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A new reverse Mulholland's inequality with one partial sum in the kernel

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Abstract

By means of the techniques of real analysis, applying some basic inequalities and formulas, a new reverse Mulholland's inequality with one partial sum in the kernel is given. We obtain the equivalent conditions of the parameters related to the best value in the new inequality. As applications, we reduce to the equivalent forms and a few inequalities for particular parameters.

Mathematics Subject Classification: 26D15; 26D10

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1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$ are such that $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following Hardy–Hilbert's inequality with the best value $\frac{\pi}{\sin(\pi/p)}$ (cf. [1, Theorem 315]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

With regards to a similar assumption, the well-known Mulholland's inequality was given as follows (cf. [1, Theorem 343]):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{mn\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty} \frac{a_m^p}{m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty} \frac{b_n^q}{n}\right)^{\frac{1}{q}}. \tag{2}$$

For $\lambda_i \in (0,2]$ $(i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda \in (0,4]$, a generalization of (1) was obtained (see [2]) in 2016 as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{3}$$

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where the constant $B(\lambda_1, \lambda_2)$ is the best value and

$$B(u,v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad (u,v>0)$$

is the Beta function related to the Gamma function. For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (3) reduces to (1); for $p = q = 2$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (3) reduces to an inequality published in [3].

In 2019, by means of (3), Adiyasuren et al. [4] gave a generalization of (3) as follows: For $\lambda_i \in (0,1] \cap (0,\lambda)$ ($\lambda \in (0,2]$; $i = 1,2$), $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geq 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left(\sum_{m=1}^\infty m^{-p\lambda_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q\lambda_2-1}B_n^q\right)^{\frac{1}{q}}, \tag{4}$$

where $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best value, and two partial sums $A_m := \sum_{i=1}^m a_i$ and $B_n := \sum_{k=1}^n b_k$ ($m,n \in \{1,2,\ldots\}$) satisfy

$$0 < \sum_{m=1}^\infty m^{-p\lambda_1-1}A_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{-q\lambda_2-1}B_n^q < \infty.$$

Some generalizations of (1) and (2) were obtained in [5–15]. In 2021, Gu and Yang [16] gave an improvement of (4) with the kernel $\frac{1}{(m^\alpha+n^\beta)^\lambda}$. But we find that the constant is not the best possible unless $\alpha = \beta = 1$. In 2016, Hong et al. [17] gave a few equivalent conditions of the parameters related to the best value in the general form of (1). Some further works were provided in [18–29].

In this article, following the methods of [16, 17], by means of the techniques of analysis, several basic inequalities and formulas, a new reverse Mulholland's inequality with one partial sum in the kernel is given. The equivalent conditions of the parameters related to the best value in the new inequality are obtained. We also deduce the equivalent forms and a few equivalent inequalities for particular parameters.

2 Some lemmas

In what follows, we assume that $0 < p < 1$ ($q < 0$), $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\lambda_i \in (0,2] \cap (0,\lambda)$ ($i = 1,2$), $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, $\mathrm{N} = \{1,2,\ldots\}$, $m,n \in \mathrm{N}\backslash\{1\}$, $a_m, b_n \geq 0$, $A_m := \sum_{k=2}^m a_k = o(e^{t\ln m})$ ($t > 0$; $m \to \infty$), and

$$0 < \sum_{m=2}^\infty \frac{\ln^{p(1-\hat{\lambda}_1)-1} m}{m^{1-p}} a_m^p < \infty, \qquad 0 < \sum_{n=2}^\infty \frac{\ln^{q(1-\hat{\lambda}_2)-1} n}{n^{1-q}} b_n^q < \infty.$$

Lemma 1 (cf. [5, (2.2.3)]) *(i) If* $(-1)^i \frac{d^i}{dt^i}h(t) > 0$, $t \in [m,\infty)$ ($m \in \mathrm{N}$), $h^{(i)}(\infty) = 0$ ($i = 0,1,2,3$), $P_i(t)$, B_i ($i \in \mathrm{N}$) *are the Bernoulli functions and numbers, then*

$$\int_m^\infty P_{2q-1}(t)h(t)\,dt = -\varepsilon_q \frac{B_{2q}}{2q}h(m) \quad (0 < \varepsilon_q < 1; q = 1,2,\ldots). \tag{5}$$

For $q = 1$, $B_2 = \frac{1}{6}$, *we have*

$$-\frac{1}{12}h(m) < \int_m^\infty P_1(t)h(t)\,dt < 0. \tag{6}$$

If $(-1)^i \frac{d^i}{dt^i}h(t) > 0$, $t \in [m,\infty)$, $h^{(i)}(\infty) = 0$ $(i = 0,1)$, *then we still have* (*cf.* [5, (2.2.13)])

$$-\frac{1}{8}h(m) < \int_m^\infty P_1(t)h(t)\,dt < 0. \tag{7}$$

(*ii*) (*cf.* [5, (2.1.14)]) *If* $n > m \in \mathrm{N}$, $f(t)(> 0) \in C^1[m,\infty)$, $f^{(i)}(\infty) = 0$ $(i = 0,1)$, *then the following Euler–Maclaurin summation formulas are valid*:

$$\sum_{i=m}^{n} f(i) = \int_m^n f(t)\,dt + \frac{1}{2}\big(f(m) + f(n)\big) + \int_m^n P_1(t)f'(t)\,dt, \tag{8}$$

$$\sum_{i=m}^{\infty} f(i) = \int_m^\infty f(t)\,dt + \frac{1}{2}f(m) + \int_m^\infty P_1(t)f'(t)\,dt. \tag{9}$$

Lemma 2 *If* $s > 0$, $s_2 \in (0,2] \cap (0,s)$, $K_s(s_2) := B(s_2, s-s_2)$, *and the weight coefficient is defined as follows*:

$$\varpi_s(s_2,m) := \ln^{s-s_2} m \sum_{n=2}^{\infty} \frac{\ln^{s_2-1} n}{n(\ln mn)^s} \quad \big(m \in \mathrm{N}\backslash\{1\}\big), \tag{10}$$

then we have the following inequalities:

$$0 < K_s(s_2)\left(1 - O\left(\frac{1}{\ln^{s_2} m}\right)\right) < \varpi_s(s_2,m) < K_s(s_2) \quad \big(m \in \mathrm{N}\backslash\{1\}\big), \tag{11}$$

where $O(\frac{1}{\ln^{s_2} m}) := \frac{1}{k_s(s_2)} \int_0^{\frac{\ln 2}{\ln m}} \frac{v^{s_2-1}}{(1+v)^s}\,dv > 0$.

Proof For a fixed $m \in \mathrm{N}\backslash\{1\}$, we set $g_m(t)$ as follows: $g_m(t) := \frac{\ln^{s_2-1} t}{(\ln m + \ln t)^s t}$ $(t > 1)$.

(i) For $s_2 \in (0,1] \cap (0,s)$, in view of the decreasingness property of series, we have

$$\int_2^\infty g_m(t)\,dt < \sum_{n=2}^{\infty} g_m(n) < \int_1^\infty g_m(t)\,dt. \tag{12}$$

(ii) For $s_2 \in (1,2] \cap (0,s)$, in view of (9), we have

$$\sum_{n=2}^{\infty} g_m(n) = \int_2^\infty g_m(t)\,dt + \frac{1}{2}g_m(2) + \int_2^\infty P_1(t)g_m'(t)\,dt = \int_1^\infty g_m(t)\,dt - h(m),$$

$$h(m) := \int_1^2 g_m(t)\,dt - \frac{1}{2}g_m(2) - \int_2^\infty P_1(t)g_m'(t)\,dt.$$

We obtain $-\frac{1}{2}g_m(2) = \frac{-\ln^{s_2-1} 2}{4(\ln m + \ln 2)^s}$. Setting $v = \ln t$ and using integration by parts, we find

$$\begin{aligned}\int_1^2 g_m(t)\,dt &= \int_0^{\ln 2} \frac{v^{s_2-1}}{(\ln m + v)^s}\,dv = \int_0^{\ln 2} \frac{dv^{s_2}}{s_2(\ln m + v)^s} \\ &= \frac{v^{s_2}}{s_2(\ln m + v)^s}\Big|_0^{\ln 2} - \int_0^{\ln 2} \frac{v^{s_2}}{s_2}\,d\frac{1}{(\ln m + v)^s}\end{aligned}$$

$$= \frac{\ln^{s_2} 2}{s_2(\ln m + \ln 2)^s} + \frac{s}{s_2(s_2+1)} \int_0^{\ln 2} \frac{dv^{s_2+1}}{(\ln m + v)^{s+1}}$$
$$> \frac{\ln^{s_2} 2}{s_2(\ln m + \ln 2)^s} + \frac{s}{s_2(s_2+1)} \frac{\ln^{s_2+1} 2}{(\ln m + \ln 2)^{s+1}}.$$

Since $\frac{\ln t}{t^2} > 0$, $(\frac{\ln t}{t^2})' = \frac{1-2\ln t}{t^3} < 0$ $(t > 2)$, by (7), we have

$$g_m'(t) = \frac{(s_2-1)\ln^{s_2-2} t}{(\ln m + \ln t)^s t^2} - \frac{s \ln^{s_2-2} t}{(\ln m + \ln t)^{s+1}} \frac{\ln t}{t^2} - \frac{\ln^{s_2-2} t}{(\ln m + \ln t)^s} \frac{\ln t}{t^2},$$
$$-\int_2^\infty P_1(t) \frac{(s_2-1)\ln^{s_2-2} t}{(\ln m + \ln t)^s t^2}\, dt$$
$$= (1-s_2) \int_2^\infty P_1(t) \frac{\ln^{s_2-2} t}{(\ln m + \ln t)^s t^2}\, dt > 0 \quad (s_2 \in (1,2]),$$
$$\int_2^\infty P_1(t) \left[\frac{s \ln^{s_2-2} t}{(\ln m + \ln t)^{s+1}} \frac{\ln t}{t^2} + \frac{\ln^{s_2 2} t}{(\ln m + \ln t)^s} \frac{\ln t}{t^2} \right] dt$$
$$> -\frac{1}{8} \left[\frac{s \ln^{s_2-1} 2}{4(\ln m + \ln 2)^{s+1}} + \frac{\ln^{s_2-1} 2}{4(\ln m + \ln 2)^s} \right],$$
$$-\int_2^\infty P_1(t) g_m'(t)\, dt > -\frac{s \ln^{s_2-1} 2}{32(\ln m + \ln 2)^{s+1}} - \frac{\ln^{s_2-1} 2}{32(\ln m + \ln 2)^s}.$$

Hence, for $s_2 \in (1,2] \cap (0,s)$, we obtain

$$h(m) > \frac{\ln^{s_2} 2}{s_2(\ln m + \ln 2)^s} + \frac{s}{s_2(s_2+1)} \frac{\ln^{s_2+1} 2}{(\ln m + \ln 2)^{s+1}}$$
$$- \frac{\ln^{s_2-1} 2}{4(\ln m + \ln 2)^s} - \frac{s \ln^{s_2-1} 2}{32(\ln m + \ln 2)^{s+1}} - \frac{\ln^{s_2-1} 2}{32(\ln m + \ln 2)^s}$$
$$= \frac{\ln^{s_2-1} 2}{(\ln m + \ln 2)^s} \left(\frac{\ln 2}{s_2} - \frac{1}{4} - \frac{1}{32} \right) + \frac{s \ln^{s_2-1} 2}{(\ln m + \ln 2)^{s+1}} \left[\frac{\ln^2 2}{s_2(s_2+1)} - \frac{1}{32} \right]$$
$$\geq \frac{\ln^{s_2-1} 2}{(\ln m + \ln 2)^s} \left(\frac{\ln 2}{2} - \frac{9}{32} \right) + \frac{s \ln^{s_2-1} 2}{(\ln m + \ln 2)^{s+1}} \left(\frac{\ln^2 2}{6} - \frac{1}{32} \right)$$
$$> 0 \quad \left(\ln 2 = 0.6931^+\right).$$

Therefore, we have $h(m) > 0$. We still have

$$\sum_{n=2}^\infty g_m(n) = \int_2^\infty g_m(t)\, dt + \frac{1}{2} g_m(2) + \int_2^\infty P_1(t) g_m'(t)\, dt$$
$$= \int_2^\infty g_m(t)\, dt + h_1(m),$$
$$h_1(m) := \frac{1}{2} g_m(2) + \int_2^\infty P_1(t) g_m'(t)\, dt.$$

For $s_2 \in (1,2] \cap (0,s)$, in view of (7), we find

$$\int_2^\infty P_1(t) \frac{(s_2-1)\ln^{s_2-2} t}{(\ln m + \ln t)^s t^2}\, dt > -\frac{s_2-1}{32} \frac{\ln^{s_2-2} 2}{(\ln m + \ln 2)^s},$$

$$-\int_2^\infty P_1(t)\left[\frac{s\ln^{s_2-2}t}{(\ln m+\ln t)^{s+1}}\frac{\ln t}{t^2}+\frac{s\ln^{s_2-2}t}{(\ln m+\ln t)^s}\frac{\ln t}{t^2}\right]dt>0,$$
$$\int_2^\infty P_1(t)g_m'(t)\,dt>-\frac{s_2-1}{32}\frac{\ln^{s_2-2}2}{(\ln m+\ln 2)^s},$$
$$h_1(m)>\frac{\ln^{s_2-1}2}{4(\ln m+\ln 2)^s}-\frac{s_2-1}{32}\frac{\ln^{s_2-2}2}{(\ln m+\ln 2)^s}\geq\frac{\ln^{s_2-2}2}{4(\ln m+\ln 2)^s}\left(\ln 2-\frac{1}{8}\right)>0.$$

Hence, we have (12).

(iii) For $s_2\in(0,2]\cap(0,s)$, by (12), setting $\nu=\frac{\ln t}{\ln m}$, it follows that

$$\varpi_s(s_2,m)=\ln^{s-s_2}m\sum_{n=2}^\infty g_m(n)<\ln^{s-s_2}m\int_1^\infty g_m(t)\,dt$$
$$=\int_0^\infty\frac{\nu^{s_2-1}\,d\nu}{(1+\nu)^s}=B(s_2,s-s_2)=k_s(s_2),$$
$$\varpi_s(s_2,m)>\ln^{s-s_2}m\int_2^\infty g_m(t)\,dt=\int_{\frac{\ln 2}{\ln m}}^\infty\frac{\nu^{s_2-1}\,d\nu}{(1+\nu)^s}=k_s(s_2)\left(1-O\left(\frac{1}{\ln^{s_2}m}\right)\right)>0,$$

where we indicate that $O(\frac{1}{\ln^{s_2}m})=\frac{1}{k_s(s_2)}\int_0^{\frac{\ln 2}{\ln m}}\frac{\nu^{s_2-1}}{(1+\nu)^s}\,d\nu$, satisfying

$$0<\int_0^{\frac{\ln 2}{\ln m}}\frac{\nu^{s_2-1}}{(1+\nu)^s}\,d\nu\leq\int_0^{\frac{\ln 2}{\ln m}}\nu^{s_2-1}\,d\nu=\frac{1}{s_2}\left(\frac{\ln 2}{\ln m}\right)^{s_2}.$$

Therefore, inequalities (11) follow.

This proves the lemma. □

Lemma 3 *If* $a\in(-1,1)$, $m\in\mathrm{N}\backslash\{1\}$, *then there exists a constant* C *such that*

$$\sum_{k=2}^m\frac{\ln^a k}{k}=\frac{\ln^{a+1}m}{a+1}+C+O\left(\frac{1}{m}\ln^a m\right)\quad(m\to\infty).\tag{13}$$

Proof We set $f(t):=\frac{1}{t}\ln^a t$ $(t\geq 2)$. Then we find

$$f'(t)=\frac{a}{t^2}\ln^{a-1}t-\frac{1}{t^2}\ln^a t=ag_1(t)-g_2(t),$$

where $g_1(t)=\frac{1}{t^2}\ln^{a-1}t$, $g_2(t)=\frac{1}{t^2}\ln^a t$ $(t\geq 2)$.

We obtain $(-1)^i g_1^{(i)}(t)>0$ $(t\geq 2; i=0,1)$. Since

$$a<1<2\ln 2\leq 2\ln t, g_2'(t)=\frac{a-2\ln t}{t^3}\ln^{a-1}t<0\quad(t\geq 2),$$

it follows that $(-1)^i g_2^{(i)}(t)>0$ $(i=0,1)$. In view of (2.2.12) in [5], we have

$$\int_2^m P_1(t)g_j(t)\,dt=\frac{\varepsilon_j}{8}g_j(t)|_2^m\quad(0<\varepsilon_j<1; j=0,1).$$

By (8), we have

$$\begin{aligned}\sum_{k=2}^{m} f(k) &= \int_2^m f(t)\,dt + \frac{1}{2}\big(f(m)+f(2)\big) + \int_2^m P_1(t)f'(t)\,dt \\ &= \int_2^m f(t)\,dt + \frac{1}{2}\big(f(m)+f(2)\big) + a\int_2^m P_1(t)g_1(t)\,dt - \int_2^m P_1(t)g_2(t)\,dt.\end{aligned}$$

By simplification, we obtain (13), where

$$\begin{aligned}&C := -\frac{1}{a+1}\ln^{a+1}2 + \left(\frac{1}{4}+\frac{\varepsilon_2}{32}\right)\ln^a 2 - \frac{\varepsilon_1 a}{32}\ln^{a-1}2 \quad \text{and} \\ &O\left(\frac{1}{m}\ln^a m\right) := \frac{\ln^a m}{2m} + \frac{\varepsilon_1 a}{8m^2}\ln^{a-1}m - \frac{\varepsilon_2}{8m^2}\ln^a m \quad (m\to\infty).\end{aligned}$$

This proves the lemma. □

Lemma 4 *For $t>0$, the following inequality is valid*:

$$\sum_{m=2}^{\infty} e^{-t\ln m}m^{-1}A_m \ge \frac{1}{t}\sum_{m=2}^{\infty} e^{-t\ln m}a_m. \tag{14}$$

Proof Since $A_m e^{-t\ln m} = o(1)$ $(m\to\infty)$, by Abel's summation by parts formula, it follows that

$$\begin{aligned}\sum_{m=2}^{\infty} e^{-t\ln m}a_m &= \lim_{m\to\infty} A_m e^{-t\ln m} + \sum_{m=2}^{\infty} A_m\big[e^{-t\ln m} - e^{-t\ln(m+1)}\big] \\ &= \sum_{m=2}^{\infty} A_m\big[e^{-t\ln m} - e^{-t\ln(m+1)}\big].\end{aligned}$$

For a fixed $m\in \mathrm{N}\backslash\{1\}$, we set $f(x) := e^{-t\ln x}$, $x\in[m,m+1]$. Since $f'(x) = -th(x)$, where $h(x) := x^{-1}e^{-t\ln x}$ is decreasing in $[m,m+1]$, by the differentiation intermediate value theorem, there exists a constant $\theta\in(0,1)$ such that

$$\begin{aligned}\sum_{m=2}^{\infty} e^{-t\ln m}a_m &= -\sum_{m=2}^{\infty} A_m\big(f(m+1)-f(m)\big) \\ &= -\sum_{m=2}^{\infty} A_m f'(m+\theta) = t\sum_{m=2}^{\infty} h(m+\theta)A_m \\ &\le t\sum_{m=2}^{\infty} h(m)A_m = t\sum_{m=2}^{\infty} m^{-1}e^{-t\ln m}A_m,\end{aligned}$$

namely, inequality (14) follows.

This proves lemma. □

Lemma 5 *For* $0 < p < 1$ $(q < 0)$, *the following reverse inequality is valid*:

$$I_\lambda := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln mn)^\lambda} > \left(k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}$$
$$\times\left[\sum_{m=2}^{\infty}\left(1-O\left(\frac{1}{\ln^{\lambda_2} m}\right)\right)\frac{\ln^{p(1-\hat{\lambda}_1)-1} m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\hat{\lambda}_2)-1} n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{15}$$

Proof In view of the symmetry, for $s_1 \in (0,2] \cap (0,s)$, $s > 0$, we set and obtain the next weight coefficient as follows:

$$0 < k_s(s_1)\left(1-O\left(\frac{1}{\ln^{s_1} n}\right)\right) < \omega_s(s_1,n) := \ln^{s-s_1} n \sum_{m=2}^{\infty}\frac{\ln^{s_1-1} m}{m(\ln mn)^s}$$
$$< k_s(s_1) = B(s_1, s-s_1) \quad \left(n \in \mathrm{N}\backslash\{1\}\right), \tag{16}$$

where $O(\frac{1}{\ln^{s_1} n}) := \frac{1}{k_s(s_1)}\int_0^{\frac{\ln 2}{\ln n}}\frac{v^{s_1-1}}{(1+v)^s}\,dv > 0$.

In view of the reverse Hölder's inequality (cf. [30]), we find

$$I_\lambda = \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{(\ln mn)^\lambda}\left[\frac{m^{1/q}\ln^{(\lambda_2-1)/p} n}{n^{1/p}\ln^{(\lambda_1-1)/q} m}a_m\right]\left[\frac{n^{1/p}\ln^{(\lambda_1-1)/q} m}{m^{1/q}\ln^{(\lambda_2-1)/p} n}b_n\right]$$
$$\geq \left[\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{(\ln mn)^\lambda}\frac{m^{p-1}\ln^{\lambda_2-1} n}{n\ln^{(\lambda_1-1)(p-1)} m}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{(\ln mn)^\lambda}\frac{n^{(q-1)}\ln^{\lambda_1-1} m}{m\ln^{(\lambda_2-1)(q-1)} n}b_n^q\right]^{\frac{1}{q}}$$
$$= \left(\sum_{m=2}^{\infty}\varpi_\lambda(\lambda_2,m)\frac{\ln^{p(1-\hat{\lambda}_1)-1} m}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\omega_\lambda(\lambda_1,n)\frac{n^{q(1-\hat{\lambda}_2)-1}n}{n^{1-q}}b_n^q\right)^{\frac{1}{q}}.$$

By (11) and (16) (for $s = \lambda$, $s_i = \lambda_i \in (0,2] \cap (0,\lambda)$ $(i = 1,2)$), we obtain (15).

This proves the lemma. □

3 Main results

Theorem 1 *The following reverse Mulholland's inequality with* A_m *in the kernel is valid*:

$$I := \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{A_m b_n}{(\ln mn)^{\lambda+1} m} > \frac{1}{\lambda}\left(k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}$$
$$\times\left[\sum_{m=2}^{\infty}\left(1-O\left(\frac{1}{\ln^{\lambda_2} m}\right)\right)\frac{\ln^{p(1-\hat{\lambda}_1)-1} m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\hat{\lambda}_2)-1} n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{17}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, *we have*

$$0 < \sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}}a_m^p < \infty, \qquad 0 < \sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}}b_n^q < \infty,$$

and the following reverse inequality:

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{A_m b_n}{(\ln mn)^{\lambda+1}m} > \frac{1}{\lambda}B(\lambda_1,\lambda_2)\left[\sum_{m=2}^{\infty}\left(1-O\left(\frac{1}{\ln^{\lambda_2} m}\right)\right)\frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}} \times\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{18}$$

Proof In view of the following expression related to the Gamma function:

$$\frac{1}{(\ln m+\ln n)^{\lambda+1}} = \frac{1}{\Gamma(\lambda+1)}\int_0^{\infty} t^{\lambda}e^{-(\ln m+\ln n)t}\,dt,$$

by (14), it follows that

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda+1)}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{m}A_m b_n\int_0^{\infty} t^{\lambda}e^{-(\ln m+\ln n)t}\,dt\\
&= \frac{1}{\Gamma(\lambda+1)}\int_0^{\infty} t^{\lambda}\left(\sum_{m=2}^{\infty}e^{-t\ln m}\frac{1}{m}A_m\right)\sum_{n=2}^{\infty}e^{-t\ln n}b_n\,dt\\
&\ge \frac{1}{\Gamma(\lambda+1)}\int_0^{\infty} t^{\lambda}\left(\frac{1}{t}\sum_{m=2}^{\infty}e^{-t\ln m}a_m\right)\sum_{n=2}^{\infty}e^{-t\ln n}b_n\,dt\\
&= \frac{1}{\Gamma(\lambda+1)}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}a_m b_n\int_0^{\infty} t^{\lambda-1}e^{-(\ln m+\ln n)t}\,dt\\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda+1)}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{(\ln mn)^{\lambda}}a_m b_n.
\end{aligned}$$

Then by (15), in view of $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$, we have (17). For $\lambda_1+\lambda_2=\lambda$ in (17), we have (18).

This proves the theorem. □

Theorem 2 *Assume that* $\lambda_1\in(0,2)\cap(0,\lambda)$, $\lambda_2\in(0,2)\cap(0,\lambda)$. *If* $\lambda_1+\lambda_2=\lambda$, *then the constant* $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in* (17) *is the best possible.*

Proof We now show that $\frac{1}{\lambda}B(\lambda_1,\lambda_2)$ in (18) is the best value under the assumptions of this theorem.

For any $0<\varepsilon<\min\{p\lambda_1,|q|(2-\lambda_2)\}$, we set

$$\tilde{a}_m := \frac{1}{m}\ln^{(\lambda_1-\frac{\varepsilon}{p})-1} m,\qquad \tilde{b}_n := \frac{1}{n}\ln^{(\lambda_2-\frac{\varepsilon}{q})-1} n\quad\bigl(m,n\in\mathrm{N}\backslash\{1\}\bigr).$$

For $a=\lambda_1-\frac{\varepsilon}{p}-1\in(-1,1)$, by (13), we have

$$\tilde{A}_m := \sum_{k=2}^{m}\tilde{a}_k = \sum_{k=2}^{m}\frac{\ln^{\lambda_1-\frac{\varepsilon}{p}-1} k}{k} = \frac{\ln^{\lambda_1-\frac{\varepsilon}{p}} m}{\lambda_1-\frac{\varepsilon}{p}} + C + O\left(\frac{1}{m}\ln^{\lambda_1-\frac{\varepsilon}{p}} m\right)\quad(m\to\infty),$$

satisfying $\tilde{A}_m = o(e^{t\ln m})$ $(t>0;\ m\to\infty)$.

If there exists a constant $M(\geq \frac{1}{\lambda}B(\lambda_1,\lambda_2))$ such that (18) is valid when we replace $\frac{1}{\lambda}B(\lambda_1,\lambda_2)$ by M, then for $a_m = \tilde{a}_m$, $b_n = \tilde{b}_n$, and $A_m = \tilde{A}_m$, we have

$$\tilde{I} := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\tilde{A}_m\tilde{b}_n}{(\ln mn)^{\lambda+1}m}$$
$$> M\Bigg[\sum_{m=2}^{\infty}\bigg(1-O\bigg(\frac{1}{\ln^{\lambda_2} m}\bigg)\bigg)\frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}}\tilde{a}_m^p\Bigg]^{\frac{1}{p}}\Bigg[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}}\tilde{b}_n^q\Bigg]^{\frac{1}{q}}.$$

We obtain

$$\tilde{I} > M\Bigg[\sum_{m=2}^{\infty}\bigg(1-O\bigg(\frac{1}{\ln^{\lambda_2} m}\bigg)\bigg)\frac{\ln^{-\varepsilon-1} m}{m}\Bigg]^{\frac{1}{p}}\Bigg(\sum_{n=2}^{\infty}\frac{\ln^{-\varepsilon-1} n}{n}\Bigg)^{\frac{1}{q}}$$
$$= M\Bigg(\sum_{m=2}^{\infty}\frac{\ln^{-\varepsilon-1} m}{m}-\sum_{m=2}^{\infty}\frac{1}{m}O\bigg(\frac{1}{\ln^{\lambda_2+\varepsilon+1} m}\bigg)\Bigg)^{\frac{1}{p}}\Bigg(\frac{\ln^{-\varepsilon-1} 2}{2}+\sum_{m=3}^{\infty}\frac{\ln^{-\varepsilon-1} m}{m}\Bigg)^{\frac{1}{q}}$$
$$> M\bigg(\int_2^{\infty}\frac{\ln^{-\varepsilon-1} x}{x}\,dx - O(1)\bigg)^{\frac{1}{p}}\bigg(\frac{\ln^{-\varepsilon-1} 2}{2}+\int_2^{\infty}\frac{\ln^{-\varepsilon-1} x}{x}\,dx\bigg)^{\frac{1}{q}}$$
$$> \frac{M}{\varepsilon}\big(\ln^{-\varepsilon} 2-\varepsilon O(1)\big)^{\frac{1}{p}}\bigg(\frac{\ln^{-\varepsilon-1} 2}{2}\varepsilon+\ln^{-\varepsilon} 2\bigg)^{\frac{1}{q}}.$$

In view of (11) (for $s = \lambda + 1 > 0$, $s_2 = \lambda_2 - \frac{\varepsilon}{q} \in (0,2)\cap(0,\lambda)$), we obtain

$$\tilde{I} = \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\ln^{(\lambda_2-\frac{\varepsilon}{q})-1} n}{(\ln mn)^{\lambda=1}mn}\Bigg[\frac{1}{\lambda_1-\frac{\varepsilon}{p}}\ln^{\lambda_1-\frac{\varepsilon}{p}} m + C + O\bigg(\frac{1}{m}\ln^{\lambda_1-\frac{\varepsilon}{p}} m\bigg)\Bigg]$$
$$= \frac{1}{\lambda_1-\frac{\varepsilon}{p}}\sum_{m=2}^{\infty}\frac{\ln^{-\varepsilon-1} m}{m}\Bigg[\ln^{(\lambda_1+1+\frac{\varepsilon}{q})} m\sum_{n=2}^{\infty}\frac{\ln^{(\lambda_2-\frac{\varepsilon}{q})-1} m}{(\ln mn)^{\lambda+1}n}\Bigg]+\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{C\ln^{(\lambda_2-\frac{\varepsilon}{q})-1} n}{(\ln mn)^{\lambda+1}mn}$$
$$+\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\ln^{(\lambda_2-\frac{\varepsilon}{q})-1} n}{(\ln mn)^{\lambda+1}mn}O\bigg(\frac{1}{m}\ln^{\lambda_1-\frac{\varepsilon}{p}} m\bigg)$$
$$< \frac{1}{\lambda_1-\frac{\varepsilon}{p}}k_{\lambda+1}\bigg(\lambda_2-\frac{\varepsilon}{q}\bigg)\sum_{m=2}^{\infty}\frac{\ln^{-\varepsilon-1} m}{m}+\sum_{n=2}^{\infty}\frac{\ln^{(\lambda_2-\frac{\varepsilon}{q})-1} n}{(\ln n)^{\lambda_2+1}n}\sum_{m=2}^{\infty}\frac{C}{(\ln m)^{\lambda_1}m}$$
$$+\sum_{n=2}^{\infty}\frac{\ln^{(\lambda_2-\frac{\varepsilon}{q})-1} n}{(\ln n)^{\lambda_2+\frac{\varepsilon}{p}+1}n}\sum_{m=2}^{\infty}\frac{1}{(\ln m)^{\lambda_1-\frac{\varepsilon}{p}}m}O\bigg(\frac{1}{m}\ln^{\lambda_1-\frac{\varepsilon}{p}} m\bigg)$$
$$= \frac{1}{\lambda_1-\frac{\varepsilon}{p}}k_{\lambda+1}\bigg(\lambda_2-\frac{\varepsilon}{q}\bigg)\Bigg(\frac{\ln^{-\varepsilon-1} 2}{2}+\sum_{m=3}^{\infty}\frac{\ln^{-\varepsilon-1} m}{m}\Bigg)+\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\frac{\varepsilon}{q}+2}n}\cdot\sum_{m=2}^{\infty}\frac{C}{(\ln m)^{\lambda_1}m}$$
$$+\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\varepsilon+2}n}\cdot\sum_{m=2}^{\infty}O\bigg(\frac{1}{m^2}\bigg)$$
$$< \frac{1}{\lambda_1-\frac{\varepsilon}{p}}k_{\lambda+1}\bigg(\lambda_2-\frac{\varepsilon}{q}\bigg)\bigg(\frac{\ln^{-\varepsilon-1} 2}{2}+\int_2^{\infty}\frac{\ln^{-\varepsilon-1} x}{x}\,dx\bigg)+O_1(1)+O_2(1)$$
$$= \frac{1}{\varepsilon(\lambda_1-\frac{\varepsilon}{p})}k_{\lambda+1}\bigg(\lambda_2-\frac{\varepsilon}{q}\bigg)\bigg(\varepsilon\frac{\ln^{-\varepsilon-1} 2}{2}+\ln^{-\varepsilon} 2\bigg)+O_1(1)+O_2(1).$$

Then we have

$$\begin{aligned}&\frac{1}{\lambda_1-\frac{\varepsilon}{p}}k_{\lambda+1}\left(\lambda_2-\frac{\varepsilon}{q}\right)\left(\varepsilon\frac{\ln^{-\varepsilon-1}2}{2}+\ln^{-\varepsilon}2\right)+\varepsilon O_1(1)+\varepsilon O_2(1)\\&\quad>\varepsilon\tilde{I}>M\left(\ln^{-\varepsilon}2-\varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{\ln^{-\varepsilon-1}2}{2}\varepsilon+\ln^{-\varepsilon}2\right)^{\frac{1}{q}}.\end{aligned}$$

Setting $\varepsilon\to 0^+$, in view of the continuity of the Beta function, we obtain

$$\frac{1}{\lambda}B(\lambda_1,\lambda_2)=\frac{1}{\lambda_1}B(\lambda_1+1,\lambda_2)\geq M.$$

Therefore, $M=\frac{1}{\lambda}B(\lambda_1,\lambda_2)$ is the best value in (18).

This proves the theorem. □

Theorem 3 *Assume that* $\lambda>0$, $\lambda_1\in(0,2]\cap(0,\lambda)$, $\lambda_2\in(0,2)\cap(0,\lambda)$. *If the constant* $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in* (17) *is the best possible, then for*

$$\lambda-\lambda_1-\lambda_2\in\left(-p\lambda_1,p(\lambda-\lambda_1)\right)\cap\left[q(2-\lambda_2),p(2-\lambda_1)\right],\tag{19}$$

we have $\lambda_1+\lambda_2=\lambda$.

Proof Since $\hat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\frac{\lambda-\lambda_1-\lambda_2}{p}+\lambda_1$, $\hat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda-\lambda_1-\lambda_2}{q}+\lambda_2$, we find $\hat{\lambda}_1+\hat{\lambda}_2=\lambda$. In view of (19), for $\lambda-\lambda_1-\lambda_2\in(-p\lambda_1,p(\lambda-\lambda_1))$, we have $\hat{\lambda}_1\in(0,\lambda)$, $\hat{\lambda}_2=\lambda-\hat{\lambda}_1\in(0,\lambda)$, and then $B(\hat{\lambda}_1,\hat{\lambda}_2)\in\mathrm{R}_+$; for $\lambda-\lambda_1-\lambda_2\leq p(2-\lambda_1)$, we have $\hat{\lambda}_1\leq 2$; for $\lambda-\lambda_1-\lambda_2\geq q(2-\lambda_2)$, we have $\hat{\lambda}_2\leq 2$. Then, for $\lambda_i=\hat{\lambda}_i$ $(i=1,2)$ in (18), we still have

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{A_m b_n}{(\ln mn)^{\lambda+1}m}&>\frac{1}{\lambda}B(\hat{\lambda}_1,\hat{\lambda}_2)\left[\sum_{m=2}^{\infty}\left(1-O\left(\frac{1}{\ln^{\hat{\lambda}_2}m}\right)\right)\frac{\ln^{p(1-\hat{\lambda}_1)-1}m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}\\&\quad\times\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\hat{\lambda}_2)-1}n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{20}$$

In view of the reverse Hölder's inequality (cf. [30]), we find

$$\begin{aligned}B(\hat{\lambda}_1,\hat{\lambda}_2)&=k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)\\&=\int_0^\infty\frac{1}{(1+u)^\lambda}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}\,du=\int_0^\infty\frac{1}{(1+u)^\lambda}\left(u^{\frac{\lambda-\lambda_2-1}{p}}\right)\left(u^{\frac{\lambda_1-1}{q}}\right)du\\&\geq\left[\int_0^\infty\frac{1}{(1+u)^\lambda}u^{\lambda-\lambda_2-1}\,du\right]^{\frac{1}{p}}\left[\int_0^\infty\frac{1}{(1+u)^\lambda}u^{\lambda_1-1}\,du\right]^{\frac{1}{q}}\\&=\left[\int_0^\infty\frac{1}{(1+v)^\lambda}v^{\lambda_2-1}\,dv\right]^{\frac{1}{p}}\left[\int_0^\infty\frac{1}{(1+u)^\lambda}u^{\lambda_1-1}\,du\right]^{\frac{1}{q}}\\&=\left(k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}.\end{aligned}\tag{21}$$

If the constant $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (17) is the best possible, then, comparing with the constants in (17) and (20), we have

$$\frac{1}{\lambda}\left(k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \geq \frac{1}{\lambda}B(\hat{\lambda}_1,\hat{\lambda}_2)(\in \mathrm{R}_+),$$

namely, $B(\hat{\lambda}_1,\hat{\lambda}_2) \leq (k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$, and then (21) attains the form of an equality.

Inequality (21) becomes an equality if and only if there exist constants A and B such that they are not both zero and (cf. [30]) $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ *a.e.* in R_+. Supposing that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ *a.e.* in R_+. It follows that $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

This proves the theorem. □

4 Equivalent forms and some particular inequalities

Theorem 4 *The following reverse inequality equivalent to* (17) *is valid*:

$$\begin{aligned} J &:= \left\{\sum_{n=2}^{\infty}\frac{\ln^{p\hat{\lambda}_2-1} n}{n}\left[\sum_{m=2}^{\infty}\frac{A_m}{(\ln mn)^{\lambda+1}m}\right]^p\right\}^{\frac{1}{p}} \\ &> \frac{1}{\lambda}\left(k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}\left[\sum_{m=2}^{\infty}\left(1-O\left(\frac{1}{\ln^{\lambda_2} m}\right)\right)\frac{\ln^{p(1-\hat{\lambda}_1)-1} m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{22}$$

Particularly, for $\lambda_1 + \lambda_2 = \lambda$, *the following reverse inequality equivalent to* (18) *is valid*:

$$\begin{aligned} &\left\{\sum_{n=2}^{\infty}\frac{\ln^{p\lambda_2-1} n}{n}\left[\sum_{m=2}^{\infty}\frac{A_m}{(\ln mn)^{\lambda+1}m}\right]^p\right\}^{\frac{1}{p}} \\ &\quad > \frac{1}{\lambda}B(\lambda_1,\lambda_2)\left[\sum_{m=2}^{\infty}\left(1-O\left(\frac{1}{\ln^{\lambda_2} m}\right)\right)\frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{23}$$

Proof Assuming that (23) is valid, by the reverse Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty}\left[\frac{\ln^{-\frac{1}{p}+\hat{\lambda}_2} n}{n^{\frac{1}{p}}}\sum_{m=2}^{\infty}\frac{A_m}{(\ln mn)^{\lambda+1}m}\right]\left(\frac{\ln^{\frac{1}{p}-\hat{\lambda}_2} n}{n^{-1/p}}b_n\right) \geq J\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\hat{\lambda}_2)-1} n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{24}$$

In view of (23), we have (17). Assuming that (17) is valid, we set

$$b_n := \frac{\ln^{p\hat{\lambda}_2-1} n}{n}\left[\sum_{m=2}^{\infty}\frac{A_m}{(\ln mn)^{\lambda+1}m}\right]^{p-1}, \quad n \in \mathrm{N}\backslash\{1\}.$$

Then we find

$$\sum_{n=2}^{\infty}\frac{\ln^{q(1-\hat{\lambda}_2)-1} n}{n^{1-q}}b_n^q = J^p = I. \tag{25}$$

If $J = \infty$, then (23) is valid; if $J = 0$, then it is impossible that makes (23) valid, namely, $J > 0$. Assuming that $0 < J < \infty$, by (17), it follows that

$$J^p = I > \frac{1}{\lambda}\big(k_\lambda(\lambda_2)\big)^{\frac{1}{p}}\big(k_\lambda(\lambda_1)\big)^{\frac{1}{q}}\left[\sum_{m=2}^{\infty}\left(1 - O\left(\frac{1}{\ln^{\lambda_2} m}\right)\right)\frac{\ln^{p(1-\hat{\lambda}_1)-1} m}{m^{1-p}} a_m^p\right]^{\frac{1}{p}} J^{p-1},$$

$$J > \frac{1}{\lambda}\big(k_\lambda(\lambda_2)\big)^{\frac{1}{p}}\big(k_\lambda(\lambda_1)\big)^{\frac{1}{q}}\left[\sum_{m=2}^{\infty}\left(1 - O\left(\frac{1}{\ln^{\lambda_2} m}\right)\right)\frac{\ln^{p(1-\hat{\lambda}_1)-1} m}{m^{1-p}} a_m^p\right]^{\frac{1}{p}}.$$

Hence, (23) is valid, which is equivalent to (17).

This proves the theorem. □

Theorem 5 *Assume that* $\lambda_1 \in (0,2) \cap (0,\lambda)$, $\lambda_2 \in (0,2] \cap (0,\lambda)$. *If* $\lambda_1 + \lambda_2 = \lambda$, *then the constant* $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in* (23) *is the best possible. On the other hand, if the same constant in* (23) *is the best possible, then for* $\lambda - \lambda_1 - \lambda_2 \in [q(2-\lambda_2), p(2-\lambda_1)]$, *we have* $\lambda_1 + \lambda_2 = \lambda$.

Proof We show that the constant $\frac{1}{\lambda}B(\lambda_1,\lambda_2)$ in (24) is the best possible. Otherwise, by (25) (for $\lambda_1 + \lambda_2 = \lambda$), we would reach a contradiction that the same constant in (18) is not the best possible.

On the other hand, if the constant in (23) is the best possible, then the same constant in (17) is also the best possible. Otherwise, by (26) (for $\lambda_1 + \lambda_2 = \lambda$), we would reach a contradiction that the same constant in (24) is not the best possible.

This proves the theorem. □

Remark 1 For $\lambda \in (0,4)$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2} (< 2)$ in (18) and (24), we have the following equivalent forms with the best value $\frac{1}{\lambda}B(\frac{\lambda}{2},\frac{\lambda}{2})$:

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{A_m b_n}{(\ln mn)^{\lambda+1} m} > \frac{1}{\lambda}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[\sum_{m=2}^{\infty}\left(1 - O\left(\frac{1}{\ln^{\lambda/2} m}\right)\right)\frac{\ln^{p(1-\frac{\lambda}{2})-1} m}{m^{1-p}} a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\frac{\lambda}{2})-1} n}{n^{1-q}} b_n^q\right]^{\frac{1}{q}}, \tag{26}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\ln^{\frac{p\lambda}{2}-1} n}{n}\left[\sum_{m=2}^{\infty}\frac{A_m}{(\ln mn)^{\lambda+1} m}\right]^p\right\}^{\frac{1}{p}} > \frac{1}{\lambda}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[\sum_{m=2}^{\infty}\left(1 - O\left(\frac{1}{\ln^{\lambda/2} m}\right)\right)\frac{\ln^{p(1-\frac{\lambda}{2})-1} m}{m^{1-p}} a_m^p\right]^{\frac{1}{p}}. \tag{27}$$

Particularly, for $\lambda = 1$, we have the following equivalent inequalities with the best value π:

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{A_m b_n}{(\ln mn)^2 m} > \pi\left[\sum_{m=2}^{\infty}\left(1 - O\left(\frac{1}{\ln^{1/2} m}\right)\right)\frac{\ln^{-\frac{p}{2}-1} m}{m^{1-p}} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{\frac{q}{2}-1} n}{n^{1-q}} b_n^q\right]^{\frac{1}{q}}, \tag{28}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}n}{n}\left[\sum_{m=2}^{\infty}\frac{A_m}{(\ln mn)^2 m}\right]^p\right\}^{\frac{1}{p}} > \pi\left[\sum_{m=2}^{\infty}\left(1-O\left(\frac{1}{\ln^{1/2}m}\right)\right)\frac{\ln^{\frac{p}{2}-1}m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}. \quad (29)$$

5 Conclusions

In this article, by means of the techniques of analysis, applying the basic inequalities and formulas, a new reverse Mulholland's inequality with one partial sum in the kernel is given in Theorem 1. The equivalent conditions of the best value related to parameters are obtained in Theorems 2 and 3. As applications, we deduce the equivalent forms in Theorems 4 and 5, and some new inequalities for particular parameters in Remark 1.

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Data availability
The data used to support the findings of this study are included within the article.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
B.Y. and L.R. carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. X.Y.H. and X.S.H. participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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