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Weighted estimates for fractional bilinear Hardy operators on variable exponent Morrey–Herz space

Muhammad Asim[1], Irshad Ayoob[2], Amjad Hussain[1] and Nabil Mlaiki[2*]

*Correspondence:
nmlaiki@psu.edu.sa;
nmlaiki2012@gmail.com
[2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this article, we analyze the boundedness for the fractional bilinear Hardy operators on variable exponent weighted Morrey–Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$. Similar estimates are obtained for their commutators when the symbol functions belong to *BMO* space with variable exponents.

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1 Introduction

Back in 1920, Hardy [1] defined an operator for a locally integrable $g \in \mathbb{R}^n$ known as the Hardy operator,

$$\mathscr{H} g(y) = y^{-1} \int_0^y g(z)\,dz, \quad y > 0, \tag{1}$$

and established the following inequality:

$$\|\mathscr{H} g\|_{L^p(\mathbb{R}^+)} \leq p' \|g\|_{L^p(\mathbb{R}^+)}, \quad \infty > p > 1, \tag{2}$$

where $p' = p/(p-1)$ is shown to be the best possible constant. At a later stage, Faris [2] gave an n-dimensional extension of (1) of which the equivalent form is given by

$$Hg(y) = \left|B\left(0, |y|\right)\right|^{-1} \int_{B(0,|y|)} g(z)\,dz, \tag{3}$$

where $|B(0,|y|)|$ is the Lebesgue measure of the ball $B(0,|y|)$ in n-dimensional Euclidean space $\mathbb{R}^n$. Recently [3], it was shown that H satisfies

$$\|Hg\|_{L^p(\mathbb{R}^n)} \leq p' \|g\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq \infty, \tag{4}$$

where p' is declared a sharp constant. Inequalities (2) and (4) were recently extended to power weighted Lebesgue spaces in [4] and [5] where sharp constants depend upon the weight indices. Inequalities (2) and (4) are known as strong-type (p,p) Hardy inequalities because in these inequalities the Hardy operator maps L^p to L^p. The authors in [6] have established the weak-type (p,p) Hardy inequalities in which the Hardy operator maps L^p to $L^{p,\infty}$. However, it was shown that the optimal constant for weak-type Hardy inequalities is 1, which is less than $p/(p-1)$. Subsequently, the sharp constants for weak-type Hardy inequalities on Morrey-type spaces were obtained in [7] and [8]. Likewise the sharp constant for the high-dimensional fractional Hardy operator [9]

$$H_\beta g(y) = \left|B\left(0,|y|\right)\right|^{\frac{\beta}{n}-1} \int_{B(0,|y|)} g(z)\,dz, \quad 0 \leq \beta < n, \tag{5}$$

on Lebesgue spaces was not fixed until 2015. Zhao and Lu [10] solved this problem by extending the Bliss results for the one-dimensional fractional Hardy operator. The boundedness of the Hardy operator H_β has been shown in [10], and the following inequality has been established:

$$\|H_\beta g\|_{L^q(\mathbb{R}^n)} \leq A\|g\|_{L^p(\mathbb{R}^n)}, \tag{6}$$

where

$$A = \left(\frac{p'}{q}\right)^{1/q} \left(\frac{n}{q\beta} \cdot B\left(\frac{n}{q\beta}, \frac{n}{q'\beta}\right)\right)^{-\beta/n}.$$

For $g_1, g_2, \ldots, g_m \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $m \in \mathbb{N}$, the m-linear Hardy operator was introduced by Z. Fu and L. Grafakos in [4], written as follows:

$$H(g_1, \ldots, g_m) = \frac{1}{|x|^{nm}} \int_{|(y_1,\ldots,y_m)|<|x|} \prod_{i=1}^{m} g_i(y_i)\,dy_1, \ldots, dy_m.$$

They also developed the sharp bounds for the m-linear Hardy operator. The 2-linear operator is known as the bilinear operator. The authors of [11] used the commutator of the bilinear Hardy operator

$$\left[b_i, H^i\right](g_1, \ldots, g_m)(z) = b_i(z)H(g_1, \ldots, g_m)(z) - H(g_1, \ldots, g_{i-1}, g_i b_i, g_{i+1}, \ldots, g_m)(z)$$

and obtained the boundedness of bilinear commutators generated by the bilinear Hardy operator. Later on, in [12] A. Hussain defined the fractional m-linear p-adic Hardy operator. Now, in the present paper, we introduce the definition of the fractional m-linear Hardy operators as

$$H_\beta(g_1, \ldots, g_m) = \frac{1}{|x|^{nm-\beta}} \int_{|(y_1,\ldots,y_m)|<|x|} \prod_{i=1}^{m} g_i(y_i)\,dy_1, \ldots, dy_m,$$

$$H_\beta^*(g_1, \ldots, g_m) = \int_{|(y_1,\ldots,y_m)|>|x|} \frac{1}{|y|^{nm-\beta}} \prod_{i=1}^{m} g_i(y_i)\,dy_1, \ldots, dy_m,$$

where $y = (y_1, y_2, \ldots, y_m)$. We also introduce a definition for the commutator of fractional m-linear Hardy operators

$$\begin{aligned}
&[b, H_\beta](g_1, \ldots, g_m)(z) = \sum_{i=1}^{m} \bigl[b_i, H_\beta^i\bigr](g_1, \ldots, g_m)(z), \\
&\bigl[b, H_\beta^*\bigr](g_1, \ldots, g_m)(z) = \sum_{i=1}^{m} \bigl[b_i, H_\beta^{*i}\bigr](g_1, \ldots, g_m)(z), \\
&\bigl[b_i, H_\beta^i\bigr](g_1, \ldots, g_m)(z) = b_i(z) H_\beta(g_1, \ldots, g_m)(z) - H_\beta(g_1, \ldots, g_{i-1}, g_i b_i, g_{i+1}, \ldots, g_m)(z), \\
&\bigl[b_i, H_\beta^{*i}\bigr](g_1, \ldots, g_m)(z) = b_i(z) H_\beta^*(g_1, \ldots, g_m)(z) - H_\beta^*(g_1, \ldots, g_{i-1}, g_i b_i, g_{i+1}, \ldots, g_m)(z).
\end{aligned}$$

Hardy inequalities have been the main focus of interest in various monographs [13, 14]. The optimal bounds for Hardy-type inequalities are established only in a few cases and the research in this area is an active part of modern analysis. Some recent publications in this area include [15, 16]. Besides this, sharp constants for Hardy-type inequalities on the product of some function spaces have also been exposed in [17]. Essential reviews of Hardy operators on various function spaces include [4, 5, 18–20].

The work in [21] sparked the idea of generalizing function spaces. The variable Lebesgue space $L^{p(\cdot)}$ was initially presented by Rákosník and Kováčik in [22]. Following that, the creation of variable exponent Lebesgue spaces began, as did the exploration of the boundedness of numerous operators, notably the maximum operator on the variable exponent Lebesgue space $L^{p(\cdot)}$ [23, 24]. In recent times, the theory of generalized function spaces has piqued the interest of researchers working in several domains of mathematical analysis, including image processing [25], electrorheological fluid modeling [26], and partial differential equations [27].

Furthermore, Izuki proposed variable exponent Herz spaces $\dot{K}_q^{\alpha,p(\cdot)}$ in [28]. After this, Drihemn and Almeida [29] proposed a revised definition of Herz spaces that included α as a variable exponent. However, in [30], the Herz space with all exponents as variables was developed and investigated. Variable exponent Morrey–Herz spaces $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}$ appeared for the first time in [31]. The generalized concept of Morrey–Herz spaces was provided in [31] by substituting the exponent α with $\alpha(\cdot)$. A few significant thoughts in this regard were given in [32, 33]. Its weighted theory formulation based upon the Muckenhoupt weights [34] is a recent accomplishment in the field of variable exponent function spaces. Cruz-Uribe in [35] introduced the boundedness for the Hardy–Littlewood maximal operator M,

$$Mg(t) = \sup_{E:\text{ball}, t \in E} \frac{1}{|E|} \int_E \bigl|g(z)\bigr| \, dz,$$

on the variable exponent weighted Lebesgue space $L^{p(\cdot)}(w)$. Hästö and Diening demonstrated in [36] the equivalence between the continuity criteria of M on $L^{p(\cdot)}(w)$ and the Muckenhoupt condition. However, weighted variable exponent Morrey–Herz spaces are set out and investigated in [11, 37].

In the present paper, we will study the boundedness of the m-linear fractional Hardy operator on variable exponent weighted Herz–Morrey space. Furthermore, we also discuss the boundedness of commutators generated by the m-linear fractional Hardy operator on variable exponent weighted Herz–Morrey space. It is worth mentioning that we study Banach spaces and Muckenhoupt weights. We consequently expand a few results introduced in [18]. To control the continuity criteria of the m-linear fractional Hardy operator, we will utilize the boundedness of the fractional integral defined as

$$I_\beta(g)(t)=\int_{\mathbb{R}^n}\frac{g(z)}{|t-z|^{n-\beta}}\,dz.$$

The boundedness of the Riesz potential on variable exponent Lebesgue spaces is reported in [38]. The boundedness of the fractional integral operator on weighted Herz spaces is introduced by Noi and Izuki [39].

This article contains four sections. The next section includes some lemmas and definitions. In the third section, we provide some important lemmas which are used in Sect. 4 to get our main results.

2 Notations and definitions

The letter C is used throughout this paper to represent a constant, and the value of the constant may differ from one line to the next. We consider a set S which is nonempty and measurable in $\mathbb{R}^n$, and χ_S stands for the characteristic function of S, where $|S|$ denotes the Lebesgue measure. Let us begin by defining variable exponent Lebesgue spaces using the basic articles and books [22, 24, 40, 41].

Definition 2.1 Suppose we have a measurable function $q(\cdot):\mathbb{R}^n\to[1,\infty]$. Moreover, $L^{q(\cdot)}(\mathbb{R}^n)$ represents a Lebesgue space where the variable exponent is a set of every measurable function g, i.e.,

$$F_q(g)=\int_{\mathbb{R}^n}\big(|g(x)|\big)^{q(x)}\,dx<\infty.$$

The space $L^{q(\cdot)}(\mathbb{R}^n)$ becomes a Banach space with the following norm:

$$\|g\|_{L^{q(\cdot)}}=\inf\left\{\sigma>0:F_q\left(\frac{g}{\sigma}\right)=\int_{\mathbb{R}^n}\left(\frac{|g(x)|}{\sigma}\right)^{q(x)}dx\le 1\right\}.$$

Definition 2.2 We denote by $P(\mathbb{R}^n)$ the set of all measurable functions $q(\cdot):\mathbb{R}^n\to(1,\infty)$ such that

$$1<q_-\le q(x)\le q_+<\infty,$$

where

$$q_-:=\operatorname*{essinf}_{x\in\mathbb{R}^n}q(x),\qquad q_+:=\operatorname*{esssup}_{x\in\mathbb{R}^n}q(x).$$

Definition 2.3 Let $q(\cdot)$ be a real-valued function on $\mathbb{R}^n$. We state that:

(i) $\mathscr{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ is a set consisting of all local log Hölder continuous functions $q(\cdot)$ fulfilling

$$\big|q(x)-q(y)\big|\lesssim\frac{-C}{\log(|x-y|)},\qquad |y-x|<\frac{1}{2},\quad x,y\in\mathbb{R}^n;$$

(ii) if $q(\cdot) \in \mathscr{C}_0^{\log}(\mathbb{R}^n)$, then it satisfies the following condition at the origin:

$$\left|q(x) - q(0)\right| \lesssim \frac{C}{\log(|\frac{1}{|x|} + e|)}, \quad x \in \mathbb{R}^n;$$

(iii) if $q(\cdot) \in \mathscr{C}_\infty^{\log}(\mathbb{R}^n)$, then it fulfills the following inequality at infinity:

$$\left|q(x) - q_\infty\right| \leq \frac{C_\infty}{\log(|x| + e)}, \quad x \in \mathbb{R}^n;$$

(iv) $\mathscr{C}^{\log} = \mathscr{C}_{\mathrm{loc}}^{\log} \cap \mathscr{C}_\infty^{\log}$ represents the set of every global log Hölder continuous function $q(\cdot)$.

It was shown in [42] that if $q(\cdot) \in P(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, then M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

Assume that $w(x)$ is a weight function on $\mathbb{R}^n$ that is both nonnegative and locally integrable. Let $L^{q(\cdot)}(w)$ be the space of all complex-valued functions f on $\mathbb{R}^n$ such that $fw^{\frac{1}{q(\cdot)}} \in L^{q(\cdot)}(\mathbb{R}^n)$. The space $L^{q(\cdot)}(w)$ is a Banach function space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(w)} = \left\|fw^{\frac{1}{q(\cdot)}}\right\|_{L^{q(\cdot)}}.$$

In [34], Benjamin Muckenhoupt proposed A_p weights theory with $1 < p < \infty$ on $\mathbb{R}^n$. Noi and Izuki recently expanded the Muckenhoupt A_p class by using p as a variable in [39, 43].

Definition 2.4 Assume $q(\cdot) \in P(\mathbb{R}^n)$. A weight w is said to be an $A_{q(\cdot)}$ weight if

$$\sup_B |B|^{-1} \left\|w^{-1/q(\cdot)} \chi_B\right\|_{L^{q'(\cdot)}} \left\|w^{1/q(\cdot)} \chi_B\right\|_{L^{q(\cdot)}} < \infty.$$

It was proved in [44] that $w \in A_{q(\cdot)}$ if and only if M is bounded on $L^{q(\cdot)}$ space.

Remark 2.5 [39] If $q(\cdot), p(\cdot) \in P(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$ and $q(\cdot) \leq p(\cdot)$, then we have

$$A_1 \subset A_{q(\cdot)} \subset A_{p(\cdot)}.$$

Definition 2.6 Suppose $\beta \in (0, n)$ and $p_1(\cdot), p_2(\cdot) \in P(\mathbb{R}^n)$ in such a way that $\frac{1}{p_1(x)} = \frac{1}{p_2(x)} + \frac{\beta}{n}$. A weight w is called $A(p_1(\cdot), p_2(\cdot))$ weight if

$$|B|^{\frac{\beta}{n}-1} \|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C.$$

Definition 2.7 [39] If $\beta \in (0, n)$, $p_1(\cdot), p_2(\cdot) \in P(\mathbb{R}^n)$, and $\frac{1}{p_1(x)} = \frac{1}{p_2(x)} + \frac{\beta}{n}$, then $w \in A_{(p_1(\cdot), p_2(\cdot))}$ if and only if $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$.

The variable exponent weighted Morrey–Herz space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w)$ is defined now. Let $\chi_k = \chi_{A_k}$, $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$.

Definition 2.8 [19] Let $0 < q < \infty$, $0 \leq \lambda < \infty$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot) \in P(\mathbb{R}^n)$, and let w be a weight on $\mathbb{R}^n$. Then $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w)$ is defined by

$$M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w) = \left\{f \in L_{\mathrm{loc}}^{p(\cdot)}\left(\mathbb{R}^n \setminus \{0\}, w\right) : \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w)} < \infty\right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)} = \sup_{k_0\in Z} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q}\|f\chi_k\|^q_{L^{p(\cdot)}(w)}\right)^{1/q}.$$

When $\lambda = 0$, we have a weighted Herz space with variable exponent $\dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(w)$.

3 Main lemmas

This section presents several relevant lemmas that will aid in the proof of our main boundedness result.

Lemma 3.1 [45] *If Y is a Banach function space, then*:

(1) *The associated space Y' should also be a Banach function space*;
(2) $\|\cdot\|_Y$ *and* $\|\cdot\|_{(Y')'}$ *are parallel*;
(3) (*generalized Hölder inequality*) *if $f \in Y'$ and $g \in Y$, then*

$$\int_{\mathbb{R}^n} |g(x)f(x)| \le \|f\|_{Y'}\|g\|_Y.$$

Lemma 3.2 [46] *Consider a Banach function space Y. M is weakly bounded on Y, such that*

$$\|\chi_{\{Mf>\rho\}}\|_Y \lesssim \rho^{-1}\|f\|_Y$$

is true for $\rho > 0$ and every $f \in Y$. Then we have

$$\sup_{B:\mathrm{ball}} \frac{1}{|B|}\|\chi_B\|_Y\|\chi_B\|_{Y'} < \infty.$$

Lemma 3.3 [39] *For all balls B and for a Banach function space Y, we have*

$$1 \le \frac{1}{|B|}\|\chi_B\|_Y\|\chi_B\|_{Y'}.$$

Lemma 3.4 [39] *If Y is a Banach function space and M is bounded on Y', then for any $E \subset \mathbb{R}^n$ and $S \subset E$, there exists a constant $\delta \in (0,1)$ such that*

$$\frac{\|\chi_S\|_Y}{\|\chi_E\|_Y} \lesssim \left(\frac{|S|}{|E|}\right)^\delta.$$

Lemma 3.5 [47] *We have*:

(1) *$Y(\mathbb{R}^n, W)$ is a Banach function space with respect to the norm*

$$\|f\|_{Y(\mathbb{R}^n,W)} = \|fw\|_Y,$$

where

$$Y(\mathbb{R}^n, W) = \{f \in \mathbb{M} : fW \in Y\};$$

(2) *$Y'(\mathbb{R}^n, W^{-1})$ is also a Banach function space.*

Remark 3.6 Consider $p(\cdot)\in P(\mathbb{R}^n)$. When we compare Lebesgue space $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ with $Y(\mathbb{R}^n, W)$, we get:

1: if $w = W$ and $Y = L^{p(\cdot)}(\mathbb{R}^n)$, then $L^{p(\cdot)}(w^{p(\cdot)}) = L^{p(\cdot)}(\mathbb{R}^n, w)$;

2: if $w^{-1} = W$ and $Y = L^{p'(\cdot)}(\mathbb{R}^n)$, then $L^{p'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p'(\cdot)}(w^{-p'(\cdot)})$.

By the result of Lemma 3.5, we obtain

$$L^{p'(\cdot)}\big(\mathbb{R}^n, w^{-1}\big) = \big(L^{p(\cdot)}\big(\mathbb{R}^n, w\big)\big)' = L^{p'(\cdot)}\big(w^{-p'(\cdot)}\big) = \big(L^{p(\cdot)}\big(w^{p(\cdot)}\big)\big)'.$$

Lemma 3.7 [48] *Let $p(\cdot)\in P(\mathbb{R}^n)\cap \mathscr{C}^{\log}(\mathbb{R}^n)$ and $w^{p_2(\cdot)}\in A_{p_2(\cdot)}$ admit $w^{-p_2'(\cdot)}\in A_{p_2'(\cdot)}$. Then there exist $\delta_{11},\delta_{22}\in(0,1)$ such that*

$$\frac{\|\chi_S\|_{(L^{p_2(\cdot)}w^{p_2(\cdot)})'}}{\|\chi_E\|_{(L^{p_2(\cdot)}w^{p_2(\cdot)})'}} \lesssim \left(\frac{|S|}{|E|}\right)^{\delta_{22}}, \qquad \frac{\|\chi_S\|_{(L^{p_1(\cdot)}w^{p_1(\cdot)})'}}{\|\chi_E\|_{(L^{p_1(\cdot)}w^{p_1(\cdot)})'}} \lesssim \left(\frac{|S|}{|E|}\right)^{\delta_{11}} \tag{7}$$

for every ball E and for each measurable set $S\subset E$.

Lemma 3.8 [39] *Let $p(\cdot)\in P(\mathbb{R}^n)\cap \mathscr{C}^{\log}(\mathbb{R}^n)$, $0<\beta<\frac{n}{p_+}$, and $\frac{1}{q(\cdot)}=\frac{1}{p(\cdot)}-\frac{\beta}{n}$. Then I^β is bounded from $L^{p(\cdot)}(w^{P(\cdot)})$ to $L^{q(\cdot)}(w^{q(\cdot)})$ for $w\in A(p(\cdot),q(\cdot))$.*

Lemma 3.9 [49] *Assume that $q(\cdot)\in P(\mathbb{R}^n)$. Then for all $b\in BMO$ and all $j,i\in Z$ with $j>i$ we have*

$$C^{-1}\|b\|_{BMO}\le \sup_{\mathrm{B:Ball}} \frac{1}{\|\chi_{\mathrm{B}}\|_{L^{q(\cdot)}}}\big\|(b-b_{\mathrm{B}})\chi_{\mathrm{B}}\big\|_{L^{q(\cdot)}}\le C\|b\|_{BMO}, \tag{8}$$

$$\big\|(b-b_{B_i})\chi_{B_j}\big\|_{L^{q(\cdot)}}\le C(j-i)\|b\|_{BMO}\|\chi_{\mathrm{B}_j}\|_{L^{q(\cdot)}}. \tag{9}$$

4 Main results

Lemma 4.1 *If $w^{p_2(\cdot)}, w^{p_1(\cdot)}\in A_1$, $q(\cdot)\in P(\mathbb{R}^n)\cap \mathscr{C}^{\log}(\mathbb{R}^n)$, and $p(\cdot)$ is such that $\frac{1}{q(\cdot)}=\frac{1}{p(\cdot)}-\frac{\beta}{n}$ with $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, then*

$$\|\chi_{B_k}\|_{L^{q(\cdot)}(w^{q(\cdot)})}\le C2^{k(2n-\beta)}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}.$$

Proof We assume that $f=\chi_{B_k}$ and use the definition of I_β

$$I_\beta(\chi_{B_k})(x)\ge C2^{k\beta}\chi_{B_k}(x),$$
$$\chi_{B_k}(x)\le C2^{-k\beta}I_\beta(\chi_{B_k})(x).$$

Applying the norm on both sides and using the results of Lemmas 3.2 and 3.8, we obtain

$$\begin{aligned}\|\chi_{B_k}\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C2^{-k\beta}\|I_\beta\chi_{B_k}\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\ &\le C2^{-k\beta}\|\chi_{B_k}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\ &\le C2^{-k\beta}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_k}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\ &\le C2^{k(2n-\beta)}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}.\end{aligned} \tag{10}$$

□

Proposition 4.2 [11] *Let $p(\cdot) \in P(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$. If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, then*

$$\begin{aligned}
&\big\| H_\beta(f_1,f_2) \cdot \chi_k \big\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w^{p(\cdot)})} \\
&= \sup_{k_0 \in Z} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q} \big\| H_\beta(f_1,f_2) \cdot \chi_k \big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \\
&\approx \max \Bigg\{ \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \big\| H_\beta(f_1,f_2) \cdot \chi_k \big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})}, \\
&\quad \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0\lambda q} \Bigg(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \big\| H_\beta(f_1,f_2) \cdot \chi_k \big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \\
&\quad + \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q} \big\| H_\beta(f_1,f_2) \cdot \chi_k \big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Bigg) \Bigg\}.
\end{aligned}$$

Theorem 4.3 *Let $0 < q, q_1, q_2 < \infty$, $q(\cdot) \in P(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, and $p(\cdot)$ be such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\beta}{n}$ with $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Also, let $w^{p_2(\cdot)}, w^{p_1(\cdot)} \in A_1$, let $\lambda = \lambda_1 + \lambda_2$, and let $\alpha(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a log Hölder continuous function at the origin satisfying $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha(\infty) = \alpha_1(\infty) + \alpha_2(\infty)$ with $\alpha(0) \leq \alpha(\infty) < n\delta_{11} + \lambda + n\delta_{22}$, where $\delta_{22}, \delta_{11} \in (0,1)$ are the constants arising in (7). Then*

$$\big\| H_\beta(f_1,f_2) \big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})} \leq C \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Proof For each $f_1, f_2 \in M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})$, if we express $f_{1j} = f_1 \cdot \chi_j = f_1 \cdot \chi_{A_j}$ and $f_{2j} = f_2 \cdot \chi_j = f_2 \cdot \chi_{A_j}$ for any $j \in \mathbb{Z}$, then we have

$$\begin{aligned}
f_1(x) &= \sum_{j=-\infty}^{\infty} f_1(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_{1j}(x), \\
f_2(x) &= \sum_{j=-\infty}^{\infty} f_2(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_{2j}(x).
\end{aligned}$$

The generalized Hölder inequality gives

$$\begin{aligned}
\big| H_\beta(f_1,f_2)(x) \cdot \chi_k(x) \big| &\leq \frac{1}{|x|^{2n-\beta}} \int_{B_k} \int_{B_k} \big| f_1(t_1) \big| \big| f_2(t_2) \big| \, dt_1 \, dt_2 \cdot \chi_k(x) \\
&= \frac{1}{|x|^{2n-\beta}} \int_{B_k} \big| f_1(t_1) \big| | \, dt_1 \int_{B_k} f_2(t_2) \, dt_2 \cdot \chi_k(x) \\
&\leq C 2^{-2kn} \sum_{j=-\infty}^{k} \|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\quad \times \|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'} 2^{k\beta} \chi_k(x).
\end{aligned} \tag{11}$$

Utilizing Lemmas 3.7 and 3.2, we acquire

$$\begin{aligned}
&\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\quad\le C2^{k\beta}\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\qquad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}2^{-2kn}\|\chi_k\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\quad\le C\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\qquad\times\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}2^{-k(2n-\beta)}\|\chi_k\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}\tag{12}$$

To move forward, we use Lemma 4.1 and (12), and we obtain

$$\begin{aligned}
&\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\quad\le C\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\\
&\qquad\times\|\chi_k\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\\
&\quad\le C\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\frac{\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_k\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\frac{\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_k\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\\
&\quad\le C\sum_{j=-\infty}^{k}2^{n\delta_{11}(j-k)}2^{n\delta_{22}(j-k)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\le C\sum_{j=-\infty}^{k}2^{(n\delta_{11}+n\delta_{22})(j-k)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}\tag{13}$$

In the remainder of the proof, in order to calculate $\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$ and $\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$, we consider the following two cases.

Case 1: For $j<0$,

$$\begin{aligned}
\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}&=2^{-j\alpha_1(0)}\left(2^{j\alpha_1(0)q_1}\|f_{1j}\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le 2^{-j\alpha_1(0)}\left(\sum_{i=-\infty}^{j}2^{i\alpha_1(0)q_1}\|f_{1i}\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le 2^{j(\lambda_1-\alpha_1(0))}2^{-j\lambda_1}\left(\sum_{i=-\infty}^{j}2^{i\alpha_1(\cdot)q_1}\|f_{1i}\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le C2^{j(\lambda_1-\alpha_1(0))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Similarly

$$\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\le C2^{j(\lambda_2-\alpha_2(0))}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Here we are using $\alpha(0) = \alpha_1(0) + \alpha_2(0)$ and $\lambda = \lambda_1 + \lambda_2$:

$$\begin{aligned}
&\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\quad\leq C2^{j(\lambda_1-\alpha_1(0))}2^{j(\lambda_2-\alpha_2(0))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\quad= C2^{j(\lambda-\alpha(0))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

Case 2: For $j \geq 0$,

$$\begin{aligned}
\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &= 2^{-j\alpha_1(\infty)}\left(2^{j\alpha_1(\infty)q_1}\|f_{1j}\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}} \\
&\leq 2^{-j\alpha_1(\infty)}\left(\sum_{i=0}^{j}2^{i\alpha_1(\infty)q_1}\|f_{1i}\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}} \\
&\leq 2^{j(\lambda_1-\alpha_1(\infty))}2^{-j\lambda_1}\left(\sum_{i=-\infty}^{j}2^{i\alpha_1(\cdot)q_1}\|f_{1i}\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}} \\
&\leq C2^{j(\lambda_1-\alpha_1(\infty))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}. \qquad (14)
\end{aligned}$$

Similarly

$$\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C2^{j(\lambda_2-\alpha_2(\infty))}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Here we are using $\alpha(\infty) = \alpha_1(\infty) + \alpha_2(\infty)$ and $\lambda = \lambda_1 + \lambda_2$:

$$\begin{aligned}
&\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\quad\leq C2^{j(\lambda_1-\alpha_1(\infty))}2^{j(\lambda_2-\alpha_2(\infty))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\quad= C2^{j(\lambda-\alpha(\infty))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

With the use of the definition of variable exponent Herz–Morrey space and Proposition 4.2, we can get the following inequality:

$$\begin{aligned}
&\left\|H_\beta(f_1,f_2)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})} \\
&\quad= \sup_{k_0\in Z}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(\cdot)q}\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}} \\
&\quad\leq \max\left\{\sup_{\substack{k_0\in Z\\ k_0<0}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q}\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}},\right. \qquad (15) \\
&\qquad \sup_{\substack{k_0\in Z\\ k_0\geq 0}}2^{-k_0\lambda}\left(\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\sum_{k=0}^{k_0}2^{k\alpha(\infty)q}\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}}\Bigg)\Bigg\}$$

$$=\max\{Y_1,Y_2+Y_3\},$$

where

$$Y_1=\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q}\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}},$$

$$Y_2=\sup_{\substack{k_0\in Z\\k_0\ge0}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}},$$

$$Y_3=\sup_{\substack{k_0\in Z\\k_0\ge0}}2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\alpha(\infty)q}\left\|H_\beta(f_1,f_2)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}}.$$

First, we will find the estimate of Y_1. Since $\alpha(0)\le\alpha(\infty)<n\delta_{11}+n\delta_{22}+\lambda$,

$$\begin{aligned}
Y_1&\le C\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k}2^{(-n\delta_{22}-n\delta_{11})(k-j)}\|f_1\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\right)^q\right)^{\frac{1}{q}}\\
&\le C\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k}2^{(-n\delta_{11}-n\delta_{22})(k-j)}2^{j(\lambda-\alpha(0))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\right.\right.\\
&\quad\left.\left.\times\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\right)^q\right)^{\frac{1}{q}}\\
&\le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\times\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\lambda q}\left(\sum_{j=-\infty}^{k}2^{(n\delta_{11}+n\delta_{22}+\lambda-\alpha(0))(j-k)}\right)^q\right)^{\frac{1}{q}}\\
&\le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

The estimate of Y_2 is the same as that of Y_1. Finally, we approximate Y_3:

$$\begin{aligned}
Y_3&\le C\sup_{\substack{k_0\in Z\\k_0\ge0}}2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\alpha(\infty)q}\left(\sum_{j=-\infty}^{k}2^{(-n\delta_{22}-n\delta_{11})(k-j)}\|f_1\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\right)^q\right)^{\frac{1}{q}}\\
&\le C\sup_{\substack{k_0\in Z\\k_0\ge0}}2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\alpha(\infty)q}\left(\sum_{j=-\infty}^{k}2^{(-n\delta_{11}-n\delta_{22})(k-j)}2^{j(\lambda-\alpha(\infty))}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\right.\right.\\
&\quad\left.\left.\times\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\right)^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\times\sup_{\substack{k_0\in Z\\k_0\geq 0}}2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\lambda q}\left(\sum_{j=-\infty}^{k}2^{(n\delta_{11}+n\delta_{22}+\lambda-\alpha(\infty))(j-k)}\right)^q\right)^{\frac{1}{q}}\\
&\leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

Putting the approximations of Y_1, Y_2, and Y_3 into (15) yields the required outcome. □

Theorem 4.4 *Let q_1, q_2, q, $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, w, $\alpha(\cdot)$, and β be as in Theorem 4.3. Additionally, if $\alpha(\infty)\geq\alpha(0)>\lambda-n\delta$, where $\delta\in(0,1)$ is a constant arising in (3.4), then*

$$\left\|H^*_\beta(f_1,f_2)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})}\leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Proof By utilizing Hölder's inequality, we obtain

$$\begin{aligned}
\left|H^*_\beta(f_1,f_2)(x)\cdot\chi_k(x)\right|&\leq\int_{R^n\setminus B_k}\int_{R^n\setminus B_k}\frac{1}{|t|^{2n-\beta}}\left|f_1(t_1)\right|\left|f_2(t_2)\right|dt_1\,dt_2\cdot\chi_k(x)\\
&\leq C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\times\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\chi_k(x),\\
\left\|H^*_\beta(f_1,f_2)(x)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}&\leq C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\times\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\|\chi_k\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}\tag{16}$$

In the light of inequality (10), we acquire

$$\begin{aligned}
\left\|H^*_\beta(f_1,f_2)(x)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}&\leq C\sum_{j=k+1}^{\infty}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\times\|\chi_j\|^{-1}_{L^{q(\cdot)}(w^{q}(\cdot))}\|\chi_k\|_{L^{q(\cdot)}(w^{q}(\cdot))}\\
&\leq C\sum_{j=k+1}^{\infty}2^{n\delta(k-j)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})},
\end{aligned}\tag{17}$$

where we used the result of Lemma 3.7.

We will use the same procedure as in Theorem 4.3 to obtain

$$\left\|H^*_\beta(f_1,f_2)\right\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})}=\max\{Z_1,Z_2+Z_3\},\tag{18}$$

where

$$Z_1=\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q}\left\|H^*_\beta(f_1,f_2)(x)\cdot\chi_k\right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})},$$

$$Z_2 = \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left\| H_\beta^*(f_1,f_2)(x)\cdot \chi_k \right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})},$$

$$Z_3 = \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q} \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q} \left\| H_\beta^*(f_1,f_2)(x)\cdot \chi_k \right\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}.$$

The boundedness of Z_l ($l = 1,2,3$) is alike to that of Y_l ($l = 1,2,3$) of Theorem 4.3. Here we are close to our result. □

Theorem 4.5 *Let $0 < q, q_1, q_2 < \infty$, $q(\cdot) \in P(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, and $p(\cdot)$ be such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\beta}{n}$ with $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Also, let $w^{p_2(\cdot)}, w^{p_1(\cdot)} \in A_1$, let $\lambda = \lambda_1 + \lambda_2$, and let $\alpha(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a log Hölder continuous function at the origin satisfying $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha(\infty) = \alpha_1(\infty) + \alpha_2(\infty)$ with $\alpha(0) \leq \alpha(\infty) < n\delta_{11} + \lambda + n\delta_{22}$, where $\delta_{22}, \delta_{11} \in (0,1)$ are the constants arising in (7). Then $[b, H_\beta]$ is bounded from $M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)}) \times M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})$ to $M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})$, where $b = (b_1, b_2)$ and $b_1, b_2 \in BMO$.*

Proof We have

$$\begin{aligned}
&\left|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k(z)\right| \\
&\quad\leq \frac{1}{|z|^{2n-\beta}} \int_{B_k}\int_{B_k} \left|f_1(t_1)f_1(t_2)\big(b_1(z)-b(t_1)\big)\right| dt_1\, dt_2 \cdot \chi_k(z) \\
&\quad\leq C2^{-k(2n-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k} \left|f_1(t)f_2(t)\big(b_1(z)-(b_1)_{B_j}+(b_1)_{B_j}-b_1(t_1)\big)\right| dt_1\, dt_2 \cdot \chi_k(z) \\
&\quad\leq C2^{-k(2n-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k} \left|f_2(t_2)f_1(t_1)\big(b_1(z)-(b_1)_{B_j}\big)\right| dt_1\, dt_2 \cdot \chi_k(z) \\
&\qquad + C2^{-k(2n-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k} \left|f_2(t_2)f_1(t_1)\big(b_1(t_1)-(b_1)_{B_j}\big)\right| dt_1\, dt_2 \cdot \chi_k(z) \\
&\quad= I + II,
\end{aligned}$$

$$I = C2^{-k(2n-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k} \left|f_2(t_2)f_1(t_1)\big(b_1(z)-(b_1)_{B_j}\big)\right| dt_1\, dt_2 \cdot \chi_k(z).$$

By utilizing the Hölder inequality, we obtain

$$\begin{aligned}
I &\leq C2^{-k(2n-\beta)} \sum_{j=-\infty}^{k} \left|\big(b_1(z)-(b_1)_{B_j}\big)\cdot\chi_k(z)\right| \|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\quad\times \|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}.
\end{aligned}$$

Utilizing Lemmas 3.7, 3.2, and 3.9, we acquire

$$\begin{aligned}
&\|I\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\quad\le C2^{k\beta}\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\qquad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}2^{-2kn}\big\|\big(b_1(z)-(b_1)_{B_j}\big)\chi_k\big\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\quad\le C2^{-k(2n-\beta)}\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\\
&\qquad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}(k-j)\|b_1\|_{BMO}\|\chi_k\|_{L^{q(\cdot)}(w^{q(\cdot)})},
\end{aligned}\tag{19}$$

$$\begin{aligned}
II&=2^{-k(2n-\beta)}\sum_{j=-\infty}^{k}\int_{B_k}\int_{B_k}\big|f_2(t_2)f_1(t_1)\big(b_1(t_1)-(b_1)_{B_j}\big)\big|\,dt_1\,dt_2\cdot\chi_k(z)\\
&=2^{-k(2n-\beta)}\sum_{j=-\infty}^{k}\int_{B_k}\bigg|f_2(t_2)\,dt_2\int_{B_k}f_1(t_1)\big(b_1(t_1)-(b_1)_{B_j}\big)\bigg|\,dt_1\cdot\chi_k(z)\\
&\le C2^{-k(2n-\beta)}\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\big\|\big(b_1(t_1)-(b_1)_{B_j}\big)\chi_j\big\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\cdot\chi_k(z),
\end{aligned}$$

$$\begin{aligned}
\|II\|_{L^{q(\cdot)}(w^{q(\cdot)})}&\le C2^{-k(2n-\beta)}\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\big\|\big(b_1(t_1)-(b_1)_{B_j}\big)\chi_j\big\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\\
&\quad\times\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\big\|\chi_k(x)\big\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\le C2^{-k(2n-\beta)}\sum_{j=-\infty}^{k}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|b_1\|_{BMO}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\\
&\quad\times\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\big\|\chi_k(x)\big\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}\tag{20}$$

From inequalities (19) and (20), we have

$$\begin{aligned}
&\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\quad\le C2^{-k(2n-\beta)}\sum_{j=-\infty}^{k}(k-j)\|b_1\|_{BMO}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\\
&\qquad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\|\chi_k\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}$$

To move forward, we use Theorem 4.3:

$$\begin{aligned}
&\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\quad\le C\sum_{j=-\infty}^{k}2^{(n\delta_{11}+n\delta_{22})(j-k)}(k-j)\|b_1\|_{BMO}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}\tag{21}$$

With the use of the definition of variable exponent Herz–Morrey space and Proposition 4.2 we get the following inequality:

$$\begin{aligned}
&\big\|[b_1,H_\beta](f_1,f_2)(z)\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})}\\
&= \sup_{k_0\in Z} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q}\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}}\\
&\le \max\left\{\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}},\right.\\
&\quad \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda}\left(\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}}\right.\\
&\quad \left.\left.+\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}}\right)\right\}\\
&= \max\{A_1, A_2+A_3\},
\end{aligned} \tag{22}$$

where

$$A_1 = \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}},$$

$$A_2 = \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}},$$

$$A_3 = \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\big\|[b_1,H_\beta](f_1,f_2)(z)\cdot\chi_k\big\|^q_{L^{q(\cdot)}(w^{q(\cdot)})}\right)^{\frac{1}{q}}.$$

In the remainder of the proof, we use the same calculation as in Theorem 4.3. We obtain the following result:

$$\big\|[b_1,H_\beta](f_1,f_2)\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})} \le C\|b_1\|_{BMO}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Similarly, we can easily estimate the following result:

$$\big\|[b_2,H_\beta](f_1,f_2)\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})} \le C\|b_2\|_{BMO}\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}. \qquad \square$$

Theorem 4.6 *Let q_1, q_2, q, $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, w, $\alpha(\cdot)$, and β be as in Theorem 4.3. Additionally, if $\alpha(\infty)\ge\alpha(0)>\lambda-n\delta$, where $\delta\in(0,1)$ is a constant arising in (3.4), then $[b,H^*_\beta]$ is bounded from $M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})\times M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})$ to $M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(w^{q(\cdot)})$, where $b=(b_1,b_2)$ and $b_1,b_2\in BMO$.*

Proof We have

$$\begin{aligned}
&\big|\big[b_1,H_\beta^*\big](f_1,f_2)(z)\cdot\chi_k(z)\big| \\
&\quad\le \int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\frac{1}{|t|^{2n-\beta}}\big|f_1(t_1)f_1(t_2)\big(b_1(z)-b(t_1)\big)\big|\,dt_1\,dt_2\cdot\chi_k(z) \\
&\quad\le C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\big|f_1(t_1)f_2(t_2)\big(b_1(z)-(b_1)_{B_j}+(b_1)_{B_j}-b_1(t_1)\big)\big|\,dt_1\,dt_2 \\
&\qquad\cdot\chi_k(z) \\
&\quad\le C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\big|f_2(t_2)f_1(t_1)\big(b_1(z)-(b_1)_{B_j}\big)\big|\,dt_1\,dt_2\cdot\chi_k(z) \\
&\qquad+C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\big|f_2(t_2)f_1(t_1)\big(b_1(t_1)-(b_1)_{B_j}\big)\big|\,dt_1\,dt_2\cdot\chi_k(z) \\
&\quad= III+IV, \\
&III = C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\big|f_2(t_2)f_1(t_1)\big(b_1(z)-(b_1)_{B_j}\big)\big|\,dt_1\,dt_2\cdot\chi_k(z).
\end{aligned}$$

By utilizing the Hölder inequality, we obtain

$$\begin{aligned}
III &\le C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\big|\big(b_1(z)-(b_1)_{B_j}\big)\cdot\chi_k(z)\big|\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\quad\times\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}.
\end{aligned}$$

Utilizing Lemmas 3.7, 3.2, and 3.9, we acquire

$$\begin{aligned}
&\|III\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\quad\le C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\qquad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\big\|\big(b_1(z)-(b_1)_{B_j}\big)\chi_k\big\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\quad\le C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\qquad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}(k-j)\|b_1\|_{BMO}\|\chi_k\|_{L^{q(\cdot)}(w^{q(\cdot)})}, \qquad (23) \\
&IV = \sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\big|f_2(t_2)f_1(t_1)\big(b_1(t_1)-(b_1)_{B_j}\big)\big|\,dt_1\,dt_2\cdot\chi_k(z) \\
&\quad= \sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\big|f_2(t_2)f_1(t_1)\big(b_1(t_1)-(b_1)_{B_j}\big)\big|\,dt_2\,dt_1\cdot\chi_k(z) \\
&\quad\le C\sum_{j=k+1}^{\infty}2^{-j(2n-\beta)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\big\|\big(b_1(t_1)-(b_1)_{B_j}\big)\chi_j\big\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\qquad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\cdot\chi_k(x),
\end{aligned}$$

$$
\begin{aligned}
\|IV\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\leq C\sum_{j=k+1}^{\infty} 2^{-j(2n-\beta)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\left\|\left(b_1(t_1)-(b_1)_{B_j}\right)\chi_j\right\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\quad\times \|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\left\|\chi_k(x)\right\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\leq C\sum_{j=k+1}^{\infty} 2^{-j(2n-\beta)}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|b_1\|_{BMO}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\quad\times \|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\left\|\chi_k(x)\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}
\tag{24}
$$

From inequalities (23) and (24), we have

$$
\begin{aligned}
&\left\|\left[b_1,H_\beta^*\right](f_1,f_2)(z)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\quad\leq C\sum_{j=k+1}^{\infty} 2^{-j(2n-\beta)}(k-j)\|b_1\|_{BMO}\|f_{1j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{2j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\quad\times\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\|\chi_k\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}
$$

With the use of Proposition 4.2 we get the following inequality:

$$
\begin{aligned}
&\left\|\left[b_1,H_\beta^*\right](f_1,f_2)\right\|_{M\dot{K}_{q,q(\cdot)}^{\alpha(\cdot),\lambda}(w^{q(\cdot)})} \\
&\quad= \sup_{k_0\in Z} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q}\left\|\left[b_1,H_\beta^*\right](f_1,f_2)(z)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}^q\right)^{\frac{1}{q}} \\
&\quad\leq \max\left\{\sup_{\substack{k_0\in Z\\k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\left\|\left[b_1,H_\beta^*\right](f_1,f_2)(z)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}^q\right)^{\frac{1}{q}},\right. \\
&\quad\sup_{\substack{k_0\in Z\\k_0\geq 0}} 2^{-k_0\lambda}\left(\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\left\|\left[b_1,H_\beta^*\right](f_1,f_2)(z)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.\left.+\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\left\|\left[b_1,H_\beta^*\right](f_1,f_2)(z)\cdot\chi_k\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}^q\right)^{\frac{1}{q}}\right)\right\} \\
&\quad= \max\{B_1,B_2+B_3\}.
\end{aligned}
\tag{25}
$$

We can easily find the estimates of B_i $(i=1,2,3)$ by similar methods to the ones in Theorem 4.5. □

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Data availability
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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
M.A: conceptualization, supervision, writing—original draft; I.A.: writing—original draft, methodology; A.H.: investigation, writing—review and editing; N.M.: conceptualization, supervision, writing—original draft. All authors reviewed the manuscript.

Author details
[1]Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan. [2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia.

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