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Inertial Halpern-type iterative algorithm for the generalized multiple-set split feasibility problem in Banach spaces

Mohammad Eslamian[1*]

*Correspondence:
mhmdeslamian@gmail.com
[1]Department of Mathematics, University of Science and Technology of Mazandaran, Behshahr, Iran

Abstract

In this paper, we study the generalized multiple-set split feasibility problem including the common fixed-point problem for a finite family of generalized demimetric mappings and the monotone inclusion problem in 2-uniformly convex and uniformly smooth Banach spaces. We propose an inertial Halpern-type iterative algorithm for obtaining a solution of the problem and derive a strong convergence theorem for the algorithm. Then, we apply our convergence results to the convex minimization problem, the variational inequality problem, the multiple-set split feasibility problem and the split common null-point problem in Banach spaces.

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Keywords: Monotone inclusion problem; Multiple-set split feasibility problem; Generalized demimetric mappings; 2-uniformly convex Banach space

1 Introduction

Let E be a real Banach space with a dual space E^*. We consider the monotone inclusion problem, which involves a single-valued monotone operator $A : E \to E^*$ and a multivalued monotone operator $B : E \to 2^{E^*}$. The problem aims to find a solution $x^* \in E$ such that $0 \in (A + B)x^*$. The set of solutions to this problem is denoted by $(A + B)^{-1}0$. This problem has practical applications in various fields such as image recovery, signal processing, and machine learning. It also encompasses several mathematical problems as special cases, including variational inequalities, split feasibility problems, minimization problems, and Nash equilibrium problems in noncooperative games, see, for instance, [1–4] and the references therein. The forward–backward splitting algorithm, introduced in [1, 2], is a well-known method for approximating solutions to the monotone inclusion problem. In this method, defined in a Hilbert space $\mathcal{H}$, we start with an initial iterate $x_1 \in \mathcal{H}$ and update subsequent iterates as follows:

$$x_{n+1} = (I + \lambda B)^{-1}(I - \lambda A)x_n,$$

where $n \geq 1$ and λ is a step-size parameter. Over the past few decades, the convergence properties and modified versions of the forward–backward splitting algorithm have been extensively studied in the literature. In [5] (see also [6]), Takahashi et al. introduced and analyzed a Halpern-type modification of the forward–backward splitting algorithm in real Hilbert spaces. They established the strong convergence of the sequence generated by their algorithm to a solution of the monotone inclusion problem. In 2019, Kimura and Nakajo [7] proposed a modified forward–backward splitting method and proved a strong convergence theorem for solutions of the monotone inclusion problem in a real 2-uniformly convex and uniformly smooth Banach space. Their convergence result is as follows:

Theorem 1.1 *Let C be a nonempty, closed, and convex subset of a real 2-uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be an α-inverse strongly monotone and $B : E \to 2^{E^*}$ be a maximal monotone. Suppose that $\Omega = (A + B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$, $u \in E$ and by*:

$$x_{n+1} = \Pi_C J^B_{\lambda_n} J_E^{-1}\big(\gamma_n J_E u + (1 - \gamma_n) J_E x_n - \lambda_n A x_n\big), \quad n \geq 0,$$

where Π_C is the generalized projection of E onto C, $J^B_{\lambda_n}$ is the resolvent of B, J_E is the normalized duality mapping of E, $\{\lambda_n\} \subset (0, \infty)$ and $\{\gamma_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then, if $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < 2\alpha c$, (where c is the constant in Lemma 2.2), the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega u$.

The inertial technique has gained significant interest among researchers due to its favorable convergence characteristics and the ability to improve algorithm performance. It was first considered in [8] for solving smooth, convex minimization problems. The key idea behind these methods is to utilize two previous iterates to update the next iterate, resulting in accelerated convergence. In [9], Lorenz and Pock proposed an inertial forward–backward splitting algorithm in real Hilbert spaces, formulated as follows:

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ y_n = x_n + \theta_n (x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n B)^{-1}(y_n - \lambda_n A y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1}$$

where $\theta_n \in [0, 1)$ is an extrapolation factor, λ_n is a step-size parameter, and $\mathcal{H}$ represents the Hilbert space. They proved that the sequence x_n generated by this algorithm converges weakly to a zero of $A + B$.

In addition to the monotone inclusion problem, we consider the fixed-point problem for a nonlinear mapping $U : E \to E$. A point $x \in E$ such that $Ux = x$ is called a fixed point of U. The set of fixed points of the nonlinear mapping U is denoted by $\mathrm{Fix}(U)$.

Let E and F be Banach spaces and $T : E \to F$ be a bounded linear operator. Let $\{C_i\}_{i=1}^p$ be a family of nonempty, closed, and convex subsets in E and $\{Q_j\}_{j=1}^r$ be a family of nonempty, closed, and convex subsets in F. The multiple-set split feasibility problem (MSSFP) is formulated as finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{p} C_i \quad \text{and} \quad Tx^\star \in \bigcap_{j=1}^{r} Q_j.$$

The multiple-set split feasibility problem with $p = r = 1$ is known as the split feasibility problem (SFP). The MSSFP was first introduced by Censor et al. [10] in the framework of Hilbert spaces for modeling inverse problems that arise from phase retrievals and medical image reconstruction. Such models were successfully developed, for instance, in radiation-therapy treatment planning, sensor networks, resolution enhancement, and so on [11, 12]. Initiated by SFP, several split-type problems have been investigated and studied. For example, the split common fixed-point problem [13], the split monotone variational inclusions problem [14], and the split common null-point problem [15]. Algorithms for solving these problems have received great attention, (see, e.g., [16–31] and some of the references therein).

In 2018, Kawasaki and Takahashi [32], introduced a new general class of mappings, called generalized demimetric mappings as follows:

Definition 1.2 Let E be a smooth, strictly convex, and reflexive Banach space. Let ζ be a real number with $\zeta \neq 0$. A mapping $U : E \to E$ with $\mathrm{Fix}(U) \neq \emptyset$ is called generalized demimetric, if

$$\zeta \langle x - x^\star, J_E(x - Ux) \rangle \geq \|x - Ux\|^2,$$

for all $x \in E$ and $x^\star \in \mathrm{Fix}(U)$. This mapping U is called ζ-generalized demimetric.

Such a class of mappings is fundamental because it includes many types of nonlinear mappings arising in applied mathematics and optimization, see [32–34] for details.

In this paper, we study the following generalized multiple-set split feasibility problem: let E be a 2-uniformly convex and uniformly smooth Banach space and let F be a smooth, strictly convex, and reflexive Banach space. Let for $i = 1, 2, \dots, m$, $U_i : F \to F$ be a finite family of generalized demimetric mappings, $A_i : E \to E^*$ be an inverse strongly monotone and $B_i : E \to 2^{E^*}$ be maximal monotone mappings. Let for each $i = 1, 2, \dots, m$, $T_i : E \to F$ be a bounded linear operator such that $T_i \neq 0$. Consider the following problem:

$$\text{Find} \quad x^* \in \bigcap_{i=1}^{m} (A_i + B_i)^{-1} 0 \quad \text{such that } T_i x^* \in \mathrm{Fix}(U_i), \quad i = 1, 2, \dots, m. \tag{2}$$

By combining the concepts and techniques from the inertial algorithm, forward–backward splitting algorithm, and the Halpern method, we introduce a new and efficient iterative method for solving the generalized multiple-set split feasibility problem. We establish strong convergence theorems for the proposed method under standard and mild conditions. The iterative scheme does not require prior knowledge of the operator norm. Finally, we apply our convergence results to the convex minimization problem, variational inequality problem, multiple-set split feasibility problem, and split common null-point problem in Banach spaces. The main results presented in this paper improve and generalize the previous results obtained by Kimura and Nakajo [7], Takahashi et al. [17] and others.

2 Preliminaries

Let E be a real Banach space with norm $\|.\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in S(E)$ with $x \neq y$. The Banach space E is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for each $x, y \in S(E)$. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S(E), \|x - y\| \geq \epsilon\right\}.$$

The space E is called uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$, and 2-uniformly convex if there is a $c > 0$ so that $\delta_E(\epsilon) \geq c\epsilon^2$ for any $\epsilon \in (0, 2]$. We know that a uniformly convex Banach space is strictly convex and reflexive. The modulus of smoothness of E is defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S(E)\right\}, \quad \forall \tau > 0.$$

The space E is called uniformly smooth if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$, and 2-uniformly smooth if there exists a $C > 0$ so that $\rho_E(\tau) \leq C\tau^2$ for any $\tau > 0$. It is observed that every 2-uniformly convex (2-uniformly smooth) space is a uniformly convex (uniformly smooth) space. It is known that E is 2-uniformly convex (2-uniformly smooth) if and only if its dual E^* is 2-uniformly smooth (2-uniformly convex). It is known that all Hilbert spaces and Lebesgue spaces L_p for $1 < p \leq 2$ are uniformly smooth and 2-uniformly convex. See [35, 36] for more details.

The normalized duality mapping of E into 2^{E^*} is defined by

$$J_E x = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}$$

for all $x \in E$. The normalized duality mapping J_E has the following properties (see, e.g., [36]):

- if E is a smooth, strictly convex, and reflexive Banach space, then J_E is a single-valued bijection and in this case, the inverse mapping J_E^{-1} coincides with the duality mapping J_{E^*};
- if E is uniformly smooth, then J_E is uniformly norm-to-norm continuous on each bounded subset of E.

The following lemma was proved by Xu [37].

Lemma 2.1 *Let E be a 2-uniformly smooth Banach space. Then, there exists a constant $\gamma > 0$ such that for every $x, y \in E$ there holds the following inequality*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J_E(x) \rangle + \gamma \|y\|^2.$$

Lemma 2.2 ([37]) *Let E be a smooth Banach space. Then, E is 2-uniformly convex if and only if there exists a constant $c > 0$ such that for each $x, y \in E$, $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, J_E(x)\rangle + c\|y\|^2$ holds.*

Lemma 2.3 ([38]) *If E is a 2-uniformly convex Banach space, then there exists a constant $d > 0$ such that for all $x, y \in E$,*

$$\|x - y\| \leq \frac{2}{d^2} \left\| J_E(x) - J_E(y) \right\|.$$

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, J_E y \rangle + \|y\|^2, \quad x, y \in E.$$

Observe that, in Hilbert spaces, $\phi(x, y)$ reduces to $\|x - y\|^2$. It is obvious from the definition of function ϕ that

$$\left(\|y\| - \|x\|\right)^2 \leq \phi(x, y) \leq \left(\|y\| + \|x\|\right)^2 \quad \forall x, y \in E. \tag{3}$$

We also know that if E is strictly convex and smooth, then, for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$; see [39, 40]. We will use the following mapping $V : E \times E^\star \to \mathbb{R}$ studied in [39]

$$V\left(x, x^*\right) = \|x\|^2 - 2\left\langle x, x^* \right\rangle + \left\| x^* \right\|^2,$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J_E^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.4 [39] *Let E be a reflexive, strictly convex, and smooth Banach space with E^* as its dual. Then,*

$$V\left(x, x^*\right) + 2\left\langle J_E^{-1}\left(x^*\right) - x, y^* \right\rangle \leq V\left(x, x^* + y^*\right)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.5 ([41]) *Let E be a uniformly convex and smooth real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$ then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty, closed, and convex subset of a strictly convex, reflexive, and smooth Banach space E and let $x \in E$. Then, there exists a unique element $\overline{x} \in C$ such that

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x).$$

We denote $\overline{x}$ by $\Pi_C x$ and call Π_C the generalized projection of E onto C; see [39, 41]. We have the following well-known result [39, 41] for the generalized projection.

Lemma 2.6 *Let C be a nonempty, convex subset of a smooth Banach space E, $x \in E$ and $\overline{x} \in C$. Then, $\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x)$ if and only if $\langle \overline{x} - y, J_E x - J_E \overline{x} \rangle \geq 0$ for every $y \in C$, or equivalently $\phi(y, \overline{x}) + \phi(\overline{x}, x) \leq \phi(y, x)$ for all $y \in C$.*

Definition 2.7 Let E be a Banach space and $T : E \to E$ be a nonlinear mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then, $I - T$ is said to be demiclosed at zero if $\{x_n\}$ is a sequence in E converges weakly to x and $(I - T)x_n$ converges strongly to zero, then $(I - T)x = 0$.

Lemma 2.8 ([32]) *Let E be a smooth, strictly convex, and reflexive Banach space and let ζ be a real number with $\zeta \neq 0$. Let $T : E \to E$ be a ζ-generalized demimetric mapping. Then, the fixed point set* $\mathrm{Fix}(T)$ *of T is closed and convex.*

Definition 2.9 An operator $A : E \to E^*$ is called an inverse, strongly monotone operator, if there exist $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in E.$$

In this case, we say that A is an α-inverse strongly monotone.

Let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in E : Bx \neq \emptyset\}$. A multivalued mapping B on E is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$ and $v \in By$. A monotone mapping B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone mapping on E. The set of null points of B is denoted by $B^{-1}0 = \{x \in E : 0 \in Bx\}$. Let E be a reflexive, strictly convex, and smooth real Banach space and let $B : E \to 2^{E^*}$ be a maximal monotone operator. Then, for any $\lambda > 0$ and $u \in E$, there exists a unique element $u_\lambda \in D(B)$ such that $J_E(u) \in J_E(u_\lambda) + \lambda B(u_\lambda)$. We define J_λ^B by $J_\lambda^B u = u_\lambda$ for every $u \in E$ and $\lambda > 0$ and such J_λ^B is called the resolvent of B. Alternatively, $J_\lambda^B = (J_E + \lambda B)^{-1} J_E$. Let E be a uniformly convex and smooth Banach space and B be a maximal monotone operator. Then, $B^{-1}0 = \mathrm{Fix}(J_\lambda^B)$ for all $\lambda > 0$. See [36, 42] for more details.

The lemma that follows is stated and proven in [[7], Lemma 3.1].

Lemma 2.10 *Let E be a 2-uniformly convex and uniformly smooth real Banach space. Let $A : E \to E^*$ be an α-inverse strongly monotone and $B : E \to 2^{E^*}$ be a maximal monotone. Let $T_\lambda(x) = J_\lambda^B J_E^{-1}(J_E x - \lambda Ax)$ for all $\lambda > 0$ and $x \in E$. Then, the following hold:*

(i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0$ *and* $(A + B)^{-1}0$ *is closed and convex;*

(ii) $\phi(x^*, T_\lambda(x)) \leq \phi(x^*, x) - (c - \lambda\beta)\|x - T_\lambda(x)\|^2 - \lambda(2\alpha - \frac{1}{\beta})\|Ax - Ax^*\|^2$ *for every $\beta > 0$ and $x^* \in (A + B)^{-1}0$, where c is the constant in Lemma 2.2.*

The following lemmas are very helpful for the convergence analysis of the algorithm.

Lemma 2.11 ([43]) *Let $\{\gamma_n\}$ be a sequence in $(0,1)$ and $\{\delta_n\}$ be a sequence in $\mathbb{R}$ satisfying*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

If $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n,$$

for each $n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 ([44]) *Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} \leq s_{n_i+1}$ for all $i \geq 0$. For every $n \in \mathbb{N}$, (sufficiently large) define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}.$$

3 Main results

We first prove the following lemma.

Lemma 3.1 *Let E be a Banach space and F be a smooth, strictly convex, and reflexive Banach space. Let J_F be the duality mappings on F. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Let $\zeta \neq 0$ and $U : F \to F$ be a ζ-generalized demimetric mapping. If $A^{-1}(\mathrm{Fix}(U)) \neq \emptyset$, then*

$$Ax - UAx = 0 \quad \Leftrightarrow \quad A^* J_F(Ax - UAx) = 0, \quad \forall x \in E.$$

Proof It is clear that for each $x \in E$, $Ax - UAx = 0$ implies that $A^* J_F(Ax - UAx) = 0$. To see the converse, let $x \in E$ such that $A^* J_F(Ax - UAx) = 0$. Taking $x^* \in A^{-1}(\mathrm{Fix}(U))$ we have

$$0 = \zeta \langle x - x^*, A^* J_F(Ax - UAx) \rangle = \zeta \langle Ax - Ax^*, J_F(Ax - UAx) \rangle \geq \|Ax - UAx\|^2,$$

which implies that $Ax - UAx = 0$. □

Now, in this position, we give our algorithm and its convergence analysis for the generalized multiple-set split feasibility problem in Banach spaces.

Theorem 3.2 *Let E be a 2-uniformly convex and uniformly smooth Banach space and let F be a smooth, strictly convex, and reflexive Banach space. Let J_E and J_F be the duality mappings on E and F, respectively. Let for $i = 1, 2, \ldots, m$, $\zeta_i \neq 0$ and $U_i : F \to F$ be a finite family of ζ_i-generalized demimetric mappings such that $U_i - I$ is demiclosed at 0. Let for each $i = 1, 2, \ldots, m$, $A_i : E \to E^*$ be an η_i-inverse strongly monotone and $B_i : E \to 2^{E^*}$ be a maximal monotone. Let for each $i = 1, 2, \ldots, m$, $T_i : E \to F$ be a bounded linear operator such that $T_i \neq 0$ and let $T_i^* : F^* \to E^*$ be the adjoint of T_i. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m (A_i + B_i)^{-1} 0 : T_i x^* \in \mathrm{Fix}(U_i), i = 1, 2, \ldots, m\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $v, x_0, x_1 \in E$ and by*:

$$\begin{cases} w_n = J_E^{-1}(J_E(x_n) + \theta_n (J_E(x_{n-1}) - J_E(x_n))), \\ z_{n,i} = J_E^{-1}(J_E w_n - r_{n,i} l_i T_i^* J_F(T_i w_n - U_i T_i w_n)), \\ y_{n,i} = J_{\lambda_{n,i}}^{B_i} J_E^{-1}(J_E z_{n,i} - \lambda_{n,i} A_i z_{n,i}), \quad i = 1, 2, \ldots, m, \\ x_{n+1} = J_E^{-1}(\alpha_n J_E v + \sum_{i=1}^m \gamma_{n,i} J_E y_{n,i}), \quad \forall n \geq 0, \end{cases}$$

where $l_i = \frac{\zeta_i}{|\zeta_i|}$, $0 \leq \theta_n \leq \overline{\theta}_n$ *and* $\theta^* \in (0,1)$ *such that*

$$\overline{\theta}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|J_E(x_n)-J_E(x_{n-1})\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & otherwise. \end{cases} \tag{4}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \frac{\frac{2l_i}{\zeta_i}\|T_iw_n - U_iT_iw_n\|^2}{\gamma\|T_i^*J_F(T_iw_n - U_iT_iw_n)\|^2} - \epsilon\right), \quad \text{if } n \in \Lambda = \{k : T_iw_k - U_iT_iw_k \neq 0\},$$

otherwise $r_{n,i} = r_i$ *is any nonnegative real number* (*where* γ *is the constant in Lemma* 2.1). *Suppose that the following conditions are satisfied*:

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \inf_{n\in N} \lambda_{n,i} \leq \sup_{n\in N} \lambda_{n,i} < 2\eta_i c$, (*c is the constant in Lemma* 2.2);

(iii) $\gamma_{n,i} \in (0,1)$, $\alpha_n + \sum_{i=1}^{m} \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$;

(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

Then, $\{x_n\}$ *converges strongly to* $\Pi_\Omega v$, *where* Π_Ω *is the generalized projection of E onto* Ω.

Proof From Lemma 3.1, we have that the stepsizes $r_{n,i}$ are well defined. We show that $\{x_n\}$ is bounded. Indeed, let $x^\star \in \Omega$, since for each $i = 1,2,\ldots,m$, U_i is a ζ_i-generalized demimetric mapping and by Lemma 2.1 we have

$$\begin{aligned}
\phi(x^\star, z_{n,i}) &= \phi\big(x^\star, J_E^{-1}\big(J_Ew_n - r_{n,i}l_iT_i^*J_F(T_iw_n - U_iT_iw_n)\big)\big) \\
&= \|x^\star\|^2 - 2\langle x^\star, J_Ew_n - r_{n,i}l_iT_i^*J_F(T_iw_n - U_iT_iw_n)\rangle \\
&\quad + \|J_Ew_n - l_ir_{n,i}T_i^*J_F(T_iw_n - U_iT_iw_n)\|^2 \\
&\leq \|x^\star\|^2 - 2\langle x^\star, J_Ew_n\rangle + 2l_ir_{n,i}\langle x^\star, T_i^*J_F(T_iw_n - U_iT_iw_n)\rangle \\
&\quad + \|w_n\|^2 - 2l_ir_{n,i}\langle w_n, T_i^*J_F(T_iw_n - U_iT_iw_n)\rangle \\
&\quad + \gamma(r_{n,i})^2\|T_i^*J_F(T_iw_n - U_iT_iw_n)\|^2 \\
&= \phi(x^\star, w_n) + 2l_ir_{n,i}\langle T_ix^\star, J_F(T_iw_n - U_iT_iw_n)\rangle \\
&\quad - 2l_ir_{n,i}\langle T_iw_n, J_F(T_iw_n - U_iT_iw_n)\rangle \\
&\quad + \gamma(r_{n,i})^2\|T_i^*J_F(T_iw_n - U_iT_iw_n)\|^2 \\
&= \phi(x^\star, w_n) - 2l_ir_{n,i}\langle T_iw_n - T_ix^\star, J_F(T_iw_n - U_iT_iw_n)\rangle \\
&\quad + \gamma(r_{n,i})^2\|T_i^*J_F(T_iw_n - U_iT_iw_n)\|^2 \\
&\leq \phi(x^\star, w_n) - 2r_{n,i}\frac{l_i}{\zeta_i}\|T_iw_n - U_iT_iw_n\|^2 \\
&\quad + \gamma(r_{n,i})^2\|T_i^*J_F(T_iw_n - U_iT_iw_n)\|^2 \\
&= \phi(x^\star, w_n) \\
&\quad + r_{n,i}\left(\gamma(r_{n,i})\|T_i^*J_F(T_iw_n - U_iT_iw_n)\|^2 - \frac{2l_i}{\zeta_i}\|T_iw_n - U_iT_iw_n\|^2\right).
\end{aligned} \tag{5}$$

For $n \in \Lambda$, from the condition of $r_{n,i}$ we have

$$\gamma \left\| T_i^* J_F(T_i w_n - U_i T_i w_n) \right\|^2 (\epsilon + r_{n,i}) \leq \frac{2l_i}{\zeta_i} \| T_i w_n - U_i T_i w_n \|^2,$$

hence

$$\gamma \epsilon \left\| T_i^* J_F(T_i w_n - U_i T_i w_n) \right\|^2 \leq \frac{2l_i}{\zeta_i} \| T_i w_n - U_i T_i w_n \|^2 - \gamma r_{n,i} \left\| T_i^* J_F(T_i w_n - U_i T_i w_n) \right\|^2.$$

This implies that

$$\begin{aligned} \gamma r_{n,i} \epsilon \left\| T_i^* J_F(T_i w_n - U_i T_i w_n) \right\|^2 \leq r_{n,i} \bigg(&\frac{2l_i}{\zeta_i} \| T_i w_n - U_i T_i w_n \|^2 \\ &- \gamma r_{n,i} \left\| T_i^* J_F(T_i w_n - U_i T_i w_n) \right\|^2 \bigg). \end{aligned} \tag{6}$$

Utilizing Lemma 2.10 we have

$$\phi\big(x^*, y_{n,i}\big) \leq \phi\big(x^*, z_{n,i}\big) - (c - \lambda_{n,i}\beta) \| y_{n,i} - z_{n,i} \|^2 - \lambda_{n,i} \left(2\eta_i - \frac{1}{\beta} \right) \left\| A_i z_{n,i} - A_i x^* \right\|^2 \tag{7}$$

for each $\beta > 0$ and $n \in \mathbb{N}$. Since $0 < \inf_{n\in\mathbb{N}} \lambda_{n,i} \leq \sup_{n\in\mathbb{N}} \lambda_{n,i} < 2c\eta_i$, for each $i \in \{1, 2, \dots, m\}$ there exists $\beta_i > 0$ such that $\inf_{n\in\mathbb{N}}(c - \lambda_{n,i}\beta_i) > 0$ and $\inf_{n\in\mathbb{N}} \lambda_{n,i}(2\eta_i - \frac{1}{\beta_i}) > 0$. From inequalities (5), (6), and (7), we obtain

$$\phi\big(x^*, y_{n,i}\big) \leq \phi\big(x^*, w_n\big). \tag{8}$$

From the definition of w_n, we obtain

$$\begin{aligned} \phi\big(x^*, w_n\big) &= \phi\big(x^*, J_E^{-1}\big(J_E(x_n) + \theta_n\big(J_E(x_{n-1}) - J_E(x_n)\big)\big) \\ &\leq (1 - \theta_n)\phi\big(x^*, x_n\big) + \theta_n \phi\big(x^*, x_{n-1}\big). \end{aligned} \tag{9}$$

From inequalities (8) and (9) we obtain

$$\begin{aligned} \phi\big(x^\star, x_{n+1}\big) &= \phi\left(x^\star, J_E^{-1}\left(\alpha_n J_E v + \sum_{i=1}^m \gamma_{n,i} J_E y_{n,i}\right)\right) \\ &= \left\|x^\star\right\|^2 - 2\left\langle x^\star, \left(\alpha_n J_E v + \sum_{i=1}^m \gamma_{n,i} J_E y_{n,i}\right)\right\rangle \\ &\quad + \left\| \alpha_n J_E v + \sum_{i=1}^m \gamma_{n,i} J_E y_{n,i} \right\|^2 \\ &\leq \left\|x^\star\right\|^2 - 2\alpha_n \langle x^\star, J_E v\rangle - 2\sum_{i=1}^m \gamma_{n,i} \langle x^\star, J_E y_{n,i}\rangle \\ &\quad + \alpha_n \|v\|^2 + \sum_{i=1}^m \gamma_{n,i} \|y_{n,i}\|^2 \end{aligned}$$

$$\begin{aligned}
&= \alpha_n \phi\big(x^\star, v\big) + \sum_{i=1}^{m} \gamma_{n,i} \phi\big(x^\star, y_{n,i}\big) \\
&\le \alpha_n \phi\big(x^\star, v\big) + (1-\alpha_n)\phi\big(x^\star, w_n\big) \\
&\le \max\big\{\phi\big(x^*, v\big), \max\big\{\phi\big(x^*, x_n\big), \phi\big(x^*, x_{n-1}\big)\big\}\big\} \\
&\le \cdots \le \max\big\{\phi\big(x^*, v\big), \max\big\{\phi\big(x^*, x_1\big), \phi\big(x^*, x_0\big)\big\}\big\}.
\end{aligned}$$

Therefore, $\phi(x^*, x_n)$ is bounded and by inequality (3), the sequence $\{x_n\}$ is also bounded. Consequently, $\{w_n\}$, $\{y_{n,i}\}$, $\{z_{n,i}\}$, and $\{T_i w_n\}$ are all bounded. We have $\theta_n \|J_E(x_n) - J_E(x_{n-1})\| \le \varepsilon_n$ for all n, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \big\|J_E(x_n) - J_E(x_{n-1})\big\| = 0. \tag{10}$$

Utilizing Lemma 2.3 we obtain that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0. \tag{11}$$

Since the sequence $\{x_n\}$ is bounded, there exists a constant $M > 0$ such that

$$\begin{aligned}
\phi\big(x^*, x_{n-1}\big) - \phi\big(x^*, x_n\big) &= \|x_{n-1}\|^2 - 2\big\langle x^*, J_E(x_{n-1})\big\rangle + \big\|x^*\big\|^2 \\
&\quad - \big(\|x_n\|^2 - 2\big\langle x^*, J_E(x_n)\big\rangle + \big\|x^*\big\|^2\big) \\
&= \big(\|x_{n-1}\|^2 - \|x_n\|^2\big) + 2\big\langle x^*, J_E(x_n) - J_E(x_{n-1})\big\rangle \\
&\le M\|x_{n-1} - x_n\| + 2\big\|J_E(x_n) - J_E(x_{n-1})\big\| \big\|x^*\big\|.
\end{aligned}$$

Hence, from (11) and (10) we obtain

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \big(\big|\phi\big(x^*, x_{n-1}\big) - \phi\big(x^*, x_n\big)\big|\big) = 0. \tag{12}$$

From inequalities (5), (6), (7), and (8) we obtain

$$\begin{aligned}
\phi\big(x^\star, x_{n+1}\big) &\le \alpha_n \phi\big(x^\star, v\big) + \sum_{i=1}^{m} \gamma_{n,i} \phi\big(x^\star, y_{n,i}\big) \\
&\le \alpha_n \phi\big(x^\star, v\big) + \sum_{i=1}^{m} \gamma_{n,i} \phi\big(x^\star, z_{n,i}\big) \\
&\quad - \sum_{i=1}^{m} \gamma_{n,i}(c - \lambda_{n,i}\beta_i)\|y_{n,i} - z_{n,i}\|^2 - \sum_{i=1}^{m} \gamma_{n,i}\lambda_{n,i}\left(2\eta_i - \frac{1}{\beta_i}\right)\big\|A_i z_{n,i} - A_i x^*\big\|^2 \\
&\le \alpha_n \phi\big(x^\star, v\big) + (1-\alpha_n)\phi\big(x^\star, w_n\big) - \sum_{i=1}^{m} \gamma_{n,i}\gamma r_{n,i}\epsilon \big\|T_i^* J_F(T_i w_n - U_i T_i w_n)\big\|^2 \\
&\quad - \sum_{i=1}^{m} \gamma_{n,i}(c - \lambda_{n,i}\beta_i)\|y_{n,i} - z_{n,i}\|^2 - \sum_{i=1}^{m} \gamma_{n,i}\lambda_{n,i}\left(2\eta_i - \frac{1}{\beta_i}\right)\big\|A_i z_{n,i} - A_i x^*\big\|^2 \\
&\le \alpha_n \phi\big(x^\star, v\big) + (1-\alpha_n)\big[(1-\theta_n)\phi\big(x^*, x_n\big) + \theta_n \phi\big(x^*, x_{n-1}\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m} \gamma_{n,i}\gamma r_{n,i}\epsilon \left\| T_i^* J_F(T_i w_n - U_i T_i w_n)\right\|^2 \\
&- \sum_{i=1}^{m} \gamma_{n,i}(c - \lambda_{n,i}\beta_i)\|y_{n,i} - z_{n,i}\|^2 \\
&- \sum_{i=1}^{m} \gamma_{n,i}\lambda_{n,i}\left(2\eta_i - \frac{1}{\beta_i}\right)\left\|A_i z_{n,i} - A_i x^*\right\|^2 \\
\le{}& \phi\left(x^*, x_n\right) + \alpha_n\Bigg[\left[\phi\left(x^*, v\right) - \phi\left(x^*, x_n\right)\right] \\
&+ (1-\alpha_n)\frac{\theta_n}{\alpha_n}\left[\phi\left(x^*, x_{n-1}\right) - \phi\left(x^*, x_n\right)\right]\Bigg] \\
&- \sum_{i=1}^{m} \gamma_{n,i}\gamma r_{n,i}\epsilon \left\| T_i^* J_F(T_i w_n - U_i T_i w_n)\right\|^2 \\
&- \sum_{i=1}^{m} \gamma_{n,i}(c - \lambda_{n,i}\beta_i)\|y_{n,i} - z_{n,i}\|^2 \\
&- \sum_{i=1}^{m} \gamma_{n,i}\lambda_{n,i}\left(2\eta_i - \frac{1}{\beta_i}\right)\left\|A_i z_{n,i} - A_i x^*\right\|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\sum_{i=1}^{m} \gamma_{n,i}(c - \lambda_{n,i}\beta_i)\|y_{n,i} - z_{n,i}\|^2 + \sum_{i=1}^{m} \gamma_{n,i}\lambda_{n,i}\left(2\eta_i - \frac{1}{\beta_i}\right)\left\|A_i z_{n,i} - A_i x^*\right\|^2 \\
&\quad + \sum_{i=1}^{m} \gamma_{n,i}\gamma r_{n,i}\epsilon \left\| T_i^* J_F(T_i w_n - U_i T_i w_n)\right\|^2 \\
&\quad \le \phi\left(x^*, x_n\right) - \phi\left(x^*, x_{n+1}\right) + \alpha_n(K_1 + K_2),
\end{aligned}
\tag{13}
$$

where $K_1 = \sup_{n\in\mathbb{N}}\{|\phi(x^*, v) - \phi(x^*, x_n)|\}$ and $K_2 = \sup_{n\in\mathbb{N}}\{|\frac{\theta_n}{\alpha_n}[\phi(x^*, x_{n-1}) - \phi(x^*, x_n)]|\}$.

Utilizing Lemma 2.4, we have

$$
\begin{aligned}
\phi\left(x^*, x_{n+1}\right) &= V\left(x^*, \alpha_n J_E v + \sum_{i=1}^{m} \gamma_{n,i} J_E y_{n,i}\right) \\
&\le V\left(x^*, \alpha_n J_E v + \sum_{i=1}^{m} \gamma_{n,i} J_E y_{n,i} - \alpha_n\left(J_E v - J_E x^*\right)\right) \\
&\quad - 2\left\langle J_E^{-1}\left(\alpha_n J_E v + \sum_{i=1}^{m} \gamma_{n,i} J_E y_{n,i}\right) - x^*, -\alpha_n\left(J_E v - J_E x^*\right)\right\rangle \\
&\le V\left(x^*, \alpha_n J_E x^* + \sum_{i=1}^{m} \gamma_{n,i} J_E y_{n,i}\right) + 2\alpha_n\left\langle x_{n+1} - x^*, J_E v - J_E x^*\right\rangle \\
&= \phi\left(x^*, J_E^{-1}\left(\alpha_n J_E x^* + \sum_{i=1}^{m} \gamma_{n,i} J_E y_{n,i}\right)\right) + 2\alpha_n\left\langle x_{n+1} - x^*, J_E v - J_E x^*\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} \gamma_{n,i} \phi\big(x^*, y_{n,i}\big) + 2\alpha_n \big\langle x_{n+1} - x^*, J_E v - J_E x^* \big\rangle \\
&\leq \sum_{i=1}^{m} \gamma_{n,i} \phi\big(x^*, z_{n,i}\big) + 2\alpha_n \big\langle x_{n+1} - x^*, J_E v - J_E x^* \big\rangle \\
&\leq (1-\alpha_n)\phi\big(x^*, w_n\big) + 2\alpha_n \big\langle x_{n+1} - x^*, J_E v - J_E x^* \big\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\phi\big(x^*, x_{n+1}\big) &\leq (1-\alpha_n)\phi\big(x^*, w_n\big) + 2\alpha_n \big\langle x_{n+1} - x^*, J_E v - J_E x^* \big\rangle \\
&\leq (1-\alpha_n)\big[(1-\theta_n)\phi\big(x^*, x_n\big) + \theta_n \phi\big(x^*, x_{n-1}\big)\big] \\
&\quad + 2\alpha_n \big\langle x_{n+1} - x^*, J_E v - J_E x^* \big\rangle \\
&\leq (1-\alpha_n)\phi\big(x^*, x_n\big) + \alpha_n \chi_n,
\end{aligned}
\tag{14}
$$

where

$$
\chi_n = (1-\alpha_n)\frac{\theta_n}{\alpha_n}\big[\phi\big(x^*, x_{n-1}\big) - \phi\big(x^*, x_n\big)\big] + 2\big\langle x_{n+1} - x^*, J_E v - J_E x^* \big\rangle.
$$

From Lemma 2.10 and Lemma 2.8, we have that Ω is closed and convex. Therefore, the generalized projection Π_Ω from E onto Ω is well defined. Suppose $x^* = \Pi_\Omega v$. The next task is to prove that the sequence $\{x_n\}$ converges to the point x^*. In order to prove this, we consider two possible cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. By the boundedness of $\{\phi(x^*, x_n)\}$, we have $\{\phi(x^*, x_n)\}$ is convergent. Furthermore, we have $\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0$ as $n \to \infty$. Therefore, from (13), for each $i \in \{1, 2, \dots, m\}$ we obtain

$$
\lim_{n\to\infty} \gamma_{n,i} \gamma r_{n,i} \epsilon \big\| T_i^* J_F (T_i w_n - U_i T_i w_n) \big\|^2 = 0.
$$

This implies that

$$
\lim_{n\to\infty} \big\| T_i^* J_F (T_i w_n - U_i T_i w_n) \big\| = 0. \tag{15}
$$

From Lemma 3.1 we obtain

$$
\lim_{n\to\infty} \| T_i w_n - U_i T_i w_n \| = 0. \tag{16}
$$

In a similar way we obtain that

$$
\lim_{n\to\infty} \sum_{i=1}^{m} \gamma_{n,i}(c - \lambda_{n,i}\beta_i)\|y_{n,i} - z_{n,i}\|^2 + \sum_{i=1}^{m} \gamma_{n,i}\lambda_{n,i}\left(2\eta_i - \frac{1}{\beta_i}\right)\big\|A_i z_{n,i} - A_i x^*\big\|^2 = 0. \tag{17}
$$

By our assumption we obtain

$$
\lim_{n\to\infty} \|y_{n,i} - z_{n,i}\| = \lim_{n\to\infty} \big\|A_i z_{n,i} - A_i x^*\big\| = 0, \quad (i = 1, 2, \dots, m). \tag{18}
$$

Since J_E is uniformly continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|J_E y_{n,i} - J_E z_{n,i}\| = 0, \quad (i = 1,2,\ldots,m). \tag{19}$$

Also, we have

$$\|J_E(w_n) - J_E(x_n)\| = \theta_n \|J_E(x_{n-1}) - J_E(x_n)\| = \alpha_n \frac{\theta_n}{\alpha_n} \|J_E(x_{n-1}) - J_E(x_n)\| \to 0.$$

Furthermore, for $i = 1,2,\ldots,m$, we obtain that

$$\begin{aligned} \|J_E(z_{n,i}) - J_E(w_n)\| &\le r_{n,i} \|T_i^* J_F(T_i w_n - U_i T_i w_n)\| \\ &\le r_{n,i} \|T_i^*\| \|T_i w_n - U_i T_i w_n)\| \to 0. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|J_E(y_{n,i}) - J_E(x_n)\| \le{}& \|J_E(y_{n,i}) - J_E(z_{n,i})\| + \|J_E(z_{n,i}) - J_E(w_n)\| \\ &+ \|J_E(w_n) - J_E(x_n)\| \to 0. \end{aligned}$$

This implies that

$$\|J_E(x_{n+1}) - J_E(x_n)\| \le \alpha_n \|J_E v - J_E x_n\| + \sum_{i=1}^m \gamma_{n,i} \|J_E(y_{n,i}) - J_E(x_n)\| \to 0.$$

By uniform continuity of J_E^{-1} on bounded subset of E^*, we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{20}$$

Since $\{x_n\}$ is bounded and E is a reflexive Banach space, there exits a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$. Since $\|y_{n,i} - x_n\| \to 0$, we have $y_{n_k,i} \rightharpoonup z$, $(i = 1,2,\ldots,m)$. Thus, we obtain $z \in \bigcap_{i=1}^m (A_i + B_i)^{-1}0$ (see, e.g., [7] for this proof). Since $\|w_n - x_n\| \to 0$, we have $w_{n_k} \rightharpoonup z$. From the continuity of T_i, we have that $T_i w_{n_k} \rightharpoonup T_i z$. Now, from (16) and the demiclosedness of $I - U_i$, we obtain $T_i z \in \mathrm{Fix}(U_i)$. This implies that $z \in \Omega$. Next, we show that $\limsup_{n\to\infty} \langle x_{n+1} - x^*, J_E v - J_E x^*\rangle \le 0$. We can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - x^*, J_E v - J_E x^*\rangle = \lim_{k\to\infty} \langle x_{n_k} - x^*, J_E v - J_E x^*\rangle.$$

Since $x^* = \Pi_\Omega v$, applying Lemma 2.6 we obtain

$$\lim_{k\to\infty} \langle x_{n_k} - x^*, J_E v - J_E x^*\rangle = \langle z - x^*, J_E v - J_E x^*\rangle \le 0.$$

This, together with $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, implies that

$$\limsup_{n\to\infty} \langle x_{n+1} - x^*, J_E v - J_E x^*\rangle = \limsup_{n\to\infty} \langle x_n - x^*, J_E v - J_E x^*\rangle \le 0.$$

Applying inequality (14) and Lemma 2.11, we deduce that $\lim_{n\to\infty}\phi(x^*,x_n)=0$. Now, from Lemma 2.5 we obtain, $\lim_{n\to\infty}\|x_n-x^*\|=0$.

Case 2: Put $\Gamma_n=\phi(x^*,x_n)$ for all $n\in\mathbb{N}$. Suppose there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j}\le\Gamma_{n_j+1}$ for all $j\in\mathbb{N}$. Define a mapping $\tau:\mathbb{N}\to\mathbb{N}$ by $\tau(n)=\max\{k\le n:\Gamma_k<\Gamma_{k+1}\}$ for all $n\ge n_0$ (for some n_0 large enough). Thus, by Lemma 2.12 we have $\tau(n)\to\infty$ and $0\le\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1}$ for all $n\ge n_0$. Following the proof line in Case 1, we can show that

$$\lim_{n\to\infty}\big\|T_iw_{\tau(n)}-U_i(T_iw_{\tau(n)})\big\|=\lim_{n\to\infty}\|y_{\tau(n),i}-z_{\tau(n),i}\|=0,\quad(i=1,2,\dots,m)\tag{21}$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\|=0.\tag{22}$$

Further, we can show that

$$\limsup_{n\to\infty}\langle x_{\tau(n)}-x^*,J_Ev-J_Ex^*\rangle\le 0.\tag{23}$$

It follows from (14) that

$$\phi\big(x^*,x_{\tau(n)+1}\big)\le(1-\alpha_{\tau(n)})\phi\big(x^*,x_{\tau(n)}\big)+\alpha_{\tau(n)}\chi_{\tau(n)}.\tag{24}$$

Since $\phi(x^*,x_{\tau(n)})<\phi(x^*,x_{\tau(n)+1})$, we have

$$\alpha_{\tau(n)}\phi\big(x^*,x_{\tau(n)}\big)\le\alpha_{\tau(n)}\chi_{\tau(n)}.$$

Since $\alpha_{\tau(n)}>0$ and $\limsup_{n\to\infty}\chi_{\tau(n)}\le 0$, we deduce that

$$\lim_{n\to\infty}\phi\big(x^*,x_{\tau(n)}\big)=0.$$

Using this and inequality (24), we conclude that $\lim_{n\to\infty}\phi(x^*,x_{\tau(n)+1})=0$. Now, from Lemma 2.12, we have that $\lim_{n\to\infty}\phi(x^*,x_n)=0$. Hence, $\lim_{n\to\infty}\|x^*-x_n\|=0$. This completes the proof. □

We know that every Hilbert space $\mathcal{H}$, is a 2-uniformly convex and uniformly smooth Banach space and the normalized duality mapping is $J_{\mathcal{H}}=I$. Also, we have the following relation in Hilbert space $\mathcal{H}$:

$$\|x+y\|^2=\|x\|^2+2\langle y,x\rangle+\|y\|^2,\quad\forall x,y\in\mathcal{H}.$$

Therefore, in Lemma 2.2 and Lemma 2.1 the constant $c=\gamma=1$. Theorem 3.2 now yields the following result regarding an algorithm for solving the generalized multiple-set split feasibility problem in Hilbert spaces.

Theorem 3.3 *Let E and F be Hilbert spaces. Let for $i=1,2,\dots,m$, $\zeta_i\ne 0$ and $U_i:F\to F$ be a finite family of ζ_i-generalized demimetric mappings such that U_i-I is demiclosed at 0. Let for $i=1,2,\dots,m$, $A_i:E\to E$ be an η_i-inverse strongly monotone and $B_i:E\to 2^E$ be* *a maximal monotone. Let for each* $i = 1, 2, \dots, m$, $T_i : E \to F$ *be a bounded linear operator such that* $T_i \neq 0$ *and let* $T_i^* : F \to E$ *be the adjoint operator of* T_i. *Suppose that* $\Omega = \{x^* \in \bigcap_{i=1}^{m} (A_i + B_i)^{-1}0 : T_i x^* \in \mathrm{Fix}(U_i), i = 1, 2, \dots, m\} \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $v, x_0, x_1 \in E$ *and by*:

$$\begin{cases} w_n = x_n + \theta_n(x_{n-1} - x_n), \\ z_{n,i} = w_n - r_{n,i} l_i T_i^*(T_i w_n - U_i T_i w_n), \\ y_{n,i} = J_{\lambda_{n,i}}^{B_i}(z_{n,i} - \lambda_{n,i} A_i z_{n,i}), \quad i = 1, 2, \dots, m, \\ x_{n+1} = \alpha_n v + \sum_{i=1}^{m} \gamma_{n,i} y_{n,i}, \quad \forall n \geq 0, \end{cases}$$

where $l_i = \frac{\zeta_i}{|\zeta_i|}$, $0 \leq \theta_n \leq \overline{\theta}_n$ *and* $\theta^* \in (0,1)$ *such that*

$$\overline{\theta}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta^*\}, \quad x_n \neq x_{n-1} \\ \theta^*, \quad \textit{otherwise}. \end{cases}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \frac{\frac{2l_i}{\zeta_i}\|T_i x_n - U_i T_i x_n\|^2}{\|T_i^*(T_i x_n - U_i T_i x_n)\|^2} - \epsilon\right), \quad \textit{if}\ n \in \Lambda = \{k : T_i x_k - U_i T_i x_k \neq 0\},$$

otherwise $r_{n,i} = r_i$ *is any nonnegative real number. Suppose that the following conditions are satisfied*:

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \inf_{n\in N} \lambda_{n,i} \leq \sup_{n\in N} \lambda_{n,i} < 2\eta_i$;
(iii) $\gamma_{n,i} \in (0,1)$, $\alpha_n + \sum_{i=1}^{m} \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$;
(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

Then, $\{x_n\}$ *converges strongly to* $P_\Omega v$.

4 Applications

In this section we present some applications of our main result.

4.1 Common solutions to variational inequality problems

Let C be a nonempty, closed, and convex subset of a Banach space E and $A : E \to E^*$ be an operator. The variational inequality problem (VIP) is formulated as follows:

$$\text{Find an element } x^\star \in C \text{ such that } \langle y - x^\star, A(x^\star)\rangle \geq 0, \quad \forall y \in C. \tag{25}$$

The set of solutions of this problem is denoted by $VI(C, A)$. It is well known that the VIP is a fundamental problem in optimization theory and nonlinear analysis (see [45]).

Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Then, it is known that the subdifferential ∂f of f defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^*\rangle, \forall y \in E\},$$

for $x \in E$ is a maximal monotone operator; see [46].

Let $i_C : E \to (-\infty, +\infty]$ be the indicator function of C. We know that i_C is proper lower semicontinuous and convex, and hence its subdifferential ∂i_C is a maximal monotone operator. Let $B = \partial i_C$. Then, it is easy to see that $J^B_\lambda x = \Pi_C x$ for every $\lambda > 0$ and $x \in E$. Further, we also obtain $(A + B)^{-1}(0) = VI(C, A)$.

Now, as an application of our main result, we obtain the following theorem for finding a common element of the set of common solutions of a system of a variational inequality problem and the set of common fixed points of a finite family of generalized demimetric mappings in 2-uniformly convex and uniformly smooth Banach spaces.

Theorem 4.1 *Let E be a 2-uniformly convex and uniformly smooth Banach space. Let $\{C_i\}_{i=1}^m$ be a finite family of nonempty, closed, and convex subsets of E. Let for $i = 1, 2, \dots, m$, $\zeta_i \neq 0$ and $U_i : E \to E$ be a finite family of ζ_i-generalized demimetric mappings such that $U_i - I$ is demiclosed at 0. Let for $i = 1, 2, \dots, m$, $A_i : E \to E^*$ be an η_i-inverse strongly monotone. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m (VI(C_i, A_i) \cap \mathrm{Fix}(U_i))\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $v, x_0, x_1 \in E$ and by*:

$$\begin{cases} w_n = J_E^{-1}(J_E(x_n) + \theta_n(J_E(x_{n-1}) - J_E(x_n))), \\ z_{n,i} = J_E^{-1}(J_E w_n - r_{n,i} l_i J_E(w_n - U_i w_n)), \\ y_{n,i} = \Pi_{C_i} J_E^{-1}(J_E z_{n,i} - \lambda_{n,i} A_i z_{n,i}), \quad i = 1, 2, \dots, m, \\ x_{n+1} = J_E^{-1}(\alpha_n J_E v + \sum_{i=1}^m \gamma_{n,i} J_E y_{n,i}), \quad \forall n \geq 0, \end{cases}$$

where $l_i = \frac{\zeta_i}{|\zeta_i|}$, $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0, 1)$ such that

$$\overline{\theta}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|J_E(x_n) - J_E(x_{n-1})\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & \text{otherwise}. \end{cases}$$

Suppose that the following conditions are satisfied:

(i) $\alpha_n \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \inf_{n\in N} \lambda_{n,i} \leq \sup_{n\in N} \lambda_{n,i} < 2\eta_i c$, (*$c$ is the constant in Lemma 2.2*);

(iii) $\gamma_{n,i} \in (0, 1)$, $\alpha_n + \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$;

(iv) $r_{n,i} \in (\epsilon, \frac{2l_i}{\gamma\zeta_i} - \epsilon)$ *for some* $\epsilon > 0$, (*γ is the constant in Lemma 2.1*);

(v) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

Then, $\{x_n\}$ converges strongly to $\Pi_\Omega v$.

Proof Setting $F = E$, $T_i = I$, and $B_i = \partial i_{C_i}$, we obtain the desired result from Theorem 3.2. □

Remark 4.2 Setting $U_i = I$ in Theorem 4.1, our result generalizes the result of [47] from the problem of finding common solutions to unrelated variational inequalities in a Hilbert space to a 2-uniformly smooth and uniformly convex Banach space.

Remark 4.3 We extend the main results of Kimura and Nakajo [7] from the problem of finding a solution of the variational inequality problem to the problem of finding a common element of the set of common solutions of a system of a variational inequality problem and a common fixed-point problem.

4.2 Convex minimization problem

For a convex differentiable function $\Phi : E \to \mathbb{R}$ and a proper convex lower semicontinuous function $\Psi : E \to (-\infty, +\infty]$, the convex minimization problem is to find a point $x^\star \in E$ such that

$$\Phi(x^\star) + \Psi(x^\star) = \min_{x\in E}\{\Phi(x) + \Psi(x)\}. \tag{26}$$

If $\nabla\Phi$ and $\partial\Psi$ represent the gradient of Φ and subdifferential of Ψ, respectively, then Fermat's rule ensures the equivalence of problem (26) to the problem of finding a point $x^\star \in E$ such that

$$0 \in \nabla\Phi(x^\star) + \partial\Psi(x^\star).$$

Many optimization problems from image processing, statistical regression, and machine learning (see, e.g., [48, 49]) can be adapted into the form of (26). In this setting, we assume that Φ is Gâteaux-differentiable with derivative $\nabla\Phi$ that is an inverse strongly monotone.

Now, as an application of our main result we obtain the following theorem.

Theorem 4.4 *Let E be a 2-uniformly convex and uniformly smooth Banach space. Let $\zeta \neq 0$ and $U : E \to E$ be a ζ-generalized demimetric mapping such that $U - I$ is demiclosed at 0. Let $\Phi : E \to \mathbb{R}$ be a convex differentiable function such that its gradient $\nabla\Phi$ is an η-inverse strongly monotone and let $\Psi : E \to (-\infty, +\infty]$ be a proper function with convexity and lower semicontinuity. Suppose that $\Omega = \{x \in \mathrm{Fix}(U) : x = \mathrm{argmin}_{z\in E}\,\Phi(z) + \Psi(z)\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $v, x_0, x_1 \in E$ and by:*

$$\begin{cases} w_n = J_E^{-1}(J_E(x_n) + \theta_n(J_E(x_{n-1}) - J_E(x_n))),\\ z_n = J_E^{-1}(J_E w_n - r_n l J_E(w_n - Uw_n)),\\ y_n = \mathrm{argmin}_{z\in E}\{\Psi(z) + \frac{1}{2\lambda_n}\|z\|^2 - \frac{1}{\lambda_n}\langle y, J_E z_n - \lambda_n\nabla\Phi(z_n)\rangle\},\\ x_{n+1} = J_E^{-1}(\alpha_n J_E v + \gamma_n J_E y_n), \quad \forall n \geq 0, \end{cases}$$

where $l = \frac{\zeta}{|\zeta|}$, $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0,1)$ such that

$$\overline{\theta}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|J_E(x_n)-J_E(x_{n-1})\|}, \theta^*\}, & x_n \neq x_{n-1}\\ \theta^*, & \text{otherwise.} \end{cases}$$

Suppose that the following conditions are satisfied:

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \inf_{n\in N}\lambda_n \leq \sup_{n\in N}\lambda_n < 2\eta c$, (*$c$ is the constant in Lemma* 2.2);

(iii) $\gamma_n \in (0,1)$, $\alpha_n + \gamma_n = 1$ *and* $\liminf_{n\to\infty}\gamma_n > 0$;

(iv) $r_n \in (\epsilon, \frac{2l}{\gamma\zeta} - \epsilon)$ *for some* $\epsilon > 0$, (*γ is the constant in Lemma* 2.1);

(v) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_n} = 0$.

Then, $\{x_n\}$ converges strongly to $\Pi_\Omega v$.

Proof We know that the subdifferential mapping $\partial\Psi$ of a proper, convex, and lower semicontinuous function Ψ is a maximal monotone. Also, we have (see [50]):

$$J_{\lambda_n}^{\partial\Psi}u_n = \operatorname*{argmin}_{z\in E}\left\{\Psi(z) + \frac{1}{2\lambda_n}\|z\|^2 - \frac{1}{\lambda_n}\langle y, J_E u_n\rangle\right\}.$$

Now, setting $F = E$, $m = 1$, $T_i = I$ and $A = \nabla\Phi$, we obtain the desired result from Theorem 3.2. □

4.3 The multiple-set split feasibility problem

Let C be a nonempty, closed, and convex subset of a strictly convex and reflexive Banach space E. Then, we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C. Let E be a uniformly convex and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Then, P_C is 1-generalized demimetric and $I - P_C$ is demiclosed at zero (see [33] for details).

As another application of our main result, we obtain the following strong convergence theorem for the multiple-set split feasibility problem.

Theorem 4.5 *Let E be a 2-uniformly convex and uniformly smooth Banach space and let F be a uniformly convex and smooth Banach space. Let $\{C_i\}_{i=1}^m$ and $\{Q_i\}_{i=1}^m$ be two finite families of nonempty, closed, and convex subsets of E and F, respectively. Let for each $i = 1, 2, \ldots, m$, $T_i : E \to F$ be a bounded linear operator such that $T_i \neq 0$. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m C_i : T_i x^* \in Q_i, i = 1, 2, \ldots, m\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $v, x_0, x_1 \in E$ and by*:

$$\begin{cases} w_n = J_E^{-1}(J_E(x_n) + \theta_n(J_E(x_{n-1}) - J_E(x_n))), \\ z_{n,i} = J_E^{-1}(J_E w_n - r_{n,i} T_i^* J_F(T_i w_n - P_{Q_i} T_i w_n)), \\ y_{n,i} = \Pi_{C_i} z_{n,i}, \quad i = 1, 2, \ldots, m \\ x_{n+1} = J_E^{-1}(\alpha_n J_E v + \sum_{i=1}^m \gamma_{n,i} J_E y_{n,i}), \quad \forall n \geq 0, \end{cases}$$

where $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0, 1)$ such that

$$\overline{\theta}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|J_E(x_n) - J_E(x_{n-1})\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & \textit{otherwise}. \end{cases}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \frac{2\|T_i w_n - P_{Q_i} T_i w_n\|^2}{\gamma\|T_i^* J_F(T_i w_n - P_{Q_i} T_i w_n)\|^2} - \epsilon\right), \quad \textit{if } n \in \Lambda = \{k : T_i w_k - P_{Q_i} T_i w_k \neq 0\},$$

otherwise $r_{n,i} = r_i$ is any nonnegative real number (where γ is the constant in Lemma 2.1). Suppose that the following conditions are satisfied:

(i) $\alpha_n \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\gamma_{n,i} \in (0, 1)$, $\alpha_n + \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$;
(iii) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

Then, $\{x_n\}$ converges strongly to $\Pi_\Omega v$.

4.4 Split common null-point problem

Let E be a uniformly convex and smooth Banach space and let $G : E \to 2^{E^*}$ be a maximal monotone operator. For each $x \in E$ and $\mu > 0$, we define the metric resolvent of G for $\mu > 0$ by

$$Q^G_\mu(x) = \left(I + \mu J_E{}^{-1} G\right)^{-1}(x), \quad \forall x \in E. \tag{27}$$

It is observed that $0 \in J_E(Q^G_\mu(x) - x) + \mu G Q^G_\mu(x)$ and $G^{-1}0 = \mathrm{Fix}(Q^G_\mu)$, (see [51]). It is known that Q^G_μ is 1-generalized demimetric. Also, we know that $I - Q^G_\mu$ is demiclosed at zero (see [33] for details).

We obtain the following strong convergence result for split common null-point problem.

Theorem 4.6 *Let E be a 2-uniformly convex and uniformly smooth Banach space and let F be a uniformly convex and smooth Banach space. Let for $i = 1, 2, \dots, m$, $B_i : E \to 2^{E^*}$ and $G_i : F \to 2^{F^*}$ be maximal monotone operators. Let for $i = 1, 2, \dots, m$, $J^{B_i}_{r_i}$, be resolvent operators of B_i for $r_i > 0$ and $Q^{G_i}_{\mu_i}$, be metric resolvent operators of G_i for $\mu_i > 0$. Let for each $i = 1, 2, \dots, m$, $T_i : E \to F$ be a bounded linear operator such that $T_i \neq 0$. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m B_i{}^{-1}0 : T_i x^* \in G_i{}^{-1}0, i = 1, 2, \dots, m\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $v, x_0, x_1 \in E$ and by*:

$$\begin{cases} w_n = J_E^{-1}(J_E(x_n) + \theta_n(J_E(x_{n-1}) - J_E(x_n))), \\ z_{n,i} = J_E^{-1}(J_E w_n - r_{n,i} T_i^* J_F(T_i w_n - Q^{G_i}_{\mu_i} T_i w_n)), \\ y_{n,i} = J^{B_i}_{\lambda_{n,i}} z_{n,i}, \quad i = 1, 2, \dots, m, \\ x_{n+1} = J_E^{-1}(\alpha_n J_E v + \sum_{i=1}^m \gamma_{n,i} J_E y_{n,i}), \quad \forall n \geq 0, \end{cases}$$

where $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0,1)$ such that

$$\overline{\theta}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|J_E(x_n) - J_E(x_{n-1})\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & \text{otherwise.} \end{cases}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \frac{2\|T_i w_n - Q^{G_i}_{\mu_i} T_i w_n\|^2}{\gamma \|T_i^* J_F(T_i w_n - Q^{G_i}_{\mu_i} T_i w_n)\|^2} - \epsilon\right), \quad \text{if } n \in \Lambda = \left\{k : T_i w_k - Q^{G_i}_{\mu_i} T_i w_k \neq 0\right\},$$

otherwise $r_{n,i} = r_i$ is any nonnegative real number (where γ is the constant in Lemma 2.1). Suppose that the following conditions are satisfied:

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \inf_{n \in N} \lambda_{n,i}$;

(iii) $\gamma_{n,i} \in (0,1)$, $\alpha_n + \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$;

(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

Then, $\{x_n\}$ converges strongly to $\Pi_\Omega v$.

4.5 Split common fixed point

Let $\mathcal{H}$ be a Hilbert space. The mapping $T : \mathcal{H} \to \mathcal{H}$ is called:

- A strict pseudocontraction, if there exists a constant $\beta \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \beta \big\|(x-Tx)-(y-Ty)\big\|^2, \quad \forall x,y \in \mathcal{H}.$$

We obtain the following strong convergence result for the split common fixed-point problem.

Theorem 4.7 *Let E and F be Hilbert spaces. Let for each $i = 1,2,\dots,m$, $U_i : F \to F$ be a ς_i-strict pseudocontraction mapping and let $S_i : E \to E$ be a κ_i-strict pseudocontraction mapping. Let for each $i = 1,2,\dots,m$, $T_i : E \to F$ be a bounded linear operator such that $T_i \neq 0$. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) : T_i x^* \in \mathrm{Fix}(U_i), i = 1,2,\dots,m\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $v, x_0, x_1 \in E$ and by*:

$$\begin{cases} w_n = x_n + \theta_n(x_{n-1} - x_n), \\ z_{n,i} = w_n - r_{n,i} T_i^*(T_i w_n - U_i T_i w_n), \\ y_{n,i} = (1-\lambda_{n,i}) z_{n,i} + \lambda_{n,i} S_i z_{n,i}, \quad i = 1,2,\dots,m, \\ x_{n+1} = \alpha_n v + \sum_{i=1}^m \gamma_{n,i} y_{n,i}, \quad \forall n \geq 0, \end{cases}$$

where $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0,1)$ such that

$$\overline{\theta}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & \textit{otherwise}. \end{cases} \tag{28}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \frac{(1-\varsigma_i)\|T_i w_n - U_i T_i w_n\|^2}{\|T_i^*(T_i w_n - U_i T_i w_n)\|^2} - \epsilon\right), \quad \textit{if } n \in \Lambda = \{k : T_i w_k - U_i T_i w_k \neq 0\},$$

otherwise $r_{n,i} = r_i$ is any nonnegative real number. Suppose that the following conditions are satisfied:

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $0 < \inf_{n\in N} \lambda_{n,i} \leq \sup_{n\in N} \lambda_{n,i} < (1-\kappa_i)$;
(iii) $\gamma_{n,i} \in (0,1)$, $\alpha_n + \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$;
(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.
Then, $\{x_n\}$ converges strongly to $P_\Omega v$.

Proof Put $A_i = I - S_i$ and $B_i = 0$, for each $i = 1,2,\dots,m$. Then, A_i is $\frac{1-\kappa_i}{2}$-inverse strongly monotone and $\mathrm{Fix}(S_i) = A_i^{-1}(0)$. Since for each $i \in \{1,2,\dots,m\}$, S_i is a ς_i-strict pseudocontraction, we have S_i is a $\frac{2}{1-\varsigma_i}$-generalized demimetric mapping and $I - S_i$ is demiclosed at zero. Hence, by Theorem 3.2, we obtain the desired result. □

5 Numerical experiments

Example 5.1 We consider the following multiple-set split feasibility problem: Find an element $x^\star \in \Omega$ with

$$\Omega = \bigcap_{i=1}^{3} \big(C_i \cap T_i^{-1}(Q_i)\big),$$

where $C_i \subset \mathbb{R}^{10}$ and $Q_i \subset \mathbb{R}^{20}$ that are defined by

$$C_i = \{x \in \mathbb{R}^{10} : \langle a_i, x\rangle \le b_i\}, \quad i = 1,2,3,$$
$$Q_1 = \{x \in \mathbb{R}^{20} : \|x - (d_1, 0, 0, \dots, 0)\| \le 1\},$$
$$Q_2 = \{x \in \mathbb{R}^{20} : \|x - (0, d_2, 0, \dots, 0)\| \le 1\},$$
$$Q_3 = \{x \in \mathbb{R}^{20} : \|x - (0, 0, d_3, \dots, 0)\| \le 1\}$$

and $T_i : \mathbb{R}^{10} \to \mathbb{R}^{20}$, are bounded linear operators with the elements of the representing matrix that are randomly generated in the closed interval $[-2,2]$. We examine the convergence of the sequences $\{x_n\}$, which are defined in Theorem 4.5, where the coordinates of the points a_i, $i = 1,2,3$, are randomly generated in the closed interval $[1,4]$ and the numbers b_i, $i = 1,2,3$, are randomly generated in the closed interval $[1,2]$, and the numbers d_i, $i = 1,2,3$, are randomly generated in the closed interval $[0,1]$. The coordinates of the point u and the initial points x_0 and x_1 are randomly generated in the interval $[0,1]$. The stopping criterion is $E_n = \|x_n - x_{n-1}\| < 10^{-5}$. In this case, $x^* = 0$, is a solution of this problem. We know that for $i = 1,2,3$,

$$P_{C_i}x = \begin{cases} x - \frac{\langle a_i, x\rangle - b_i}{\|a_i\|^2} a_i, & \text{if } \langle a_i, x\rangle > b_i \\ x, & \text{if } \langle a_i, x\rangle \le b_i \end{cases} \tag{29}$$

and

$$P_{Q_1}(x) = \begin{cases} (d_1, 0, 0, \dots, 0) + \frac{x - (d_1,0,0,\dots,0)}{\|x - (d_1,0,0,\dots,0)\|}, & x \notin Q_1 \\ x, & x \in Q_1. \end{cases}$$

The numerical results that we have obtained are presented in Table 1.

The behavior of E_n in Table 1 with $\theta^* = 0.7$ is depicted in Fig. 1.

Example 5.2 In Theorem 4.7, set $E = F = \ell_2(\mathbb{R}) := \{x = (x_1, x_2, \dots, x_n, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, with inner product $\langle .,. \rangle$ and norm $\|.\|$ defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i, \qquad \|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}.$$

Let $S : E \to E$, $U : F \to F$, and $T : E \to F$ be defined by

$$Sx = \frac{-3}{2}x, \qquad Ux = -x, \qquad Tx = x.$$

Table 1 Table of numerical results for Theorem 4.5

	$\alpha_n = \frac{0.1}{n+10}$, $\beta_n = \frac{1}{(n+10)^{1.2}}$ and $\gamma_{n,i} = \frac{1-\alpha_n}{3}$			
	$\theta^* = 0.01$	$\theta^* = 0.25$	$\theta^* = 0.5$	$\theta^* = 0.7$
n	1814	1285	703	266
Time (s)	0.3614	0.3276	0.2726	0.1520

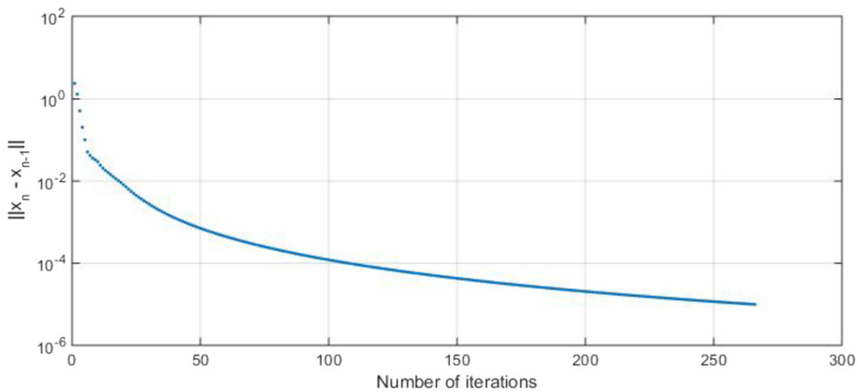

Figure 1 The graph of the error $\|x_n - x_{n-1}\|$

It is easy to verify that S is a $\frac{1}{5}$-strict pseudocontraction mapping, U is nonexpansive (a 0-strict pseudocontraction mapping) and T is a bounded linear operator. Furthermore, $\Omega = \mathrm{Fix}(U) \cap \mathrm{Fix}(S) = \{0\}$. We take $\theta_n = 0$, $r_n = \frac{2}{3}$, $\lambda_n = \frac{1}{2}$, $\nu = 0$, and $\alpha_n = \frac{1}{n+1}$. Then, the sequence $\{x_n\}$ induced by our algorithm, reduces to

$$x_{n+1} = \left(\frac{1}{n+1}\right)\left(\frac{1}{12}\right)^n x_0, \quad \forall n \geq 0.$$

Now, for $x_0 = (1, 1, 1, 0, 0, \dots)$, we obtain the following numerical results:

$$x_{10} = (1.46823257E - 12, 1.46823257E - 12, 1.46823257E - 12, 0, 0, \dots),$$

$$x_{20} = (1.24209778E - 23, 1.24209778E - 23, 1.24209778E - 23, 0, 0, \dots),$$

$$x_{50} = (2.15460430E - 56, 2.15460430E - 56, 2.15460430E - 56, 0, 0, \dots).$$

6 Conclusions

In this paper, a new iterative scheme with inertial effect is proposed for solving the generalized multiple-set split feasibility problem in a 2-uniformly convex and uniformly smooth Banach space E and a smooth, strictly convex, and reflexive Banach space F. The strong convergence of the iterative sequences generated by the presented algorithm is established without requiring the prior knowledge of operator norm. Finally, we applied our result to study and approximate the solutions of certain classes of optimization problems. The results obtained in this paper improved and extended many others known in the field. As part of our future research, we would like to extend the results in this paper to a more general space, such as the p-uniformly convex Banach space. Furthermore, we would consider the generalized multiple-set split feasibility problem involving a sum of maximal monotone and Lipschitz continuous monotone mappings. It would be interesting if the algorithm proposed in this paper could be applied to some practical optimization problems.

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