and the proof is completed. □

Remark 2.1 Note that Pavić in [19, p. 3576] Theorem 5.4 introduced a similar result to Proposition 2.1 (i) by using the convexity of a preinvex function on each invex hall in $\mathbb{R}^n$ instate of η-paths. But we do not know in general whether for each η the corresponding invex hall and η-paths are equivalent or not?

Note that by Proposition 2.1 we can construct several examples of preinvex functions. The next example illustrates how Proposition 2.1 works for particular nontrivial functions η and f.

Example 2.1 Let $S := [-3,-2] \cup [-1,2]$. It is easy to see that S is an invex set with respect to $\eta : S \times S \to \mathbb{R}$ defined by

$$\eta(x,y) := \begin{cases} x-y, & x,y \in [-3,-2], \\ x-y, & x,y \in [-1,2], \\ -3-,y & x \in [-1,2], y \in [-3,-2], \\ -1-y, & x \in [-3,-2], y \in [-1,2]. \end{cases}$$

Moreover, η satisfies condition C (see [26, p., 231]). Then

$$P_{yx} = \begin{cases} [y,x], & x,y \in [-3,-2] \text{ or } x,y \in [-1,2], y < x, \\ [x,y], & x,y \in [-3,-2] \text{ or } x,y \in [-1,2], x < y, \\ [-3,y], & x \in [-1,2], y \in [-3,-2], \\ [-1,y], & x \in [-3,2], y \in [-1,2]. \end{cases}$$

Define the function $f : S \to \mathbb{R}$ as follows:

$$f(x) := \begin{cases} e^x, & x \in [-3,-2], \\ x^2-4, & x \in [-1,2]. \end{cases}$$

We see that for every $x,y \in S, f(y+\eta(x,y)) \leq f(x)$ and η satisfies condition C. Simple computation shows that the restriction of f to any η-path P_{yx} in S is a convex function. Now, by Proposition 2.1 (ii), f is a preinvex function on S.

In the next example we obtain a preinvex function by combining Theorem 5 in [4, p. 319] and Theorem 1.2 in [24, p. 178].

Example 2.2 Let $S := \mathbb{R}$. Define the map η as follows:

$$\eta(x,y) := \begin{cases} x-y, & x \leq 0, y \leq 0, \\ x-y, & x \geq 0, y \geq 0, \\ y-x, & \text{otherwise}. \end{cases}$$

Then the function $f : S \to \mathbb{R}$ defined by $f(x) := -4|x| + 3e^{-|x|}$ is a preinvex function on S, which is not convex.

A generalization of Theorem 1.2 is given in the following theorem.

Theorem 2.1 *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$, and η satisfies conditions C. Suppose that $f : S \to \mathbb{R}$ is a preinvex function. Then:*

(i) f has finite left and right derivatives at each point of $int(S)$;

(ii) for every $x, y \in int(S)$ with $\eta(x, y) \neq 0$, we have

$$f(y) \geq f(x) + \eta(y, x) f'_+(x). \tag{6}$$

Proof (i) Let $x \in int(S)$. By the invexity of S there exist $x_1, x_2 \in S$ and $\delta > 0$ such that

$$x + t\eta(x_1, x) < x < x + t\eta(x_2, x)$$

for all $t \in [0, \delta)$. Pick

$$A_1 := \{x + t\eta(x_1, x) | t \in [0, \delta)\},$$
$$A_2 := \{x + t\eta(x_2, x) | t \in [0, \delta)\}.$$

It is easy to see that $A_1, A_2, A := A_1 \cup A_2$ are convex sets and x is an interior point of S. Since f is preinvex on S, by Proposition 2.1 (i), f is convex on A. Therefore, both $f'_-(x)$ and $f'_+(x)$ are finite by Theorem 1.2.

(ii) Let $x, y \in int(S)$ and $t \in (0, 1)$. By the preinvexity of f, we have

$$f\big(x + t\eta(y, x)\big) \leq (1 - t) f(x) + t f(y).$$

Dividing by t and taking limit as $t \to 0$ imply that

$$\begin{aligned} f(y) - f(x) &\geq \lim_{t \to 0} \frac{f(x + t\eta(y, x)) - f(x)}{t} \\ &= \eta(y, x) \lim_{t \to 0} \frac{f(x + t\eta(y, x)) - f(x)}{t\eta(y, x)} \\ &= \eta(y, x) \lim_{u \to 0} \frac{f(x + u) - f(x)}{u} \\ &= \eta(y, x) f'_+(x). \end{aligned}$$

□

The following example fulfills the conditions of Theorem 2.1.

Example 2.3 Pick $S := [-2, 2]$ and define the mapping $\eta : S \times S \to \mathbb{R}$ as follows:

$$\eta(x, y) := \begin{cases} x - y, & x \geq 0, y \geq 0, \\ x - y, & x < 0, y < 0, \\ -2 - y, & x > 0, y \leq 0, \\ 2 - y, & x \leq 0, y > 0. \end{cases}$$

Then S is an invex set with respect to $\eta(x, y)$, and η satisfies condition C. Now, the function $f : S \to \mathbb{R}$ defined as $f(x) := -2|x|$ is a preinvex function and has finite left and right derivatives at each point of $int(S) = (-2, 2)$, (see [27, p., 611]).

First of our approach is the following special case.

Theorem 2.2 *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$, and η satisfies conditions C. Suppose that $f : S \to \mathbb{R}$ is a preinvex function. Assume that $g : J \to \mathbb{R}$ is an integrable function for some interval $J \subseteq \mathbb{R}$. Let $c, d \in J$, $c < d$ be such that for every $x \in [c,d]$, $g(x) \in P_{ab}$(or P_{ba}) for $a := g(c), b := g(d)$(or $b := g(c), a := g(d)$). Then the following inequality holds*:

$$\begin{aligned} & f\left(\frac{1}{d-c}\int_c^d g(x)\,dx\right) \\ & \quad \le \frac{1}{\eta(b,a)}\int_{P_{ab}} fog(x)\,dx\left(or\ \frac{1}{\eta(a,b)}\int_{P_{ba}} fog(x)\,dx\right), \end{aligned} \tag{7}$$

provided that fog is integrable, where $\int_{P_{ab}}$ is denoted for integral over P_{ab}.

Proof Let $a, b \in S$. By Proposition 2.1, f is convex on P_{ab}(or P_{ba}). Hence, (7) is an immediate consequence of Jensen's inequality (1). □

The following example gives an application of Theorem 2.2.

Example 2.4 Pick $S := I_1 \cup I_4$, where

$$I_1 := (-5,-3], \qquad I_2 := [-1,0], \qquad I_3 := [0, \sqrt{2}-1], \qquad I_4 := I_2 \cup I_3.$$

Define the mapping $\eta : S \times S \to S$ as follows:

$$\eta(x,y) := \begin{cases} x-y, & x,y \in I_i, i = 1,2,3, \\ -1-y, & x \in I_1, y \in I_2, \\ -y, & x \in I_1, y \in I_3, \\ -3-y, & x \in I_4, y \in I_1. \end{cases}$$

It is easy to see that S is an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and η satisfies condition C. Define the integrable function $g : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ as

$$g(x) := \begin{cases} -5 - 2\sin x, & -\frac{\pi}{2} \le x < 0, \\ -1 + \sqrt{2}\sin x, & 0 \le x \le \frac{\pi}{2}. \end{cases}$$

Moreover, by using Proposition 2.1, the function $f : S \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{x^6}, & x \in I_1, \\ 1 + x^{10}, & x \in I_4, \end{cases}$$

is a preinvex function. Let $c, d \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $c < d$ and $g(c) := b, g(d) := a$. To examine Theorem 2.2, we consider four cases.

(i) If $a, b \in I_i, i = 1,2,3$, then P_{ab} is the line segment between a and b; hence we obtain (7) by using Jensen's inequality (1).

(ii) If $a \in I_4$ and $b \in I_1$, then $c \in [-\frac{\pi}{2}, 0), d \in [0, \frac{\pi}{2}]$, and $\eta(a,b) = -3 - b$, so $P_{ba} = [b, -3]$. Therefore,

$$\begin{aligned}\frac{1}{d-c}\int_c^d g(x)\,dx &= \frac{1}{d-c}\left(\int_c^0 (-5-2\sin x)\,dx + \int_0^d (-1+\sqrt{2}\sin x)\,dx\right)\\ &= \frac{1}{d-c}\{2+\sqrt{2}+5c-d-2\cos c-\sqrt{2}\cos d\} := e_1.\end{aligned} \tag{8}$$

Now, by Theorem 2.2, we have

$$f(e_1) \le \frac{1}{\eta(a,b)}\int_{P_{ba}} fog(x)\,dx = \frac{1}{-3-b}\int_b^{-3} e^{(5+2\sin x)^6}\,dx. \tag{9}$$

(iii) If $a \in I_1$ and $b \in I_2$, then $c \in [0, \frac{\pi}{4}], d \in [-\frac{\pi}{2}, 0)$, and $\eta(a,b) = -1 - b$, so $P_{ba} = [-1, b]$. Hence

$$\begin{aligned}\frac{1}{d-c}\int_c^d g(x)\,dx &= \frac{1}{d-c}\left(\int_d^0 (-5-2\sin x)\,dx + \int_0^c (-1+\sqrt{2}\sin x)\,dx\right)\\ &= \frac{1}{d-c}\{2+\sqrt{2}+5d-c-2\cos d-\sqrt{2}\cos c\} := e_2.\end{aligned} \tag{10}$$

Thus by Theorem 2.2 we get

$$f(e_2) \le \frac{1}{-1-b}\int_{-1}^{b} (-1+\sqrt{2}\sin x)^{10}\,dx. \tag{11}$$

(iv) If $a \in I_1$ and $b \in I_3$, then $c \in [\frac{\pi}{4}, \frac{\pi}{2}], d \in [-\frac{\pi}{2}, 0)$, and $\eta(a,b) = -b$, so $P_{ba} = [0, b]$. Hence by (10) we have

$$\begin{aligned}&\frac{1}{d-c}\int_c^d g(x)\,dx\\ &\quad= \frac{1}{d-c}\left(\int_d^0 (-5-2\sin x)\,dx + \int_0^c (-1+\sqrt{2}\sin x)\,dx\right)\\ &\quad= \frac{1}{d-c}\{4+5d-\sqrt{2}c-2\cos c-2\cos d\} := e_3.\end{aligned} \tag{12}$$

So using Theorem 2.2 implies that

$$f(e_3) \le -\frac{1}{b}\int_{-\sqrt{2}}^{b} (-1+\sqrt{2}\sin x)^{10}\,dx. \tag{13}$$

Motivated by [14, Theorem 1.8.1, p. 47] and [20, Theorem 2.23, p. 64], we introduce the following theorem, which is a generalization of Jensen's Theorem 1.1 in the preinvex functions setting.

Theorem 2.3 *Let (X, Σ, μ) be a finite measure space and $g : X \to \mathbb{R}$ be a μ-integrable function. Suppose that $S \subseteq \mathbb{R}$ is an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and S includes the image of g. If $f : S \to \mathbb{R}$ is a preinvex function, then*:

(*i*) $M_1(g) \in S$;

(ii) If $\psi(x) := \eta(g(x), M_1(g))$ and $\psi(x) \neq 0$ for every $x \in S$ such that $g(x) \neq M_1(g)$, then there exists $K \in \mathbb{R}$ such that the following inequality holds:

$$f\left(\frac{1}{\mu(X)}\int_X g\,d\mu\right) \leq \frac{1}{\mu(X)}\int_X (fog)\,d\mu - K\frac{1}{\mu(X)}\int_X \psi\,d\mu, \tag{14}$$

provided that ψ and fog are μ-integrable.

Proof (i) If $M_1(g) \notin S$, then $g(x) \neq M_1(g)$ for every $x \in X$, hence the function $h : X \to \mathbb{R}$ defined by $h(x) := M_1(g) - g(x)$ (or $-h$) is a positive function and

$$\int_X h\,d\mu = \int_X \bigl(M_1(g) - g(x)\bigr)\,d\mu = \mu(X)M_1(g) - \int_X g\,d\mu = 0,$$

which is a contradiction.

(ii) If $M_1(g) \in \operatorname{int}(S)$ and $K := f'_+(M_1(g))$, then by Theorem 2.1 we have

$$\begin{aligned} f\bigl(g(x)\bigr) &\geq f\bigl(M_1(g)\bigr) + \eta\bigl(g(x), M_1(g)\bigr)f'_+\bigl(M_1(g)\bigr) \\ &= f\bigl(M_1(g)\bigr) + \eta\bigl(g(x), M_1(g)\bigr)K \end{aligned} \tag{15}$$

for every $x \in X$, and (14) follows by integrating both sides of (15) over X. Now, suppose that $M_1(g) := b$ is a boundary point of S. Since $\int_X (M_1(g) - g(x))\,d\mu = 0$, so we have $g = M_1(g)$ almost everywhere. Let $A := \{x \in X | g(x) = M_1(g)\}$, then

$$f\left(\frac{1}{\mu(X)}\int_X g\,d\mu\right) = f(b)$$

and

$$\int_X (fog)\,d\mu = \int_A f(b)\,d\mu = f(b),$$

equality in (14) holds if we choose $K = 0$. □

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.1 *Suppose that the conditions of Theorem 2.3 are satisfied. Additionally, if*

$$\frac{1}{\mu(X)}\int_X \eta\bigl(g(x), M_1(g)\bigr)\,d\mu = 0, \tag{16}$$

then

$$f\left(\frac{1}{\mu(X)}\int_X g\,d\mu\right) \leq \frac{1}{\mu(X)}\int_X (fog)\,d\mu. \tag{17}$$

Note that in the trivial case if $\eta(y,x) := y - x$, then S and f will be a convex set and a convex function, respectively, and Corollary 2.1 gives us the usual Jensen's inequality presented in Theorem 1.1. In the following corollary we obtain the left-hand side of Hermite–Hadamard inequality as a consequence of Theorem 2.3.

Corollary 2.2 *Under the conditions of Theorem 2.3, if $a, b \in S$ with $\eta(b,a) \neq 0$ and $a < a + \eta(b,a)$, then we have*

$$f\left(a + \frac{1}{2}\eta(b,a)\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx. \tag{18}$$

Proof Let the function $g : P_{ab} \to P_{ab}$ be defined by $g(x) = x$. It is easy to see that

$$M_1(g) = \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} x\,dx = a + \frac{1}{2}\eta(b,a)$$

and

$$\eta\big(g(x), M_1(g)\big) = \eta\left(x, a + \frac{1}{2}\eta(b,a)\right).$$

Now, if we use the change of variable $x := a + s\eta(b,a), s \in [0,1]$, then by (4) we obtain

$$\begin{aligned} \int_a^{a+\eta(b,a)} \eta\big(g(x), M_1(g)\big)\,dx &= \int_a^{a+\eta(b,a)} \eta\left(x, a + \frac{1}{2}\eta(b,a)\right) dx \\ &= \eta(b,a) \int_0^1 \left(s - \frac{1}{2}\right) ds = 0. \end{aligned} \tag{19}$$

Therefore, by Corollary 2.1, we deduce that

$$\begin{aligned} f\left(a + \frac{1}{2}\eta(b,a)\right) &= f\left(\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} g(x)\,dx\right) \\ &\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx, \end{aligned} \tag{20}$$

which is the left-hand side of inequality (5), and the proof is completed. □

To introduce an application of Theorem 2.3, we recall the definition of a special measure from [15, p. 262] and [17, p. 469], see also [3, 55–69]..

Definition 2.1 A real Borel measure μ on $I = [a,b]$ is said to be Steffensen–Popoviciu measure provided that

(i) $\mu(I) > 0$,
(ii) $\int_a^b f(x)\,d\mu(x) \geq 0$ for every nonnegative $f \in C$.

Several examples of Steffensen–Popoviciu measures can be found in [17, p. 471].

Example 2.5 Set $X := [-1,1]$. According to [17, p. 471], $d\mu(x) := (x^2 - a)\,dx$ for every $0 < a < \frac{1}{3}$ is a Steffensen–Popoviciu measure on X. Choose $a := \frac{1}{6}$. Then we have

$$\mu(X) = \int_{[-1,1]} \left(x^2 - \frac{1}{6}\right) dx = \int_{-1}^1 \left(x^2 - \frac{1}{6}\right) dx = \frac{1}{3}.$$

Define the μ-integrable function $g : X \to \mathbb{R}$ as follows:

$$g(x) := 4x^2 - 1, \quad x \in [-1,1].$$

Note that $\mathrm{Im}(g) = [-2,2]$, and by simple computation we obtain

$$M_1(g) = \frac{1}{\mu(X)}\int_X g\,d\mu = 3\int_{-1}^{1}\left(x^2 - \frac{1}{6}\right)\left(4x^2 - 1\right)dx = \frac{22}{15}. \tag{21}$$

Pick $S := [-2,2]$ and consider the mapping η defined in Example 2.3. Since $f_1(x) := e^x$ is an increasing and convex function on $[-2,2]$ and $h(x) := -2|x|$ is a preinvex function on $[-2,2]$, so by Theorem 5 in [4, p. 319] the function

$$f(x) := e^{-2|x|} \quad \text{for all } x \in [-2,2]$$

is a preinvex function (which is not convex). Since

$$K = f'_+\left(M_1(g)\right) = f'_+\left(\frac{22}{15}\right) = -2f\left(\frac{22}{15}\right),$$

therefore by using Theorem 2.3 and equality (21), we get

$$f\left(\frac{22}{15}\right) \le 3\int_X e^{-|4x^2-2|}\,d\mu + 6f\left(\frac{22}{15}\right)\int_X \eta\left(4x^2-2, \frac{22}{15}\right)d\mu. \tag{22}$$

Taking into account that $4x^2 - 2 \ge 0$ on $[-1, -\frac{\sqrt{2}}{2}] \cup [\frac{\sqrt{2}}{2}, 1]$ and $4x^2 - 2 \le 0$ on $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$, by using the definition of η, we obtain

$$\begin{aligned}
&\int_{-1}^{1}\eta\left(4x^2-2, \frac{22}{15}\right)d\mu \\
&\quad= \int_{-1}^{-\frac{\sqrt{2}}{2}}\left(x^2-\frac{1}{6}\right)\eta\left(4x^2-2, \frac{22}{15}\right)dx + \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\left(x^2-\frac{1}{6}\right)\eta\left(4x^2-2, \frac{22}{15}\right)dx \\
&\qquad+ \int_{\frac{\sqrt{2}}{2}}^{1}\left(x^2-\frac{1}{6}\right)\eta\left(4x^2-2, \frac{22}{15}\right)dx \\
&= \int_{-1}^{-\frac{\sqrt{2}}{2}}\left(x^2-\frac{1}{6}\right)\left(4x^2-\frac{52}{15}\right)dx + \frac{8}{15}\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\left(x^2-\frac{1}{6}\right)dx \\
&\qquad+ \int_{\frac{\sqrt{2}}{2}}^{1}\left(x^2-\frac{1}{6}\right)\eta\left(4x^2-\frac{52}{15}\right)dx = -\frac{4\sqrt{2}-3}{45}.
\end{aligned} \tag{23}$$

Therefore, by combining (22) and (23), we have

$$\frac{9+8\sqrt{2}}{45}f\left(\frac{22}{15}\right) \le \int_X e^{-|4x^2-2|}\,d\mu.$$

3 Conclusions

In this paper, we have used the class of preinvex functions, which is an important generalization of the class of convex functions. Some generalized versions of integral Jensen's inequality are introduced in Theorems 2.1, 2.2, and 2.3. Theorem 2.3 is a new approach to Jensen's integral inequality that is an improvement of Theorem 1.1. A version of Hermite–Hadamard inequality is also obtained as a consequence. The study of integral Jensen's inequality for other types of generalized convex functions are our intend to explore in future works.

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Data availability
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Declarations

Competing interests
The authors declare no competing interests.

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References

1. Antczak, T.: Mean value in invexity analysis. Nonlinear Anal. **60**, 1471–1484 (2005)
2. Barani, A., Pouryayevali, M.R.: Invex sets and preinvex functions on Reimannian manifolds. J. Math. Anal. Appl. **328**, 767–779 (2007)
3. Dragomir, S.S., Pearce, C.E.M.: Selected Topics on Hermite–Hadamard Inequalities and Applications (RGMIA Monographs https://rgmia.org/monographs/hermite_hadamard.html). Victoria University (2000)
4. Giorgi, G.: Some remarks on preinvex functions and other generalized convex functions. Math. Rep. **10**(60), 317–325 (2008)
5. Hanson, M.A.: On sufficiency of the Kuhn–Tucker conditions. J. Math. Anal. Appl. **80**, 545–550 (1981)
6. Horvath, L.: Inequalities corresponding to the classical Jensen's inequality. J. Math. Inequal. Appl. **3**, 189 (2009)
7. Horvath, L.: A refinements of the integral form from Jensen's inequality. J. Inequal. Appl. **2012**, 178 (2012)
8. Khan, M.B., Srivastava, H.M., Mohammed, P.O., Nonlaopon, K., Hamed, Y.S.: Some new Jensen, Schur and Hermite–Hadamard inequalities for log convex fuzzy interval-valued functions. AIMS Math. **7**(3), 4338–4358 (2022)
9. Latif, M.A., Dragomir, S.S.: New inequalities Hermite–Hadamard and Fejér type via preinvexity. J. Comput. Anal. Appl. **19**, 1–16 (2015)
10. Liu, C., Yang, X.: Characterizations of the approximate solution sets of nonsmooth optimization problems and its applications. Optim. Lett. (2014). https://doi.org/10.1007/s11590-014-0780-4
11. Matković, A., Pečarić, J.E.: A variant of Jensen's inequality for convex functions of several variables. J. Math. Inequ. Appl. **1**, 45–51 (2007)
12. Matłoka, M.: Inequalities for h-preinvex functions. Appl. Math. Comput. **15**, 52–57 (2014)
13. Mohan, S.R., Neogy, S.K.: On invex sets and preinvex function. J. Math. Anal. Appl. **189**, 901–908 (1995)
14. Niculescu, C.P.: Convex Functions and Their Applications. A Contemporary Approach. Sringer, New York (2006)
15. Niculescu, C.P.: On a result of G. Bennett. Bull. Math. Soc. Sci. Math. Roumanie Tome **54**(102), 261–267 (2011)
16. Niculescu, C.P., Spridon, C.I.: New Jensen-type inequalities. J. Math. Anal. Appl. **401**, 343–348 (2013)
17. Niculescu, C.P., Stanescu, M.M.: The Steffensen–Popoviciu measures in the context of quasiconvex functions. J. Math. Inequal. **11**, 469–483 (2017)
18. Pavić, Z., Pečarić, J.E., Perić, I.: Integral, discrete and functional variants of Jensen's inequality. J. Math. Inequ. Appl. **5**, 253–264 (2011)
19. Pavić, Z., Wu, S., Novoselac, V.: Important inequalities for preinvex functions. J. Nonlinear Sci. Appl. **9**, 3570–3579 (2016)
20. Pečarić, J.E., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, San Diego (1992)
21. Peng, Z., Liu, Y., Long, X.J.: Remarks on new properties of preinvex functions. Mod. Appl. Sci. **3**, 11–16 (2009)
22. Sayyari, Y., Barsam, H., Sattarzadeh, A.R.: On new refinement of the Jensen inequality using uniformly convex functions with applications. Appl. Anal. https://doi.org/10.1080/00036811.2023.2171873
23. Srivastava, H.M., Mehrez, S., Sitnik, S.M.: Hermite–Hadamard-type integral inequalities for convex functions and their applications. Mathematics **10**, Article ID 3127 (2022). https://doi.org/10.3390/math10173127
24. Weir, T., Jeyakumar, V.: A class of nonconvex functions and mathematical programming. Bull. Aust. Math. Soc. **38**, 177–189 (1988)
25. Weir, T., Mond, B.: Preinvex functions in multiple objective optimization. J. Math. Anal. Appl. **136**, 29–38 (1998)
26. Yang, X.M., Li, D.: On properties of preinvex functions. J. Math. Anal. Appl. **256**, 229–241 (2001)
27. Yang, X.M., Yang, X.Q., Teo, K.L.: Generalized invexity and generalized invariant monotonocity. J. Optim. Theory Appl. **117**, 607–625 (2003)

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