Thabet and Kedim *Journal of Inequalities and Applications* (2023) 2023:162
https://doi.org/10.1186/s13660-023-03070-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

An investigation of a new Lyapunov-type inequality for Katugampola–Hilfer fractional BVP with nonlocal and integral boundary conditions

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Abstract

In this manuscript, we focus our attention on investigating new Lyapunov-type inequalities (**LTIs**) for two classes of boundary value problems (**BVPs**) in the framework of Katugampola–Hilfer fractional derivatives, supplemented by nonlocal, integral, and mixed boundary conditions. The equivalent integral equations of the proposed Katugampola–Hilfer fractional **BVPs** are established in the context of Green functions. Also, the properties of these Green functions are proved. The **LTIs** are investigated as sufficient criteria for the existence and nonexistence of nontrivial solutions for the subjected problems. Our systems are more general than in the literature, as a consequence there are many new and known specific cases included. Finally, our results are applied for estimating eigenvalues of two given **BVPs**.

Mathematics Subject Classification: 26A33; 26D10; 34B27

Keywords: Fractional boundary value problem; Katugampola–Hilfer fractional derivative; Existence and nonexistence solutions; Lyapunov-type inequality

1 Introduction

Fractional derivatives are a more flexible and formidable system for differentiation than classical integer derivatives. They can be used to model for all purposes a wider instability of phenomena in the real world; we refer the readers to some related research papers [1–11], and references cited therein. The Lyapunov-type inequalities (**LTIs**) are mathematical tools used to study the balance and behavior of dynamical systems. They can be utilized for a large range of problems in physics, engineering, and mathematics, such as the analysis of ordinary, partial, and fractional differential equations. They gained the interest of many researchers, for example, see these works [12–18]. The **LTI** for a boundary value problem (**BVP**) is one of the important tools to investigate the existence and nonexistence of nontrivial solutions, as well as estimate the eigenvalues for ordinary and fractional **BVPs**; for more details, see the surveys [19–21] and the references therein. In particular, the author [22] for the first time proved that there are nontrivial solutions for the second-order ordinary **BVP**,

$$\chi''(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, \quad \vartheta \in (\iota, \ell),$$

with boundary conditions

$$\chi(\iota) = 0 = \chi(\ell), \tag{1.1}$$

if the following **LTI** holds:

$$\int_{\iota}^{\ell} |\mathfrak{p}(\varepsilon)|\, d\varepsilon > \frac{4}{\ell - \iota}. \tag{1.2}$$

Ferreira in his work [23] introduced the following **LTI**:

$$\int_{\iota}^{\ell} |\mathfrak{p}(\varepsilon)|\, d\varepsilon > \Gamma(u)\left(\frac{4}{\ell - \iota}\right)^{u-1} \tag{1.3}$$

for the Riemann fractional **BVP**

$${}^{\mathrm{R}}\mathfrak{D}_{\iota^+}^{u}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, \quad u \in (1,2], \vartheta \in (\iota, \ell), \tag{1.4}$$

with the same boundary conditions as in (1.1). Also, Ferreira [24] established the following **LTI**:

$$\int_{\iota}^{\ell} |\mathfrak{p}(\varepsilon)|\, d\varepsilon > \frac{\Gamma(u)u^{u}}{((u-1)(\ell - \iota))^{u-1}} \tag{1.5}$$

for the Caputo fractional **BVP**

$${}^{\mathrm{C}}\mathfrak{D}_{\iota^+}^{u}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, \quad u \in (1,2], \vartheta \in (\iota, \ell), \tag{1.6}$$

under the boundary conditions given in (1.1). Additionally, Ferreira studied the **LTI** in [25] for the problem (1.4), supplemented with an integral boundary condition:

$$\chi(\iota) = 0, \qquad \chi(\ell) = \lambda \int_{\iota}^{\ell} \chi(\varepsilon)\, d\varepsilon, \quad \lambda \in \mathbb{R}.$$

Moreover, in 2016 Pathak [26] investigated the following **LTI**:

$$\frac{\Gamma(u)(\zeta + u - 2)^{\zeta + u - 2}}{(\zeta - 1)^{\zeta - 1}((u-1)(\ell - \iota))^{u-1}} \le \int_{\iota}^{\ell} |\mathfrak{p}(\varepsilon)|\, d\varepsilon, \quad \zeta = u + \nu(2 - u), \tag{1.7}$$

for a Hilfer fractional **BVP** of the form

$${}^{\mathrm{H}}\mathbb{D}_{\iota^+}^{u,\nu}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, \quad \vartheta \in \mathbb{T} = (\iota, \ell), u \in (1,2], \nu \in [0,1], \tag{1.8}$$

with boundary conditions given in (1.1). Moreover, in the same work, the author established an **LTI**

$$\frac{\Gamma(u)(\zeta - 1)}{(\ell - \iota)\max\{\zeta - u, u - 1\}} \le \int_{\iota}^{\ell} \left(\ell^{\rho} - \varepsilon^{\rho}\right)^{u-2} |\mathfrak{p}(\varepsilon)|\, d\varepsilon, \quad \zeta = u + \nu(2 - u), \tag{1.9}$$

for the same equation (1.8), under the following boundary conditions:

$$\chi(\iota) = 0, \qquad \chi'(\ell) = 0. \tag{1.10}$$

In 2021, the authors [27] proved an **LTI**

$$\frac{\iota u^u \Gamma(u)(\ln\frac{\ell}{\iota} - \lambda\sigma)}{\ln \ell/\iota + \lambda[\ln\frac{\ell}{\iota}\int_\iota^\ell \mathfrak{h}(\vartheta)\,d\vartheta - \sigma][(u-1)(\ln\ell - \ln\iota)]^{u-1}} < \int_\iota^\ell \left|\mathfrak{p}(\varepsilon)\right| d\varepsilon, \tag{1.11}$$

where $\sigma = \int_\iota^\ell \mathfrak{h}(\vartheta)\ln\frac{\vartheta}{\iota}\,d\vartheta$, for the Caputo–Hadamard fractional **BVP**

$$\begin{cases} {}^{\mathrm{H,C}}\mathbb{D}_{\iota^+}^{u}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, & \vartheta \in \mathbb{T} = (\iota,\ell), u \in (1,2], \\ \chi(\iota) = 0, \qquad \chi(\ell) = \lambda\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, & \lambda \geq 0. \end{cases}$$

Furthermore, in 2021 the authors of [28] studied **LTIs** for the following Katugampola–Hilfer fractional **BVPs** with multipoint boundary conditions:

$$\begin{cases} {}^{\rho,\mathrm{H}}\mathbb{D}_{\iota^+}^{u,\nu}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, \quad \vartheta \in \mathbb{T} = (\iota,\ell), u \in (1,2], \nu \in [0,1], \\ \quad (i) \quad \chi(\iota) = 0, \qquad \chi(\ell) = \sum_{i=1}^{m-2}\delta_i\chi(\eta_i), \quad \text{and} \\ \quad (ii) \quad \chi(\iota) = 0, \qquad \vartheta^{1-\rho}\frac{d}{d\vartheta}\chi(\vartheta)|_{\vartheta=\ell} = \sum_{i=1}^{m-2}\delta_i\chi(\eta_i). \end{cases}$$

On the other hand, nonlocal and integral boundary conditions have gained the interest of some research studies, for instance, Ahmad et al. [29] studied the qualitative properties for the coupled system of fractional **BVP** under such boundary conditions. Ntouyas [30] employed fixed point theorems to establish the existence and uniqueness results for Caputo fractional **BVP** with nonlocal and integral boundary conditions as given below:

$$\begin{cases} {}^{\mathrm{C}}\mathbb{D}_0^{u}\chi(\vartheta) = \mathcal{F}(\vartheta,\chi(\vartheta)), \quad \vartheta \in (0,1), u \in (1,2], \\ \chi(0) = \chi_0 + \mathfrak{g}(\chi), \qquad \chi(1) = u\,{}^{\mathrm{R}}\mathbb{I}_0^{p}\chi(\eta), \quad \eta \in (0,1), \end{cases}$$

where $\mathfrak{g}: \mathbb{R} \to \mathbb{R}$ is a nonlocal continuous function.

To our knowledge, no one investigated **LTIs** for fractional **BVPs** in the Katugampola–Hilfer derivative sense under nonlocal and integral boundary conditions. Thus, to close this gap and motivated by the aforementioned results, in the current work, we investigate new **LTIs** for the following Katugampola–Hilfer fractional differential equation:

$${}^{\rho,\mathrm{H}}\mathbb{D}_{\iota^+}^{u,\nu}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, \quad \vartheta \in \mathbb{T} = (\iota,\ell), u \in (1,2], \nu \in [0,1], \tag{1.12}$$

supplemented by the following:

(i) nonlocal and integral boundary conditions:

$$\chi(\iota) = \mathfrak{f}(\chi), \qquad \chi(\ell) = \int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, \tag{1.13}$$

(ii) nonlocal and mixed boundary conditions:

$$\chi(\iota) = \mathfrak{f}(\chi), \qquad \vartheta^{1-\rho}\frac{d}{d\vartheta}\chi(\vartheta)\bigg|_{\vartheta=\ell} = \int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, \tag{1.14}$$

where ${}^{\rho,\mathrm{H}}\mathbb{D}_{\iota^+}^{u,\nu}$ denotes to the Katugampola–Hilfer derivatives of fractional order $u \in (1,2]$ and type $\nu \in [0,1]$ such that $\rho > 0$. Furthermore, $\chi, \mathfrak{p}, \mathfrak{h} : \mathbb{T} \to \mathbb{R}$ are continuous functions, and the nonlocal function $\mathfrak{f} : \mathcal{C}(\mathbb{T},\mathbb{R}) \to \mathbb{R}$ is continuous and such that there exists a constant $\mu > 0$ so that $|\mathfrak{f}(\chi)| \le \mu\|\chi\|, \forall \chi \in \mathbb{R}$, where $\|\cdot\|$ is the norm of a Banach space $\mathcal{C}(\mathbb{T},\mathbb{R})$. Byszewski [31] remarked that the nonlocal condition can be useful in modeling physical phenomena with a better effect than the initial value condition. It is important to declare that the current work is more general than [28] because the integral boundary condition used in this work can be reduced to a summation of multiple points as in [28]. Also, our work contains a nonlocal function $\mathfrak{f}$ which can be described as $\mathfrak{f}(\chi) = \sum_{i=1}^{m} c_i \chi(\vartheta_i)$, $c_i \in \mathbb{R}$ $(i = 1,2,\ldots,m)$, and $\iota < \vartheta_1 < \vartheta_2 < \cdots < \vartheta_m < \ell$, to represent the diffusion phenomenon of a small amount of gas in a transparent tube; for more details, see [32]. Additionally, our results cover a lot new and existing research papers, and also can be reduced to a variety of fractional derivatives such as Hadamard–Hilfer ${}^{\mathrm{H,H}}\mathbb{D}_{\iota^+}^{u,\nu}$ for $(\rho \to 0^+)$; Hilfer ${}^{\mathrm{H}}\mathbb{D}_{\iota^+}^{u,\nu}$ for $(\rho = 1)$; Hadamard–Caputo ${}^{\mathrm{H,C}}\mathbb{D}_{\iota^+}^{u}$ for $(\rho \to 0^+$, and $\nu = 1)$; Hadamard–Riemann ${}^{\mathrm{H,R}}\mathbb{D}_{\iota^+}^{u}$ for $(\rho \to 0^+$ and $\nu = 0)$; Katugampola–Caputo ${}^{\rho,\mathrm{C}}\mathbb{D}_{\iota^+}^{u}$ for $(\nu = 1)$; Katugampola–Riemann ${}^{\rho,\mathrm{R}}\mathbb{D}_{\iota^+}^{u}$ for $(\nu = 0)$; Caputo ${}^{\mathrm{C}}\mathbb{D}_{\iota^+}^{u}$ for $(\rho = 1$ and $\nu = 1)$; Riemann ${}^{\mathrm{R}}\mathbb{D}_{\iota^+}^{u}$ for $(\rho = 1$ and $\nu = 0)$; and the usual second derivative for $(\rho = 1, \nu = 0$, and $u = 2)$.

The rest of this article is arranged as follows: Sect. 2 presents some important definitions and lemmas. Section 3 constructs the Green functions of fractional **BVPs** (1.12)–(1.13) and (1.12)–(1.14). Sections 4 and 5 concern proofs of **LTIs** for the proposed Katugampola–Hilfer fractional **BVPs**.

2 Background material

In this section, we present some relevant definitions and lemmas, which are important for our analysis. Let $\mathcal{C}((\iota,\ell),\mathbb{R})$ denote the class of all continuous functions which is a Banach space equipped with the norm $\|\chi\| = \max_{\vartheta\in(\iota,\ell)} |\chi(\vartheta)|$. Furthermore, let $AC_{\varrho}^{u}((\iota,\ell),\mathbb{R})$ denote the space of absolutely continuous functions defined by

$$AC_{\varrho}^{u}\big((\iota,\ell),\mathbb{R}\big) = \left\{\chi : (\iota,\ell) \to \mathbb{R} : \big(\varrho^{u-1}\chi\big)(\vartheta) \in AC\big((\iota,\ell),\mathbb{R}\big), \varrho = \frac{1}{\vartheta^{\rho-1}}\frac{d}{d\vartheta}\right\}.$$

Definition 2.1 ([33]) The Katugampola–Riemann fractional integral for the function $\mathfrak{f} \in L^1((\iota,\ell),\mathbb{R})$ of order u > 0 is defined as

$$\big({}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{u}\mathfrak{f}\big)(\vartheta) = \frac{1}{\Gamma(u)}\int_{\iota}^{\vartheta} \varepsilon^{\rho-1}\left(\frac{\vartheta^{\rho}-\varepsilon^{\rho}}{\rho}\right)^{u-1}\mathfrak{f}(\varepsilon)\,d\varepsilon, \quad \rho > 0,$$

with the following property:

$${}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{\sigma}\big(\vartheta^{\rho}-\iota^{\rho}\big)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\sigma)}\big(\vartheta^{\rho}-\iota^{\rho}\big)^{\mu+\sigma}, \quad \sigma \in \mathbb{R}^+, \mu > -1,$$

where Γ is the Gamma function. Additionally, the Katugampola–Riemann fractional derivatives for a function $\mathfrak{f} \in AC_{\varrho}^{u}((\iota,\ell),\mathbb{R})$ is defined by

$$\big({}^{\rho,\mathrm{R}}\mathbb{D}_{\iota^+}^{u}\mathfrak{f}\big)(\vartheta) = \varrho^{n}\big({}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{n-u}\mathfrak{f}\big)(\vartheta), \quad u \in (n-1,n], n \in \mathbb{N}, n = [u]+1, \tag{2.1}$$

where $[\cdot]$ is the integer part of u. Also, the Katugampola–Caputo fractional derivative for a function $\mathfrak{f} \in AC^{u}_{\varrho}((\iota,\ell),\mathbb{R}),\mathbb{R})$ is defined by

$$\left({}^{\rho,\mathrm{C}}\mathbb{D}^{u}_{\iota^+}\mathfrak{f}\right)(\vartheta) = {}^{\rho,\mathrm{R}}\mathbb{I}^{n-u}_{\iota^+}\left(\varrho^{n}\mathfrak{f}\right)(\vartheta), \quad u \in (n-1,n], n \in \mathbb{N}. \tag{2.2}$$

Definition 2.2 ([34]) Let $\rho > 0$. The Katugampola–Hilfer derivative for a function $\mathfrak{f} \in AC^{u}_{\varrho}((\iota,\ell),\mathbb{R})$ of fractional order $\mathrm{u} \in (n-1,n]$ with type $\mathrm{v} \in [0,1]$ is defined by

$${}^{\rho,\mathrm{H}}\mathbb{D}^{u,\nu}_{\iota^+}\mathfrak{f}(\vartheta) = {}^{\rho,\mathrm{R}}\mathbb{I}^{\mathrm{v}(n-\mathrm{u})}_{\iota^+}\varrho^{n}\,{}^{\rho,\mathrm{R}}\mathbb{I}^{(1-\mathrm{v})(n-\mathrm{u})}_{\iota^+}\mathfrak{f}(\vartheta).$$

Remark 2.3 The definition of fractional derivative (2.2) can be reduced to the following definitions:

(*i*) Hadamard–Hilfer derivative [35] for ($\rho \to 0^+$), which defined as

$${}^{\mathrm{H,H}}\mathbb{D}^{u,\nu}_{\iota^+}\mathfrak{f}(\vartheta) = {}^{\mathrm{H,R}}\mathbb{I}^{\mathrm{v}(n-\mathrm{u})}_{\iota^+}\left(\vartheta\frac{d}{d\vartheta}\right)^{n}\,{}^{\mathrm{H,R}}\mathbb{I}^{(1-\mathrm{v})(n-\mathrm{u})}_{\iota^+}\mathfrak{f}(\vartheta),$$

where ${}^{\mathcal{H}}\mathbb{I}^{\theta}_{\iota^+}$ is the Hadamard–Riemann integral of order $\theta > 0$ [36], defined by

$$\left({}^{\mathcal{H}}\mathbb{I}^{\theta}_{\iota^+}\mathfrak{f}\right)(\vartheta) = \frac{1}{\Gamma(\theta)}\int_{\iota}^{\vartheta}\left(\ln\frac{\vartheta}{\varepsilon}\right)^{\theta-1}\frac{\mathfrak{f}(\varepsilon)}{\varepsilon}\,d\varepsilon.$$

(*ii*) Hilfer derivative for ($\rho = 1$) [37];
(*iii*) Hadamard–Caputo derivative for ($\rho \to 0^+$ and $\nu = 1$) [38];
(*iv*) Hadamard–Riemann derivative for ($\rho \to 0^+$ and $\nu = 0$) [35];
(*v*) Katugampola–Riemann derivative (2.1) for ($\nu = 0$);
(*vi*) Katugampola–Caputo derivative (2.2) for ($\nu = 1$);
(*vii*) Caputo derivative for ($\rho = 1$ and $\nu = 1$) [36];
(*viii*) Riemann derivative for ($\rho = 1$ and $\nu = 0$) [36].

Lemma 2.4 ([34]) *Let* $\rho > 0$, $n-1 < \zeta \le n$, *be such that* $u \in (n-1,n]$, $\mathrm{v} \in [0,1]$ *with* $\zeta = \mathrm{u} + \mathrm{v}(n-\mathrm{u})$, *and* $\mathfrak{f} \in AC^{u}_{\varrho}((\iota,\ell),\mathbb{R})$. *Then, one has*

$${}^{\rho,\mathrm{R}}\mathbb{I}^{\zeta}_{\iota^+}\left({}^{\rho,\mathrm{R}}\mathbb{D}^{\zeta}_{\iota^+}\mathfrak{f}\right)(\vartheta) = {}^{\rho,\mathrm{R}}\mathbb{I}^{\mathrm{u}}_{\iota^+}\left({}^{\rho,\mathrm{H}}\mathbb{D}^{u,\nu}_{\iota^+}\mathfrak{f}\right)(\vartheta), \qquad {}^{\rho,\mathrm{H}}\mathbb{D}^{u,\nu}_{\iota^+}\left({}^{\rho,\mathrm{R}}\mathbb{I}^{\mathrm{u}}_{\iota^+}\mathfrak{f}(\vartheta)\right) = \mathfrak{f}(\vartheta).$$

Lemma 2.5 ([38]) *If* $\rho > 0$, $u \in (n-1,n]$, $\mathrm{v} \in [0,1]$, *and* $\mathfrak{f} \in AC^{u}_{\varrho}((\iota,\ell),\mathbb{R})$, *then*

$${}^{\rho,\mathrm{R}}\mathbb{I}^{\mathrm{u}}_{\iota^+}\left({}^{\rho,\mathrm{R}}\mathbb{D}^{\mathrm{u}}_{\iota^+}\mathfrak{f}(\vartheta)\right) = \mathfrak{f}(\vartheta) - \sum_{j=1}^{n} d_j\left(\frac{\vartheta^{\rho}-\iota^{\rho}}{\rho}\right)^{u-j}, \quad d_j \in \mathbb{R}.$$

3 The Green functions

In this section, we will derive the Green functions of the Katugampola–Hilfer fractional **BVPs** (1.12)–(1.13) and (1.12)–(1.14). Afterward, we will investigate their properties, which play a key role in reducing the dimensions of **BVPs** to one. Regarding this, let us define:

$$\Pi = 1 - \int_{\iota}^{\ell}\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\mathfrak{h}(\vartheta)\,d\vartheta > 0, \qquad \Xi = 1 - \int_{\iota}^{\ell}\frac{1}{\rho(\zeta-1)}\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-2}}\mathfrak{h}(\vartheta)\,d\vartheta > 0.$$

Lemma 3.1 *Let $\rho > 0$, $u \in (1,2]$, $\nu \in [0,1]$, $\zeta = u + \nu(2-u), u, \zeta \in (1,2]$, $\chi, \mathfrak{h} \in \mathcal{C}((\iota,\ell),\mathbb{R})$, and $\mathfrak{p}: \mathbb{T} \to \mathbb{R}$. Then, the solution of Katugampola–Hilfer fractional* **BVP** (1.12)–(1.13) *is provided by*

$$\chi(\vartheta) = \int_\iota^\ell \mathbb{H}(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \left(1 - \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\right)\mathfrak{f}(\chi) + \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\frac{1}{\Pi}\int_\iota^\ell \left(1 - \frac{(\sigma^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\right)\mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma, \tag{3.1}$$

where

$$\mathbb{H}(\vartheta,\varepsilon) = \mathbb{H}_1(\vartheta,\varepsilon) + \mathbb{H}_2(\vartheta,\varepsilon), \tag{3.2}$$

such that

$$\mathbb{H}_1(\vartheta,\varepsilon) = \frac{\rho^{1-u}\varepsilon^{\rho-1}}{\Gamma(u)(\ell^\rho - \iota^\rho)^{\zeta-1}} \begin{cases} (\vartheta^\rho - \iota^\rho)^{\zeta-1}(\ell^\rho - \varepsilon^\rho)^{u-1}, & \iota \le \vartheta \le \varepsilon \le \ell, \\ (\vartheta^\rho - \iota^\rho)^{\zeta-1}(\ell^\rho - \varepsilon^\rho)^{u-1} \\ \quad - (\ell^\rho - \iota^\rho)^{\zeta-1}(\vartheta^\rho - \varepsilon^\rho)^{u-1}, & \iota \le \varepsilon \le \vartheta \le \ell, \end{cases} \tag{3.3}$$

and

$$\mathbb{H}_2(\vartheta,\varepsilon) = \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\frac{1}{\Pi}\int_\iota^\ell \mathbb{H}_1(\sigma,\varepsilon)\mathfrak{h}(\sigma)\,d\sigma.$$

Proof We derive a solution of the **BVP** (1.12)–(1.13), by applying ${}^{\rho,\mathrm{R}}\mathbb{I}^u_{\iota^+}$ on both sides of Eq. (1.12) and using Lemma 2.5. Indeed, we have

$$\chi(\vartheta) = d_1\left(\frac{\vartheta^\rho - \iota^\rho}{\rho}\right)^{\zeta-1} + d_2\left(\frac{\vartheta^\rho - \iota^\rho}{\rho}\right)^{\zeta-2} - \left({}^{\rho,\mathrm{R}}\mathbb{I}^u_{\iota^+}\mathfrak{p}\chi\right)(\vartheta).$$

From the condition $\chi(\iota) = \mathfrak{f}(\chi)$, we find that $d_2 = (\frac{\vartheta^\rho - \iota^\rho}{\rho})^{2-\zeta}\mathfrak{f}(\chi)$, since $\zeta - 2 < 0$. So,

$$\chi(\vartheta) = \mathfrak{f}(\chi) + d_1\left(\frac{\vartheta^\rho - \iota^\rho}{\rho}\right)^{\zeta-1} - \left({}^{\rho,\mathrm{R}}\mathbb{I}^u_{\iota^+}\mathfrak{p}\chi\right)(\vartheta). \tag{3.4}$$

Also, using the condition $\chi(\ell) = \int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon$, we obtain

$$d_1 = \left(\frac{\ell^\rho - \iota^\rho}{\rho}\right)^{1-\zeta}\left({}^{\rho,\mathrm{R}}\mathbb{I}^u_{\iota^+}\mathfrak{p}\chi\right)(\ell) + \left(\frac{\ell^\rho - \iota^\rho}{\rho}\right)^{1-\zeta}\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon - \left(\frac{\ell^\rho - \iota^\rho}{\rho}\right)^{1-\zeta}\mathfrak{f}(\chi).$$

Thus, by putting the value of d_1 into Eq. (3.4), we get

$$\begin{aligned}\chi(\vartheta) &= \mathfrak{f}(\chi) + \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon + \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\left({}^{\rho,\mathrm{R}}\mathbb{I}^u_{\iota^+}\mathfrak{p}\chi\right)(\ell) \\ &\quad - \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\mathfrak{f}(\chi) - \left({}^{\rho,\mathrm{R}}\mathbb{I}^u_{\iota^+}\mathfrak{p}\chi\right)(\vartheta) \\ &= \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon\end{aligned}$$

$$+\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\frac{1}{\Gamma(u)}\int_{\iota}^{\ell}\varepsilon^{\rho-1}\left(\frac{\ell^{\rho}-\varepsilon^{\rho}}{\rho}\right)^{u-1}(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon$$
$$-\frac{1}{\Gamma(u)}\int_{\iota}^{\vartheta}\varepsilon^{\rho-1}\left(\frac{\vartheta^{\rho}-\varepsilon^{\rho}}{\rho}\right)^{u-1}(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\left(1-\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi).$$

Therefore,

$$\begin{aligned}\chi(\vartheta)&=\int_{\iota}^{\ell}\mathbb{H}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\left(1-\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\\&\quad+\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\int_{\iota}^{\ell}(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon,\end{aligned}\tag{3.5}$$

where $\mathbb{H}_1(\vartheta,\varepsilon)$ is given in Eq. (3.3). Now, we have

$$\begin{aligned}&\int_{\iota}^{\ell}(\mathfrak{h}\chi)(\vartheta)\,d\vartheta\\&=\int_{\iota}^{\ell}\left(\int_{\iota}^{\ell}\mathbb{H}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\left(1-\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\right.\\&\quad\left.+\,\delta_2\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\int_{\iota}^{\ell}(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon\right)\mathfrak{h}(\vartheta)\,d\vartheta\\&=\int_{\iota}^{\ell}\left(\int_{\iota}^{\ell}\mathbb{H}_1(\vartheta,\varepsilon)\mathfrak{h}(\vartheta)\,d\vartheta\right)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\int_{\iota}^{\ell}\left(1-\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\mathfrak{h}(\vartheta)\,d\vartheta\\&\quad+\int_{\iota}^{\ell}\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\mathfrak{h}(\vartheta)\,d\vartheta\cdot\int_{\iota}^{\ell}(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\iota}^{\ell}(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon&=\frac{1}{\Pi}\int_{\iota}^{\ell}\left(\int_{\iota}^{\ell}\mathbb{H}_1(\sigma,\varepsilon)\mathfrak{h}(\sigma)\,d\sigma\right)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon\\&\quad+\frac{1}{\Pi}\int_{\iota}^{\ell}\left(1-\frac{(\sigma^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma.\end{aligned}$$

Then, Eq. (3.5) becomes

$$\begin{aligned}\chi(\vartheta)&=\int_{\iota}^{\ell}\mathbb{H}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\left(1-\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\\&\quad+\int_{\iota}^{\ell}\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\frac{1}{\Pi}\left(\int_{\iota}^{\ell}\mathbb{H}_1(\sigma,\varepsilon)\mathfrak{h}(\sigma)\,d\sigma\right)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon\\&\quad+\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\frac{1}{\Pi}\int_{\iota}^{\ell}\left(1-\frac{(\sigma^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma\\&=\int_{\iota}^{\ell}\mathbb{H}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\int_{\iota}^{\ell}\mathbb{H}_2(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\left(1-\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\\&\quad+\frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\frac{1}{\Pi}\int_{\iota}^{\ell}\left(1-\frac{(\sigma^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\right)\mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma\end{aligned}$$

$$= \int_\iota^\ell \mathbb{H}(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \left(1 - \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\right)\mathfrak{f}(\chi)$$
$$+ \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}} \frac{1}{\Pi} \int_\iota^\ell \left(1 - \frac{(\sigma^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-1}}\right)\mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma.$$

Hence, the proof is finished. □

Lemma 3.2 *Let $\rho > 0$, $u \in (1,2]$, $\nu \in [0,1]$, $\zeta = u + \nu(2-u), u, \zeta \in (1,2]$, $\chi, \mathfrak{h} \in \mathcal{C}((\iota,\ell),\mathbb{R})$, and $\mathfrak{p} : \mathbb{T} \to \mathbb{R}$. Then, the solution of Katugampola–Hilfer fractional* **BVP** (1.12)–(1.14) *is equivalent to*

$$\chi(\vartheta) = \int_\iota^\ell \mathbb{G}(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \mathfrak{f}(\chi) \tag{3.6}$$
$$+ \frac{1}{\rho(\zeta-1)} \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-2}} \frac{1}{\Xi} \int_\iota^\ell \mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma,$$

where

$$\mathbb{G}(\vartheta,\varepsilon) = \mathbb{G}_1(\vartheta,\varepsilon) + \mathbb{G}_2(\vartheta,\varepsilon), \tag{3.7}$$

such that

$$\mathbb{G}_1(\vartheta,\varepsilon)$$
$$= \frac{\rho^{1-u}\varepsilon^{\rho-1}(\ell^\rho - \varepsilon^\rho)^{u-2}}{\Gamma(u)(\zeta-1)} \begin{cases} (u-1)(\vartheta^\rho - \iota^\rho)^{\zeta-1}(\ell^\rho - \iota^\rho)^{2-\zeta}, & \iota \le \vartheta \le \varepsilon \le \ell, \\ (u-1)(\vartheta^\rho - \iota^\rho)^{\zeta-1}(\ell^\rho - \iota^\rho)^{2-\zeta} & \\ \quad - (\zeta-1)\frac{(\vartheta^\rho - \varepsilon^\rho)^{u-1}}{(\ell^\rho - \varepsilon^\rho)^{u-2}}, & \iota \le \varepsilon \le \vartheta \le \ell, \end{cases} \tag{3.8}$$

and

$$\mathbb{G}_2(\vartheta,\varepsilon) = \frac{1}{\rho(\zeta-1)} \frac{(\vartheta^\rho - \iota^\rho)^{\zeta-1}}{(\ell^\rho - \iota^\rho)^{\zeta-2}} \frac{1}{\Xi} \int_\iota^\ell \mathbb{G}_1(\sigma,\varepsilon)\mathfrak{h}(\sigma)\,d\sigma.$$

Proof We derive an equivalent form of solution of the **BVP** (1.12)–(1.14), by applying the same method as in Lemma 3.1 up to Eq. (3.4), that is,

$$\chi(\vartheta) = \mathfrak{f}(\chi) + d_1\left(\frac{\vartheta^\rho - \iota^\rho}{\rho}\right)^{\zeta-1} - \left({}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{u}\mathfrak{p}\chi\right)(\vartheta), \tag{3.9}$$

which yields

$$\vartheta^{1-\rho}\frac{d}{d\vartheta}\chi(\vartheta)\Big|_{\vartheta=\ell} = d_1(\zeta-1)\left(\frac{\ell^\rho - \iota^\rho}{\rho}\right)^{\zeta-2} - \left({}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{u-1}\mathfrak{p}\chi\right)(\ell). \tag{3.10}$$

From the condition $\vartheta^{1-\rho}\frac{d}{d\vartheta}\chi(\vartheta)|_{\vartheta=\ell} = \int_\iota^\ell(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon$, we get

$$d_1 = \frac{1}{(\zeta-1)}\left(\frac{\ell^\rho - \iota^\rho}{\rho}\right)^{2-\zeta}\left({}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{u-1}\mathfrak{p}\chi\right)(\ell) + \frac{1}{(\zeta-1)}\left(\frac{\ell^\rho - \iota^\rho}{\rho}\right)^{2-\zeta}\int_\iota^\ell(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon.$$

So, by substituting the value of d_1 into Eq. (3.4), we find

$$\begin{aligned}\chi(\vartheta) &= \mathfrak{f}(\chi) + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon \\ &\quad + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\big({}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{u-1}\mathfrak{p}\chi\big)(\ell) - \big({}^{\rho,\mathrm{R}}\mathbb{I}_{\iota^+}^{u}\mathfrak{p}\chi\big)(\vartheta) \\ &= \mathfrak{f}(\chi) + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon \\ &\quad + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\frac{1}{\Gamma(u-1)}\int_\iota^\ell \varepsilon^{\rho-1}\left(\frac{\ell^\rho-\varepsilon^\rho}{\rho}\right)^{u-2}(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon \\ &\quad - \frac{1}{\Gamma(u)}\int_\iota^\vartheta \varepsilon^{\rho-1}\left(\frac{\vartheta^\rho-\varepsilon^\rho}{\rho}\right)^{u-1}(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon.\end{aligned}$$

Therefore,

$$\chi(\vartheta) = \int_\iota^\ell \mathbb{G}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \mathfrak{f}(\chi) + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, \tag{3.11}$$

where $\mathbb{G}_1(\vartheta,\varepsilon)$ is given in Eq. (3.8). Next,

$$\begin{aligned}&\int_\iota^\ell (\mathfrak{h}\chi)(\vartheta)\,d\vartheta \\ &\quad= \int_\iota^\ell\left(\int_\iota^\ell \mathbb{G}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \mathfrak{f}(\chi) + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon\right)\mathfrak{h}(\vartheta)\,d\vartheta \\ &\quad= \int_\iota^\ell\left(\int_\iota^\ell \mathbb{G}_1(\vartheta,\varepsilon)\mathfrak{h}(\vartheta)\,d\vartheta\right)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \int_\iota^\ell \mathfrak{f}(\chi)\mathfrak{h}(\vartheta)\,d\vartheta \\ &\qquad + \int_\iota^\ell \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\mathfrak{h}(\vartheta)\,d\vartheta\cdot\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon.\end{aligned}$$

Then,

$$\int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon = \frac{1}{\Xi}\int_\iota^\ell\left(\int_\iota^\ell \mathbb{G}_1(\sigma,\varepsilon)\mathfrak{h}(\sigma)\,d\sigma\right)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \frac{1}{\Xi}\int_\iota^\ell \mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma.$$

Hence, Eq. (3.11) becomes

$$\begin{aligned}\chi(\vartheta) &= \int_\iota^\ell \mathbb{G}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \mathfrak{f}(\chi) \\ &\quad + \int_\iota^\ell \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\frac{1}{\Xi}\left(\int_\iota^\ell \mathbb{G}_1(\sigma,\varepsilon)\mathfrak{h}(\sigma)\,d\sigma\right)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon \\ &\quad + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\frac{1}{\Xi}\int_\iota^\ell \mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma \\ &= \int_\iota^\ell \mathbb{G}_1(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \int_\iota^\ell \mathbb{G}_2(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon \\ &\quad + \mathfrak{f}(\chi) + \frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\frac{1}{\Xi}\int_\iota^\ell \mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma\end{aligned}$$

$$= \int_{\iota}^{\ell} \mathbb{G}(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \mathfrak{f}(\chi)$$
$$+ \frac{1}{\rho(\zeta-1)} \frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-2}} \frac{1}{\Xi} \int_{\iota}^{\ell} \mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma.$$

Therefore, the proof is completed. □

Next, we present properties of the Green functions $\mathbb{H}(\vartheta,\varepsilon)$ and $\mathbb{G}(\vartheta,\varepsilon)$, which are given in (3.2) and (3.7), respectively.

Lemma 3.3 *Let* $\rho > 0$, $u \in (1,2]$, $\nu \in [0,1]$, $\zeta = u + \nu(2-u), u, \zeta \in (1,2]$. *Then, the Green functions* $\mathbb{H}(\vartheta,\varepsilon)$ *and* $\mathbb{G}(\vartheta,\varepsilon)$ *satisfy the following properties*:

(*i*) $\mathbb{H}(\vartheta,\varepsilon)$ *and* $\mathbb{G}(\vartheta,\varepsilon)$ *are continuous functions for all* $(\vartheta,\varepsilon) \in \mathbb{T}^2$;

(*ii*) $|\mathbb{H}(\vartheta,\varepsilon)| < (\frac{\zeta-1}{\zeta+u-2})^{\zeta-1}(\frac{(\ell^{\rho}-\iota^{\rho})(u-1)}{\zeta+u-2})^{u-1}\frac{\rho^{1-u}\varepsilon^{\rho-1}}{\Gamma(u)}[1+\frac{1}{\Pi}\int_{\iota}^{\ell}|\mathfrak{h}(\sigma)|\,d\sigma]$;

(*iii*) $|\mathbb{G}(\vartheta,\varepsilon)| < \frac{(\ell^{\rho}-\iota^{\rho})(\ell^{\rho}-\varepsilon^{\rho})^{u-2}}{(\zeta-1)\Gamma(u)\rho^{u-1}\varepsilon^{1-\rho}}\max\{\zeta-u,u-1\}[1+\frac{(\ell^{\rho}-\iota^{\rho})}{\rho(\zeta-1)\Xi}\int_{\iota}^{\ell}|\mathfrak{h}(\sigma)|\,d\sigma]$.

Proof

(*i*) This claim is obvious.

(*ii*) Since $\mathbb{H}(\vartheta,\varepsilon) = \mathbb{H}_1(\vartheta,\varepsilon) + \mathbb{H}_2(\vartheta,\varepsilon)$, and similar to Lemma 4.4(ii) in [28], we conclude that

$$\left|\mathbb{H}_1(\vartheta,\varepsilon)\right| \le \left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^{\rho}-\iota^{\rho})(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\rho^{1-u}\varepsilon^{\rho-1}}{\Gamma(u)},$$

which implies that

$$\left|\mathbb{H}_2(\vartheta,\varepsilon)\right| \le \frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-1}}\frac{1}{\Pi}\int_{\iota}^{\ell}\left|\mathbb{H}_1(\sigma,\varepsilon)\right|\left|\mathfrak{h}(\sigma)\right|d\sigma$$
$$< \frac{1}{\Pi}\left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^{\rho}-\iota^{\rho})(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\rho^{1-u}\varepsilon^{\rho-1}}{\Gamma(u)}\int_{\iota}^{\ell}\left|\mathfrak{h}(\sigma)\right|d\sigma.$$

Hence,

$$\left|\mathbb{H}(\vartheta,\varepsilon)\right|$$
$$\le \left|\mathbb{H}_1(\vartheta,\varepsilon)\right| + \left|\mathbb{H}_2(\vartheta,\varepsilon)\right|$$
$$< \left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^{\rho}-\iota^{\rho})(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\rho^{1-u}\varepsilon^{\rho-1}}{\Gamma(u)}\left[1+\frac{1}{\Pi}\int_{\iota}^{\ell}\left|\mathfrak{h}(\sigma)\right|d\sigma\right].$$

(*iii*) We have $\mathbb{G}(\vartheta,\varepsilon) = \mathbb{G}_1(\vartheta,\varepsilon) + \mathbb{G}_2(\vartheta,\varepsilon)$, and similar to Lemma 4.4(iii) in [28], we deduce that

$$\left|\mathbb{G}_1(\vartheta,\varepsilon)\right| \le \frac{(\ell^{\rho}-\iota^{\rho})(\ell^{\rho}-\varepsilon^{\rho})^{u-2}}{(\zeta-1)\Gamma(u)\rho^{u-1}\varepsilon^{1-\rho}}\max\{\zeta-u,u-1\},$$

which yields that

$$\left|\mathbb{G}_2(\vartheta,\varepsilon)\right| \le \frac{(\vartheta^{\rho}-\iota^{\rho})^{\zeta-1}}{(\ell^{\rho}-\iota^{\rho})^{\zeta-2}}\frac{1}{\rho(\zeta-1)\Xi}\int_{\iota}^{\ell}\left|\mathbb{G}_1(\sigma,\varepsilon)\right|\left|\mathfrak{h}(\sigma)\right|d\sigma$$
$$< \frac{(\ell^{\rho}-\iota^{\rho})(\ell^{\rho}-\varepsilon^{\rho})^{u-2}}{(\zeta-1)\Gamma(u)\rho^{u-1}\varepsilon^{1-\rho}}\frac{\max\{\zeta-u,u-1\}}{(\ell^{\rho}-\iota^{\rho})^{-1}}\frac{1}{\rho(\zeta-1)\Xi}\int_{\iota}^{\ell}\left|\mathfrak{h}(\sigma)\right|d\sigma.$$

Thus,

$$\begin{aligned}
&\left|\mathbb{G}(\vartheta,\varepsilon)\right| \\
&\quad\le \left|\mathbb{G}_1(\vartheta,\varepsilon)\right| + \left|\mathbb{G}_2(\vartheta,\varepsilon)\right| \\
&\quad< \frac{(\ell^\rho-\iota^\rho)(\ell^\rho-\varepsilon^\rho)^{u-2}}{(\zeta-1)\Gamma(u)\rho^{u-1}\varepsilon^{1-\rho}}\max\{\zeta-u,u-1\}\left[1+\frac{(\ell^\rho-\iota^\rho)}{\rho(\zeta-1)\Xi}\int_\iota^\ell\left|\mathfrak{h}(\sigma)\right|d\sigma\right].
\end{aligned}$$

Hence, the desired results are proved. □

4 LTI for the problem (1.12)–(1.13)

In this section, we focus our attention on proving a new **LTI** for the Katugampola–Hilfer fractional problem (1.12)–(1.13). For end this, we let $\Lambda := \mathcal{C}(\mathbb{T},\mathbb{R})$ denote the Banach space of continuous real-valued functions equipped with the maximum norm $\|\chi\| = \max_{\vartheta\in\mathbb{T}}|\chi(\vartheta)|$.

Theorem 4.1 *Let the Katugampola–Hilfer fractional* **BVP** (1.12)–(1.13) *possess a nontrivial solution* $\chi \in \mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\begin{aligned}
&\frac{1}{1+\frac{1}{\Pi}\int_\iota^\ell|\mathfrak{h}(\sigma)|\,d\sigma} \\
&\quad< 2\mu + \left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^\rho-\iota^\rho)(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\max\{\iota^{\rho-1},\ell^{\rho-1}\}}{\rho^{u-1}\Gamma(u)}\int_\iota^\ell\left|\mathfrak{p}(\varepsilon)\right|d\varepsilon.
\end{aligned} \tag{4.1}$$

Proof In view of Lemma 3.1, we have

$$\begin{aligned}
\chi(\vartheta) ={}& \int_\iota^\ell \mathbb{H}(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon + \left(1-\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\right)\mathfrak{f}(\chi) \\
&+ \frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\frac{1}{\Pi}\int_\iota^\ell\left(1-\frac{(\sigma^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\right)\mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma.
\end{aligned}$$

By applying the maximum norm and using Lemma 3.3(2), as well as taking into account that for $\mu > 0$ one has $|\mathfrak{f}(\chi)| \le \mu\|\chi\|$, $\forall\chi\in\mathbb{R}$, we find

$$\begin{aligned}
\|\chi\| \le{}& \int_\iota^\ell\left|\mathbb{H}(\vartheta,\varepsilon)\right|\left|(\mathfrak{p}\chi)(\varepsilon)\right|d\varepsilon + \left|\left(1-\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\right)\right|\left|\mathfrak{f}(\chi)\right| \\
&+ \frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\frac{1}{\Pi}\int_\iota^\ell\left|\left(1-\frac{(\sigma^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\right)\right|\left|\mathfrak{f}(\chi)\right|\left|\mathfrak{h}(\sigma)\right|d\sigma \\
\le{}& \|\chi\|\left(\int_\iota^\ell\left|\mathbb{H}(\vartheta,\varepsilon)\right|\left|\mathfrak{p}(\varepsilon)\right|d\varepsilon + \mu\left|\left(1-\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\right)\right|\right. \\
&\left.+ \frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\frac{\mu}{\Pi}\int_\iota^\ell\left|\left(1-\frac{(\sigma^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-1}}\right)\right|\left|\mathfrak{h}(\sigma)\right|d\sigma\right) \\
<{}& \|\chi\|\left(\left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^\rho-\iota^\rho)(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\max\{\iota^{\rho-1},\ell^{\rho-1}\}}{\rho^{u-1}\Gamma(u)}\right. \\
&\left.\times\left[1+\frac{1}{\Pi}\int_\iota^\ell\left|\mathfrak{h}(\sigma)\right|d\sigma\right]\int_\iota^\ell\left|\mathfrak{p}(\varepsilon)\right|d\varepsilon + 2\mu + \frac{2\mu}{\Pi}\int_\iota^\ell\left|\mathfrak{h}(\sigma)\right|d\sigma\right).
\end{aligned}$$

Since $\|\chi\| > 0$, we obtain

$$1 < \left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^{\rho}-\iota^{\rho})(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\max\{\iota^{\rho-1},\ell^{\rho-1}\}}{\rho^{u-1}\Gamma(u)} \times\left[1+\frac{1}{\Pi}\int_{\iota}^{\ell}\left|\mathfrak{h}(\sigma)\right|d\sigma\right]\int_{\iota}^{\ell}\left|\mathfrak{p}(\varepsilon)\right|d\varepsilon+2\mu+\frac{2\mu}{\Pi}\int_{\iota}^{\ell}\left|\mathfrak{h}(\sigma)\right|d\sigma,$$

which yields that

$$\frac{1}{1+\frac{1}{\Pi}\int_{\iota}^{\ell}|\mathfrak{h}(\sigma)|\,d\sigma} < 2\mu+\left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^{\rho}-\iota^{\rho})(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\max\{\iota^{\rho-1},\ell^{\rho-1}\}}{\rho^{u-1}\Gamma(u)}\int_{\iota}^{\ell}\left|\mathfrak{p}(\varepsilon)\right|d\varepsilon.$$

Thus, the desired result is established. □

In the next corollary, we present a condition of the nonexistence of nontrivial solutions for the problem (1.12).

Corollary 4.2 *If*

$$\frac{1}{1+\frac{1}{\Pi}\int_{\iota}^{\ell}|\mathfrak{h}(\sigma)|\,d\sigma} \geq 2\mu+\left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell^{\rho}-\iota^{\rho})(u-1)}{\zeta+u-2}\right)^{u-1}\frac{\max\{\iota^{\rho-1},\ell^{\rho-1}\}}{\rho^{u-1}\Gamma(u)}\int_{\iota}^{\ell}\left|\mathfrak{p}(\varepsilon)\right|d\varepsilon,$$

then the Katugampola–Hilfer fractional **BVP** (1.12)–(1.13) *has no nontrivial solutions.*

Proof Arguing by contradiction and using Theorem 4.1, we can prove the desired result. □

The Hadamard–Hilfer version of the **BVP** (1.12)–(1.13) is given in the following corollary.

Corollary 4.3 *Suppose the following Hadamard–Hilfer fractional* **BVP**:

$$\begin{cases} {}^{\mathrm{H,H}}\mathbb{D}_{\iota^+}^{u,\nu}\chi(\vartheta)+\mathfrak{p}(\vartheta)\chi(\vartheta)=0, & \vartheta\in\mathbb{T}=(\iota,\ell), u\in(1,2], \nu\in[0,1], \\ \chi(\iota)=\mathfrak{f}(\chi), & \chi(\ell)=\int_{\iota}^{\ell}(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, \end{cases} \tag{4.2}$$

possesses a nontrivial solution $\chi\in\mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\frac{1}{1+\frac{1}{\Pi_0}\int_{\iota}^{\ell}|\mathfrak{h}(\sigma)|\,d\sigma} < 2\mu+\left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ln(\ell)-\ln(\iota))(u-1)}{\zeta+u-2}\right)^{u-1}\frac{1}{\iota\Gamma(u)}\int_{\iota}^{\ell}\left|\mathfrak{p}(\varepsilon)\right|d\varepsilon, \tag{4.3}$$

where $\Pi_0 = 1 - \int_\iota^\ell \frac{(\ln(\vartheta)-\ln(\iota))^{\zeta-1}}{(\ln(\ell)-\ln(\iota))^{\zeta-1}} \mathfrak{h}(\vartheta)\, d\vartheta > 0$. *Moreover, if* $\mathfrak{f} = \mathfrak{h} = 0$, *in the problem* (4.2), *then the inequality* (4.3) *becomes*

$$\iota\Gamma(u)\left(\frac{\zeta-1}{\zeta+u-2}\right)^{1-\zeta}\left(\frac{(\ln(\ell)-\ln(\iota))(u-1)}{\zeta+u-2}\right)^{1-u} < \int_\iota^\ell \left|\mathfrak{p}(\varepsilon)\right| d\varepsilon. \tag{4.4}$$

Proof By taking the limit when $\rho \to 0^+$ in (4.1), the inequality (4.3) follows. Also, by letting $\mathfrak{f} = \mathfrak{h} = 0$, the inequality (4.4) is established. □

Remark 4.4 We note the following:

1. The inequality (4.3) coincides with inequality (1.11), if $\nu = 1$, $\mathfrak{f} = 0$, and $\lambda = 1$.
2. The inequality (4.4) agrees with the results in [39], if $\nu = 0$.

The Hilfer version of the **BVP** (1.12)–(1.13) is given in the following corollary.

Corollary 4.5 *Suppose the following Hilfer fractional* **BVP**:

$$\begin{cases} {}^{\mathrm{H}}\mathbb{D}_{\iota^+}^{u,\nu}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, & \vartheta \in \mathbb{T} = (\iota,\ell), u \in (1,2], \nu \in [0,1], \\ \chi(\iota) = \mathfrak{f}(\chi), & \chi(\ell) = \int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\, d\varepsilon, \end{cases} \tag{4.5}$$

possesses a nontrivial solution $\chi \in \mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\begin{aligned} &\frac{1}{1 + \frac{1}{\Pi_1}\int_\iota^\ell |\mathfrak{h}(\sigma)|\, d\sigma} \\ &\quad < 2\mu + \left(\frac{\zeta-1}{\zeta+u-2}\right)^{\zeta-1}\left(\frac{(\ell-\iota)(u-1)}{\zeta+u-2}\right)^{u-1}\frac{1}{\Gamma(u)}\int_\iota^\ell \left|\mathfrak{p}(\varepsilon)\right| d\varepsilon, \end{aligned} \tag{4.6}$$

where $\Pi_1 = 1 - \int_\iota^\ell \frac{(\vartheta-\iota)^{\zeta-1}}{(\ell-\iota)^{\zeta-1}} \mathfrak{h}(\vartheta)\, d\vartheta > 0$.

Proof By putting $\rho = 1$ in the inequality (4.1), the proof is finished. □

Remark 4.6 Clearly, the inequality (4.6) coincides with inequality (1.7), if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$.

The Katugampola–Riemann version of the **BVP** (1.12)–(1.13) is given in the following corollary.

Corollary 4.7 *Suppose the following Katugampola–Riemann fractional* **BVP**:

$$\begin{cases} {}^{\rho,\mathrm{R}}\mathbb{D}_{\iota^+}^{u}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, & \vartheta \in \mathbb{T} = (\iota,\ell), u \in (1,2], \rho > 0, \\ \chi(\iota) = \mathfrak{f}(\chi), & \chi(\ell) = \int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\, d\varepsilon, \end{cases} \tag{4.7}$$

possesses a nontrivial solution $\chi \in \mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\begin{aligned} &\frac{1}{1+\frac{1}{\Pi_2}\int_\iota^\ell |\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad < 2\mu + \left(\frac{(\ell^\rho - \iota^\rho)}{4}\right)^{u-1} \frac{\rho^{1-u}\max\{\iota^{\rho-1},\ell^{\rho-1}\}}{\Gamma(u)} \int_\iota^\ell \left|\mathfrak{p}(\varepsilon)\right|d\varepsilon, \end{aligned} \tag{4.8}$$

such that $\Pi_2 = 1 - \int_\iota^\ell \frac{(\vartheta^\rho-\iota^\rho)^{u-1}}{(\ell^\rho-\iota^\rho)^{u-1}}\mathfrak{h}(\vartheta)\,d\vartheta > 0$.

Proof By putting $\nu = 0$ in the inequality (4.1), the proof is completed. □

Remark 4.8 We note the following points:

1. The inequality (4.8) coincides with the inequality in Theorem 5 of [16] if $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$.
2. The inequality (4.8) reduces to inequality (1.3) if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$.
3. The inequality (4.8) coincides with inequality (1.2) if $\rho = 1$, $\mathfrak{f} = 0$, $\mathfrak{h} = 0$, and $u = 2$.
4. The inequality (4.8) agrees with the inequality in Theorem 2.2 of [25] if $\rho = 1$, $\mathfrak{f} = 0$, $\mathfrak{h} = \lambda$, and $\nu = 0$.

The Katugampola–Caputo version of the **BVP** (1.12)–(1.13) is given in the following corollary.

Corollary 4.9 *Suppose the following Katugampola–Caputo fractional* **BVP**:

$$\begin{cases} {}^{\rho,C}\mathbb{D}^u_{\iota^+}\chi(\vartheta) + \mathfrak{p}(\vartheta)\chi(\vartheta) = 0, & \vartheta \in \mathbb{T} = (\iota,\ell), u \in (1,2], \rho > 0, \\ \chi(\iota) = \mathfrak{f}(\chi), & \chi(\ell) = \int_\iota^\ell (\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, \end{cases}$$

possesses a nontrivial solution $\chi \in \Lambda$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\begin{aligned} &\frac{1}{1+\frac{1}{\Pi_3}\int_\iota^\ell |\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad < 2\mu + \left(\frac{1}{u}\right)^{u} \left((\ell^\rho - \iota^\rho)(u-1)\right)^{u-1} \frac{\rho^{1-u}\max\{\iota^{\rho-1},\ell^{\rho-1}\}}{\Gamma(u)} \int_\iota^\ell \left|\mathfrak{p}(\varepsilon)\right|d\varepsilon, \end{aligned} \tag{4.9}$$

where $\Pi_3 = 1 - \int_\iota^\ell \frac{(\vartheta^\rho-\iota^\rho)}{(\ell^\rho-\iota^\rho)}\mathfrak{h}(\vartheta)\,d\vartheta > 0$.

Proof Letting $\nu = 1$ in the inequality (4.1), the desired result is proved. □

Remark 4.10 It is important to declare the following:

1. The inequality (4.9) coincides with inequality (1.5) if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$.
2. The inequality (4.9) coincides with inequality (1.2) if $\rho = 1$, $\mathfrak{f} = 0$, $\mathfrak{h} = 0$, and $u = 2$.

5 LTI for the problem (1.12)–(1.14)

Theorem 5.1 *Let the Katugampola–Hilfer fractional* **BVP** (1.12)–(1.14) *possess a nontrivial solution* $\chi \in \mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following* *inequality*:

$$\begin{aligned} &\frac{1}{1+\frac{(\ell^\rho-\iota^\rho)}{\rho(\zeta-1)\Xi}\int_\iota^\ell|\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad<\mu+\frac{(\ell^\rho-\iota^\rho)}{(\zeta-1)}\frac{\rho^{1-u}}{\Gamma(u)}\max\{\zeta-u,u-1\}\int_\iota^\ell\varepsilon^{\rho-1}\bigl(\ell^\rho-\varepsilon^\rho\bigr)^{u-2}\bigl|\mathfrak{p}(\varepsilon)\bigr|\,d\varepsilon. \end{aligned} \tag{5.1}$$

Proof We prove this result in the same manner as Theorem 4.1. Indeed, according to Lemma 3.2, we get

$$\chi(\vartheta)=\int_\iota^\ell\mathbb{G}(\vartheta,\varepsilon)(\mathfrak{p}\chi)(\varepsilon)\,d\varepsilon+\mathfrak{f}(\chi)+\frac{1}{\rho(\zeta-1)}\frac{(\vartheta^\rho-\iota^\rho)^{\zeta-1}}{(\ell^\rho-\iota^\rho)^{\zeta-2}}\frac{1}{\Xi}\int_\iota^\ell\mathfrak{f}(\chi)\mathfrak{h}(\sigma)\,d\sigma.$$

Now, by taking the maximum norm and applying Lemma 3.3(3), we obtain

$$\begin{aligned} \|\chi\|<\|\chi\|\biggl(&\frac{(\ell^\rho-\iota^\rho)}{(\zeta-1)}\frac{\rho^{1-u}}{\Gamma(u)}\max\{\zeta-u,u-1\} \\ &\times\biggl[1+\frac{(\ell^\rho-\iota^\rho)}{\rho(\zeta-1)\Xi}\int_\iota^\ell\bigl|\mathfrak{h}(\sigma)\bigr|\,d\sigma\biggr]\int_\iota^\ell\varepsilon^{\rho-1}\bigl(\ell^\rho-\varepsilon^\rho\bigr)^{u-2}\bigl|\mathfrak{p}(\varepsilon)\bigr|\,d\varepsilon \\ &+\mu\biggl[1+\frac{(\ell^\rho-\iota^\rho)}{\rho(\zeta-1)\Xi}\int_\iota^\ell\bigl|\mathfrak{h}(\sigma)\bigr|\,d\sigma\biggr]\biggr). \end{aligned}$$

Since $\|\chi\|>0$, the above implies that

$$\begin{aligned} &\frac{1}{1+\frac{(\ell^\rho-\iota^\rho)}{\rho(\zeta-1)\Xi}\int_\iota^\ell|\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad<\mu+\frac{(\ell^\rho-\iota^\rho)}{(\zeta-1)}\frac{\rho^{1-u}}{\Gamma(u)}\max\{\zeta-u,u-1\}\int_\iota^\ell\varepsilon^{\rho-1}\bigl(\ell^\rho-\varepsilon^\rho\bigr)^{u-2}\bigl|\mathfrak{p}(\varepsilon)\bigr|\,d\varepsilon. \end{aligned}$$

Hence, the required result is proved. □

Corollary 5.2 *If*

$$\begin{aligned} &\frac{1}{1+\frac{(\ell^\rho-\iota^\rho)}{\rho(\zeta-1)\Xi}\int_\iota^\ell|\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad\geq\mu+\frac{(\ell^\rho-\iota^\rho)}{(\zeta-1)}\frac{\rho^{1-u}}{\Gamma(u)}\max\{\zeta-u,u-1\}\int_\iota^\ell\varepsilon^{\rho-1}\bigl(\ell^\rho-\varepsilon^\rho\bigr)^{u-2}\bigl|\mathfrak{p}(\varepsilon)\bigr|\,d\varepsilon, \end{aligned}$$

then the Katugampola–Hilfer fractional **BVP** (1.12)–(1.14) *has no nontrivial solutions*.

Proof Arguing by contradiction and using Theorem 5.1, we can prove the desired result. □

The Hadamard–Hilfer version of the **BVP** (1.12)–(1.14) is given in the following corollary.

Corollary 5.3 *Suppose the following Hadamard–Hilfer fractional* **BVP**:

$$\begin{cases} {}^{\mathrm{H,H}}\mathbb{D}_{\iota^+}^{u,\nu}\chi(\vartheta)+\mathfrak{p}(\vartheta)\chi(\vartheta)=0, & \vartheta\in\mathbb{T}=(\iota,\ell), u\in(1,2], \nu\in[0,1], \\ \chi(\iota)=\mathfrak{f}(\chi), & \vartheta\frac{d}{d\vartheta}\chi(\vartheta)|_{\vartheta=\ell}=\int_\iota^\ell(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, \end{cases} \tag{5.2}$$

possesses a nontrivial solution $\chi \in \mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\begin{aligned}&\frac{1}{1+\frac{1}{\Xi_0}\int_\iota^\ell |\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad < \mu + \frac{(\ln(\ell)-\ln(\iota))}{(\zeta-1)}\frac{\max\{\zeta-u,u-1\}}{\Gamma(u)}\int_\iota^\ell \frac{1}{\varepsilon}\bigl(\ln(\ell)-\ln(\varepsilon)\bigr)^{u-2}\bigl|\mathfrak{p}(\varepsilon)\bigr|\,d\varepsilon,\end{aligned} \tag{5.3}$$

where $\Xi_0 = \frac{1}{(\ln(\ell)-\ln(\iota))} - \int_\iota^\ell \frac{(\ln(\vartheta)-\ln(\iota))^{\zeta-1}}{(\ln(\ell)-\ln(\iota))^{\zeta-1}}\mathfrak{h}(\vartheta)\,d\vartheta > 0$. *Moreover, if* $\mathfrak{f}=\mathfrak{h}=0$, *in the problem* (5.2), *then the inequality* (5.3) *becomes*

$$\frac{(\zeta-1)\Gamma(u)}{(\ln(\ell)-\ln(\iota))\max\{\zeta-u,u-1\}} < \int_\iota^\ell \frac{1}{\varepsilon}\bigl(\ln(\ell)-\ln(\varepsilon)\bigr)^{u-2}\bigl|\mathfrak{p}(\varepsilon)\bigr|\,d\varepsilon. \tag{5.4}$$

Proof By taking the limit as $\rho \to 0^+$ in the inequality (5.1), the inequality (5.3) is established. Also, by taking $\mathfrak{f}=\mathfrak{h}=0$, the inequality (5.4) follows. □

Remark 5.4 We note that the inequality (5.4) agrees with Corollary 3.4 of [40] if $\nu = 0$.

The Hilfer version of the **BVP** (1.12)–(1.14) is given in the following corollary.

Corollary 5.5 *Suppose the following Hilfer fractional* **BVP**:

$$\begin{cases} {}^{\mathrm{H}}\mathbb{D}_{\iota^+}^{u,\nu}\chi(\vartheta)+\mathfrak{p}(\vartheta)\chi(\vartheta)=0, & \vartheta\in\mathbb{T}=(\iota,\ell), u\in(1,2], \nu\in[0,1],\\ \chi(\iota)=\mathfrak{f}(\chi), & \frac{d}{d\vartheta}\chi(\vartheta)|_{\vartheta=\ell}=\int_\iota^\ell(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon,\end{cases} \tag{5.5}$$

possesses a nontrivial solution $\chi \in \mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\begin{aligned}&\frac{1}{1+\frac{(\ell-\iota)}{(\zeta-1)\Xi_1}\int_\iota^\ell |\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad < \mu + \frac{(\ell-\iota)}{(\zeta-1)\Gamma(u)}\max\{\zeta-u,u-1\}\int_\iota^\ell (\ell-\varepsilon)^{u-2}\bigl|\mathfrak{p}(\varepsilon)\bigr|\,d\varepsilon,\end{aligned} \tag{5.6}$$

where $\Xi_1 = 1-\int_\iota^\ell \frac{1}{(\zeta-1)}\frac{(\vartheta-\iota)^{\zeta-1}}{(\ell-\iota)^{\zeta-2}}\mathfrak{h}(\vartheta)\,d\vartheta > 0$.

Proof The result follows by putting $\rho = 1$ in the inequality (5.1). □

Remark 5.6 Notice that the inequality (5.6) coincides with inequality (1.9) if $\rho=1$, $\mathfrak{f}=0$, and $\mathfrak{h}=0$.

The Katugampola–Riemann version of the **BVP** (1.12)–(1.14) is given in the following corollary.

Corollary 5.7 *Suppose the Katugampola–Riemann fractional* **BVP**:

$$\begin{cases} {}^{\rho,\mathrm{R}}\mathbb{D}_{\iota^+}^{u}\chi(\vartheta)+\mathfrak{p}(\vartheta)\chi(\vartheta)=0, & \vartheta\in\mathbb{T}=(\iota,\ell), u\in(1,2], \rho>0,\\ \chi(\iota)=\mathfrak{f}(\chi), & \vartheta^{1-\rho}\frac{d}{d\vartheta}\chi(\vartheta)|_{\vartheta=\ell}=\int_\iota^\ell(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon,\end{cases} \tag{5.7}$$

possesses a nontrivial solution $\chi \in \mathcal{C}((\iota,\ell),\mathbb{R})$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\frac{1}{1+\frac{(\ell^\rho-\iota^\rho)}{\rho(u-1)\Xi_2}\int_\iota^\ell|\mathfrak{h}(\sigma)|\,d\sigma} < \mu + \frac{\rho^{1-u}(\ell^\rho-\iota^\rho)}{\Gamma(u)}\int_\iota^\ell \varepsilon^{\rho-1}\big(\ell^\rho-\varepsilon^\rho\big)^{u-2}\big|\mathfrak{p}(\varepsilon)\big|\,d\varepsilon, \tag{5.8}$$

where $\Xi_2 = 1-\int_\iota^\ell \frac{1}{\rho(u-1)}\frac{(\vartheta^\rho-\iota^\rho)^{u-1}}{(\ell^\rho-\iota^\rho)^{u-2}}\mathfrak{h}(\vartheta)\,d\vartheta > 0$.

Proof By putting $\nu = 0$ in the inequality (5.1), the proof is completed. □

Remark 5.8 We note the following points:

1. The inequality (5.8) agrees with Theorem 3.3 of [41] if $\mathfrak{f} = 0$ and $\mathfrak{h} = 0$.
2. The inequality (5.8) covers Corollary 3.4 of [41] if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$.

The Katugampola–Caputo version of the **BVP** (1.12)–(1.14) is given in the following corollary.

Corollary 5.9 *Suppose the Katugampola–Caputo fractional* **BVP**:

$$\begin{cases} {}^{\rho,\mathrm{C}}\mathbb{D}^{u}_{\iota^+}\chi(\vartheta)+\mathfrak{p}(\vartheta)\chi(\vartheta)=0, & \vartheta\in\mathbb{T}=(\iota,\ell), u\in(1,2], \rho>0,\\ \chi(\iota)=\mathfrak{f}(\chi), & \vartheta^{1-\rho}\frac{d}{d\vartheta}\chi(\vartheta)|_{\vartheta=\ell}=\int_\iota^\ell(\mathfrak{h}\chi)(\varepsilon)\,d\varepsilon, \end{cases}$$

possesses a nontrivial solution $\chi \in \Lambda$. *Then a real continuous function* $\mathfrak{p}$ *satisfies the following inequality*:

$$\begin{aligned} &\frac{1}{1+\frac{(\ell^\rho-\iota^\rho)}{\rho\Xi_3}\int_\iota^\ell|\mathfrak{h}(\sigma)|\,d\sigma} \\ &\quad< \mu + \frac{(\ell^\rho-\iota^\rho)}{\rho^{u-1}\Gamma(u)}\max\{2-u,u-1\}\int_\iota^\ell \frac{(\ell^\rho-\varepsilon^\rho)^{u-2}}{\varepsilon^{1-\rho}}\big|\mathfrak{p}(\varepsilon)\big|\,d\varepsilon, \end{aligned} \tag{5.9}$$

where $\Xi_3 = 1-\int_\iota^\ell(\vartheta^\rho-\iota^\rho)\mathfrak{h}(\vartheta)\,d\vartheta > 0$.

Proof Letting $\nu = 1$ in the inequality (5.1), the desired result is proved. □

Remark 5.10 If $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$, then the inequality (5.9) agrees with the results in [42].

6 Examples

This section is devoted to illustrating the validity of our main results when estimating the eigenvalues of two **BVPs**, which means that each eigenvalue corresponds to a nontrivial (nonzero) solution for these **BVPs**.

Example 6.1 Suppose that λ is an eigenvalue of the Katugampola–Hilfer **BVP**:

$$\begin{cases} {}^{\rho,\mathrm{H}}\mathbb{D}^{u,\nu}_{\iota^+}\chi(\vartheta)+\lambda\chi(\vartheta)=0, & \vartheta\in(0,1), u\in(1,2], \nu\in[0,1],\\ \chi(0)=\frac{1}{4}\chi(1/5)+\frac{1}{6}\chi(1/4)+\frac{1}{8}\chi(1/3), & \chi(1)=\int_0^1\varepsilon^2\chi(\varepsilon)\,d\varepsilon, \end{cases} \tag{6.1}$$

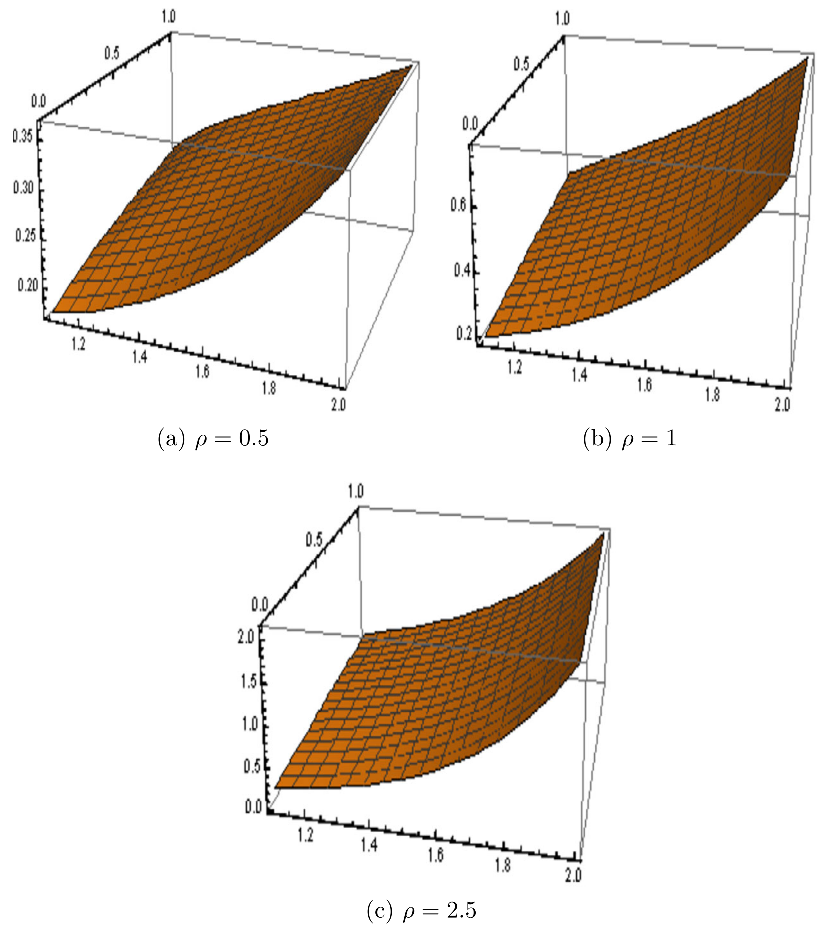


Figure 1 3D plots of lower bounds for the eigenvalue of BVP (6.1), for $u \in (1,2]$ and $v \in [01]$ at three values of ρ

Here, we have $\mathfrak{p}(\vartheta) = \lambda$, $\mathfrak{f}(\chi) = \frac{1}{4}\chi(1/5) + \frac{1}{6}\chi(1/4) + \frac{1}{8}\chi(1/3)$, $\mathfrak{h}(\varepsilon) = \varepsilon^2$, $\iota = 0$, and $\ell = 1$. Thus, we find $\mu = 1/4$, and $\Pi = 1 - \frac{1}{\rho(\zeta-1)+3} > 0$, iff $\rho(\zeta - 1) + 2 > 0$, which is true for all $\rho > 0$. Moreover, by the inequality (4.1), we have

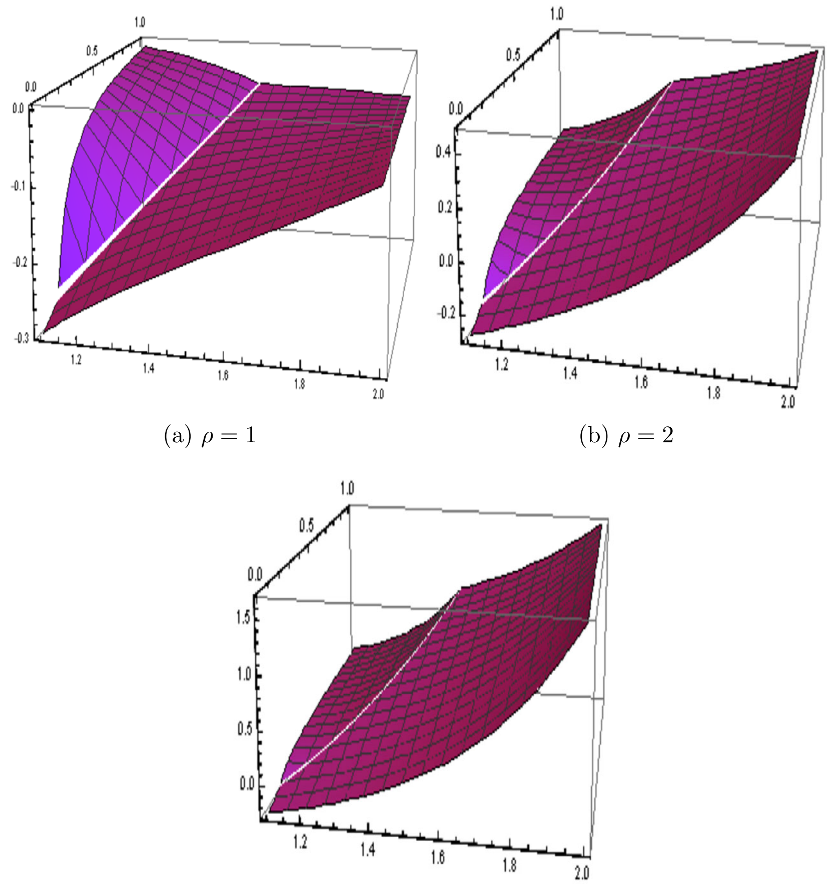
$$\rho^{u-1}\Gamma(u)\left(\frac{3\Pi}{3\Pi+1} - \frac{1}{2}\right)\left(\frac{\zeta-1}{\zeta+u-2}\right)^{1-\zeta}\left(\frac{u-1}{\zeta+u-2}\right)^{1-u} < |\lambda|.$$

Figure 1 shows the estimates of lower bounds for the eigenvalue λ of the fractional BVP (6.1), at $\rho = 0.5, 1, 1.5$, for $u \in (1,2]$ and $v \in [01]$.

Example 6.2 Let λ be an eigenvalue of the Katugampola–Hilfer **BVP**:

$$\begin{cases} {}^{\rho,\mathrm{H}}\mathbb{D}^{u,v}_{\iota^+}\chi(\vartheta) + \lambda\chi(\vartheta) = 0, & \vartheta \in (0,1), u \in (1,2], v \in [0,1], \\ \chi(0) = \frac{2}{3}\chi(0.1) + \frac{3}{5}\chi(0.5), & \vartheta^{1-\rho}\frac{d}{d\vartheta}\chi(\vartheta)|_{\vartheta=1} = \int_0^1 \frac{1}{2}\chi(\varepsilon)\,d\varepsilon. \end{cases} \tag{6.2}$$

Here, we have $\mathfrak{p}(\vartheta) = \lambda$, $\mathfrak{f}(\chi) = \frac{2}{3}\chi(0.1) + \frac{3}{5}\chi(0.5)$, $\mathfrak{h}(\varepsilon) = \frac{1}{2}$, $\iota = 0$, and $\ell = 1$. Then, one has $\mu = 2/3$, and $\Xi = 1 - \frac{1}{2\rho(\zeta-1)[\rho(\zeta-1)+1]} > 0$, iff $2\rho(\zeta - 1)[\rho(\zeta - 1) + 1] > 1$, which has solution only for $\rho(\zeta - 1) \geq 1$, and so, due to $\zeta - 1 \in (0,1)$, one finds $\rho \geq 1$, meaning that for $0 < \rho < 1$,



(a) $\rho = 1$ (b) $\rho = 2$ (c) $\rho = 3$

Figure 2 3D plots of lower bounds for the eigenvalue of BVP (6.2), for $u \in (1,2]$ and $v \in [01]$ at three values of ρ

the **BVP** (6.2) may only have trivial solutions. Additionally, according to the inequality (5.1), we get

$$\left(\frac{2\rho(\zeta-1)\Xi}{2\rho(\zeta-1)\Xi+1}-\frac{2}{3}\right)\frac{\rho^{u}(u-1)(\zeta-1)\Gamma(u)}{\max\{\zeta-u,u-1\}}<|\lambda|.$$

Figure 2 shows the estimates of lower bounds for the eigenvalue λ of the fractional BVP (6.2), at $\rho = 1, 2, 3$, for $u \in (1,2]$ and $v \in [01]$.

7 Conclusions

In modeling physical phenomena it is remarked that a nonlocal condition can be more useful than an initial value condition. This manuscript aimed to investigate new **LTIs** for two classes of the Katugampola–Hilfer fractional **BVPs** under nonlocal, integral, and mixed boundary conditions (1.12)–(1.13) and (1.12)–(1.14). A sufficient criterion of the existence and nonexistence of nontrivial solutions for the proposed problems was proved by new **LTIs**. Finally, our findings were employed for approximating the eigenvalues of two given **BVPs**. Our results are more general than those in the existing literature, so there are many new and existing specific cases included. In particular, they covered all results in the works [16, 22–28, 39–42], for instance:

- The inequality (4.3) coincides with inequality (1.11) if $\nu = 1$, $\mathfrak{f} = 0$, and $\lambda = 1$;
- The inequality (4.4) agrees with the results of [39] if $\nu = 0$;
- The inequality (4.6) coincides with inequality (1.7) if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$;
- The inequality (4.8) coincides with the inequality in Theorem 5 of [16] if $\mathfrak{f} = 0$ and $\mathfrak{h} = 0$;
- The inequality (4.8) agrees with inequality (1.3) if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$;
- The inequality (4.8) coincides with inequality (1.2) if $\rho = 1$, $\mathfrak{f} = 0$, $\mathfrak{h} = 0$, and $u = 2$;
- The inequality (4.8) agrees with the inequality in Theorem 2.2 of [25] if $\rho = 1$, $\mathfrak{f} = 0$, $\mathfrak{h} = \lambda$, and $\nu = 0$;
- The inequality (4.9) coincides with inequality (1.5) if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$;
- The inequality (4.9) coincides with inequality (1.2) if $\rho = 1$, $\mathfrak{f} = 0$, $\mathfrak{h} = 0$, and $u = 2$;
- The inequality (5.4) agrees with Corollary 3.4 of [40] if $\nu = 0$;
- The inequality (5.6) coincides with inequality (1.9) if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$;
- The inequality (5.8) agrees with Theorem 3.3 of [41] if $\mathfrak{f} = 0$ and $\mathfrak{h} = 0$;
- The inequality (5.8) gives Corollary 3.4 of [41] if $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$;
- If $\rho = 1$, $\mathfrak{f} = 0$, and $\mathfrak{h} = 0$, then the inequality (5.9) coincides with the results in [42].

In the future, inspired by this work, we express our intention to focus on studying the **LTIs** for boundary value problems involving tempered Riemann and Caputo fractional operators.

Acknowledgements
This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1445). The authors express their gratitude to dear unknown referees for their helpful suggestions which improved the final version of this paper.

Funding
No funding.

Data availability
Data sharing was not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
All authors have equal contributions. All authors read and approved the final manuscript.

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Received: 17 October 2023 Accepted: 12 December 2023 Published online: 22 December 2023

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