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Diversity of several estimates transformed on time scales

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Abstract

In this research article, we prove several generalizations of reverse Callebaut, Rogers–Hölder, and Cauchy–Schwarz inequalities via reverses of Young inequalities on time scales. Discrete, continuous, and quantum versions of the results are unified and extended on time scales.

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1 Introduction

The calculus of time scales was accomplished by Stefan Hilger [7]. A time scale is an arbitrary nonempty closed subset of the real numbers. Let $\mathbb{T}$ be a time scale, $\xi, \omega \in \mathbb{T}$ with $\xi < \omega$, and an interval $[\xi, \omega]_{\mathbb{T}}$ means the intersection of the real interval with the given time scale. The major aim of the calculus of time scales is to establish results in general, comprehensive, unified, and extended forms. This hybrid theory is also widely applied in dynamic inequalities, see [2, 8–12]. The basic ideas about time scale calculus are given in the monographs [3, 4].

We state here the different versions of reverses of Callebaut, Rogers–Hölder, and Cauchy–Schwarz inequalities, see [5].

Let $x_k > 0$, $y_k > 0$, and $w_k \geq 0$ for any $k \in \{1, 2, \dots, \eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. If there exist constants $m, M > 0$ such that $0 < m \leq \frac{x_k}{y_k} \leq M < \infty$ for any $k \in \{1, 2, \dots, \eta\}$, then

$$\begin{aligned} \sum_{k=1}^{\eta} w_k x_k^{2(1-\nu)} y_k^{2\nu} \sum_{k=1}^{\eta} w_k x_k^{2\nu} y_k^{2(1-\nu)} &\leq \sum_{k=1}^{\eta} w_k x_k^2 \sum_{k=1}^{\eta} w_k y_k^2 \\ &\leq S\left(\left(\frac{M}{m}\right)^2\right) \sum_{k=1}^{\eta} w_k x_k^{2(1-\nu)} y_k^{2\nu} \sum_{k=1}^{\eta} w_k x_k^{2\nu} y_k^{2(1-\nu)}, \end{aligned} \tag{1.1}$$

for any $\nu \in [0, 1]$ and, in particular,

$$\left(\sum_{k=1}^{\eta} w_k x_k y_k\right)^2 \leq \sum_{k=1}^{\eta} w_k x_k^2 \sum_{k=1}^{\eta} w_k y_k^2 \leq S\left(\left(\frac{M}{m}\right)^2\right) \left(\sum_{k=1}^{\eta} w_k x_k y_k\right)^2. \tag{1.2}$$

Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. If there exist constants m, M, n, N such that $0<m\leq x_k\leq M<\infty$ and $0<n\leq y_k\leq N<\infty$ for any $k\in\{1,2,\dots,\eta\}$, then we have the following reverse of Rogers–Hölder discrete inequality:

$$\left(\sum_{k=1}^{\eta} w_k x^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{\eta} w_k y^q\right)^{\frac{1}{q}}\leq S\left(\left(\frac{M}{m}\right)^p\left(\frac{N}{n}\right)^q\right)\sum_{k=1}^{\eta} w_k x_k y_k, \tag{1.3}$$

and, in particular, the reverse of Cauchy–Bunyakovsky–Schwarz inequality

$$\left(\sum_{k=1}^{\eta} w_k x^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\eta} w_k y^2\right)^{\frac{1}{2}}\leq S\left(\left(\frac{MN}{mn}\right)^2\right)\sum_{k=1}^{\eta} w_k x_k y_k. \tag{1.4}$$

2 Preliminaries

First, we present a short introduction to the diamond-α derivative as given in [1, 13].

Let $\mathbb{T}$ be a time scale and $f(\tau)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ sense. For $\tau\in\mathbb{T}$, the diamond-α dynamic derivative $f^{\diamond_\alpha}(\tau)$ is defined by

$$f^{\diamond_\alpha}(\tau)=\alpha f^{\Delta}(\tau)+(1-\alpha)f^{\nabla}(\tau),\quad 0\leq\alpha\leq 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha=1$, or the standard ∇-derivative for $\alpha=0$. It represents a weighted dynamic derivative for $\alpha\in(0,1)$.

The following definition is given in [13].

Let $\xi,\tau\in\mathbb{T}$ and $h:\mathbb{T}\to\mathbb{R}$. Then the diamond-$\alpha$ integral from ξ to τ of h is defined by

$$\int_{\xi}^{\tau} h(\lambda)\diamond_\alpha\lambda=\alpha\int_{\xi}^{\tau} h(\lambda)\Delta\lambda+(1-\alpha)\int_{\xi}^{\tau} h(\lambda)\nabla\lambda,\quad 0\leq\alpha\leq 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

The following well-known Young inequality holds:

For $\Phi,\Psi>0$ and $\nu\in[0,1]$, we have

$$\Phi^{1-\nu}\Psi^{\nu}\leq(1-\nu)\Phi+\nu\Psi. \tag{2.1}$$

The following inequalities are given in [5].

For any $\Phi,\Psi\in[m,M]\subset(0,\infty)$ and $\nu\in[0,1]$, we have

$$(1-\nu)\Phi+\nu\Psi\leq S\left(\frac{M}{m}\right)\Phi^{1-\nu}\Psi^{\nu}, \tag{2.2}$$

where Specht ratio [6, 14] is defined by

$$S(h)=\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}},$$

with $h>0$, $h\neq 1$.

Let $\nu \in [0,1]$ and $\Phi, \Psi > 0$. Then

$$(1-\nu)\Phi + \nu\Psi \le S(L)\Phi^{1-\nu}\Psi^{\nu}, \tag{2.3}$$

where $0 < L^{-1} \le \frac{\Phi}{\Psi} \le L < \infty$ and $L > 1$.

Let $\nu \in [0,1]$ and $\Phi, \Psi > 0$. Then

$$(1-\nu)\Phi + \nu\Psi \le \max\bigl\{S(l), S(L)\bigr\}\Phi^{1-\nu}\Psi^{\nu}, \tag{2.4}$$

where $0 < l^{-1} \le \frac{\Phi}{\Psi} \le L < \infty$ and $L, l > 0$, with $Ll > 1$.

In this paper, it is assumed that all considered integrals exist and are finite.

3 Main results

In the following, we give an extension of reverse Callebaut inequality on time scales. Throughout this section, we assume that neither $s \equiv 0$ nor $t \equiv 0$.

Theorem 3.1 *Let $z, s, t \in C([\xi,\omega]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions. Assume further that $0 < m \le \frac{|s(\lambda)|}{|t(\lambda)|} \le M < \infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $\nu \in [0,1]$. Then the following inequalities hold true*:

$$\begin{aligned}
&\int_\xi^\omega |z(\lambda)||s(\lambda)|^{2(1-\nu)}|t(\lambda)|^{2\nu} \diamond_\alpha \lambda \int_\xi^\omega |z(\lambda)||s(\lambda)|^{2\nu}|t(\lambda)|^{2(1-\nu)} \diamond_\alpha \lambda \\
&\quad \le \int_\xi^\omega |z(\lambda)||s(\lambda)|^2 \diamond_\alpha \lambda \int_\xi^\omega |z(\lambda)||t(\lambda)|^2 \diamond_\alpha \lambda \\
&\quad \le S\left(\left(\frac{M}{m}\right)^2\right) \int_\xi^\omega |z(\lambda)||s(\lambda)|^{2(1-\nu)}|t(\lambda)|^{2\nu} \diamond_\alpha \lambda \int_\xi^\omega |z(\lambda)||s(\lambda)|^{2\nu}|t(\lambda)|^{2(1-\nu)} \diamond_\alpha \lambda.
\end{aligned} \tag{3.1}$$

Proof For $\lambda, \zeta \in [\xi,\omega]_{\mathbb{T}}$, we observe that

$$m^2 \le \frac{|s(\lambda)|^2}{|t(\lambda)|^2}, \frac{|s(\zeta)|^2}{|t(\zeta)|^2} \le M^2. \tag{3.2}$$

Let $\Phi(\lambda) = \frac{|s(\lambda)|^2}{|t(\lambda)|^2}$ and $\Psi(\zeta) = \frac{|s(\zeta)|^2}{|t(\zeta)|^2}$, $\lambda, \zeta \in [\xi,\omega]_{\mathbb{T}}$. Then using the inequalities (2.1) and (2.2), we have

$$\begin{aligned}
\left(\frac{|s(\lambda)|^2}{|t(\lambda)|^2}\right)^{1-\nu}\left(\frac{|s(\zeta)|^2}{|t(\zeta)|^2}\right)^{\nu} &\le (1-\nu)\frac{|s(\lambda)|^2}{|t(\lambda)|^2} + \nu\frac{|s(\zeta)|^2}{|t(\zeta)|^2} \\
&\le S\left(\left(\frac{M}{m}\right)^2\right)\left(\frac{|s(\lambda)|^2}{|t(\lambda)|^2}\right)^{1-\nu}\left(\frac{|s(\zeta)|^2}{|t(\zeta)|^2}\right)^{\nu}.
\end{aligned} \tag{3.3}$$

Multiplying by $|t(\lambda)|^2|t(\zeta)|^2$, $\lambda, \zeta \in [\xi,\omega]_{\mathbb{T}}$, (3.3) takes the form

$$\begin{aligned}
&|s(\lambda)|^{2(1-\nu)}|t(\lambda)|^{2\nu}|s(\zeta)|^{2\nu}|t(\zeta)|^{2(1-\nu)} \\
&\quad \le (1-\nu)|s(\lambda)|^2|t(\zeta)|^2 + \nu|t(\lambda)|^2|s(\zeta)|^2 \\
&\quad \le S\left(\left(\frac{M}{m}\right)^2\right)|s(\lambda)|^{2(1-\nu)}|t(\lambda)|^{2\nu}|s(\zeta)|^{2\nu}|t(\zeta)|^{2(1-\nu)}.
\end{aligned} \tag{3.4}$$

Multiplying by $|z(\lambda)|$ and integrating (3.4) with respect to λ from ξ to ω, we obtain

$$\begin{aligned}
&\left(\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{2(1-\nu)}\left|t(\lambda)\right|^{2\nu}\diamond_{\alpha}\lambda\right)\left|s(\zeta)\right|^{2\nu}\left|t(\zeta)\right|^{2(1-\nu)}\\
&\quad\le(1-\nu)\left(\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{2}\diamond_{\alpha}\lambda\right)\left|t(\zeta)\right|^{2}+\nu\left(\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|t(\lambda)\right|^{2}\diamond_{\alpha}\lambda\right)\left|s(\zeta)\right|^{2}\\
&\quad\le S\left(\left(\frac{M}{m}\right)^{2}\right)\left(\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{2(1-\nu)}\left|t(\lambda)\right|^{2\nu}\diamond_{\alpha}\lambda\right)\left|s(\zeta)\right|^{2\nu}\left|t(\zeta)\right|^{2(1-\nu)}.
\end{aligned}\tag{3.5}$$

Again, multiplying by $|z(\zeta)|$ and integrating (3.5) with respect to ζ from ξ to ω, we obtain the desired inequality (3.1). □

The following reverse of Callebaut inequality holds:

Corollary 3.1 *Let $z,s,t\in C([\xi,\omega]_{\mathbb{T}},\mathbb{R})$ be $\diamond_{\alpha}$-integrable functions. Assume further that $0<m\le\frac{|s(\lambda)|}{|t(\lambda)|}\le M<\infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Then the following inequalities hold true*:

$$\begin{aligned}
&\left(\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|\left|t(\lambda)\right|\diamond_{\alpha}\lambda\right)^{2}\\
&\quad\le\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{2}\diamond_{\alpha}\lambda\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|t(\lambda)\right|^{2}\diamond_{\alpha}\lambda\\
&\quad\le S\left(\left(\frac{M}{m}\right)^{2}\right)\left(\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|\left|t(\lambda)\right|\diamond_{\alpha}\lambda\right)^{2}.
\end{aligned}\tag{3.6}$$

Proof Take $\nu=\frac{1}{2}$ in Theorem 3.1, and the result follows. □

The following another reverse of Callebaut inequality holds:

Corollary 3.2 *Let $z,s,t\in C([\xi,\omega]_{\mathbb{T}},\mathbb{R})$ be $\diamond_{\alpha}$-integrable functions. Assume further that $0<m\le\frac{|s(\lambda)|}{|t(\lambda)|}\le M<\infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $\nu\in[0,1]$. Then the following inequalities hold true*:

$$\begin{aligned}
&\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{1+\nu}\left|t(\lambda)\right|^{1-\nu}\diamond_{\alpha}\lambda\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{1-\nu}\left|t(\lambda)\right|^{1+\nu}\diamond_{\alpha}\lambda\\
&\quad\le\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{2}\diamond_{\alpha}\lambda\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|t(\lambda)\right|^{2}\diamond_{\alpha}\lambda\\
&\quad\le S\left(\left(\frac{M}{m}\right)^{2}\right)\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{1+\nu}\left|t(\lambda)\right|^{1-\nu}\diamond_{\alpha}\lambda\int_{\xi}^{\omega}\left|z(\lambda)\right|\left|s(\lambda)\right|^{1-\nu}\left|t(\lambda)\right|^{1+\nu}\diamond_{\alpha}\lambda.
\end{aligned}\tag{3.7}$$

Proof Replace ν by $\frac{1}{2}(1-\nu)$ in Theorem 3.1, and the result follows. □

The following another reverse of Callebaut inequality holds:

Corollary 3.3 *Let $z,s,t\in C([\xi,\omega]_{\mathbb{T}},\mathbb{R})$ be $\diamond_{\alpha}$-integrable functions. Assume further that $0<m\le\frac{|s(\lambda)|}{|t(\lambda)|}\le M<\infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $\nu\in[0,2]$. Then the following inequalities hold* *true*:

$$\begin{aligned}
&\int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{2-\nu} |t(\lambda)|^{\nu} \diamond_{\alpha} \lambda \int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{\nu} |t(\lambda)|^{2-\nu} \diamond_{\alpha} \lambda \\
&\quad \le \int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{2} \diamond_{\alpha} \lambda \int_{\xi}^{\omega} |z(\lambda)| |t(\lambda)|^{2} \diamond_{\alpha} \lambda \\
&\quad \le S\left(\left(\frac{M}{m}\right)^{2}\right) \int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{2-\nu} |t(\lambda)|^{\nu} \diamond_{\alpha} \lambda \int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{\nu} |t(\lambda)|^{2-\nu} \diamond_{\alpha} \lambda.
\end{aligned} \tag{3.8}$$

Proof Take $\nu = \frac{1}{2}\nu$ in Theorem 3.1, and the result follows. □

In the following, we give an extension of reverse Rogers–Hölder inequality on time scales.

Theorem 3.2 *Let $z, s, t \in C([\xi, \omega]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_{\alpha}$-integrable functions satisfying $\int_{\xi}^{\omega} |z(\lambda)| \diamond_{\alpha} \lambda = 1$. Assume further that $0 < m \le |s(\lambda)| \le M < \infty$ and $0 < n \le |t(\lambda)| \le N < \infty$ on the set $[\xi, \omega]_{\mathbb{T}}$. Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then the following inequality holds true*:

$$\begin{aligned}
&\left(\int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{p} \diamond_{\alpha} \lambda\right)^{\frac{1}{p}} \left(\int_{\xi}^{\omega} |z(\lambda)| |t(\lambda)|^{q} \diamond_{\alpha} \lambda\right)^{\frac{1}{q}} \\
&\quad \le S\left(\left(\frac{M}{m}\right)^{p} \left(\frac{N}{n}\right)^{q}\right) \int_{\xi}^{\omega} |z(\lambda)| |s(\lambda) t(\lambda)| \diamond_{\alpha} \lambda.
\end{aligned} \tag{3.9}$$

Proof Using the given conditions, for $\lambda \in [\xi, \omega]_{\mathbb{T}}$, we have

$$m^{p} \le |s(\lambda)|^{p} \le M^{p} \quad \text{and} \quad n^{q} \le |t(\lambda)|^{q} \le N^{q},$$

which imply that

$$\left(\frac{m}{M}\right)^{p} \le \frac{|s(\lambda)|^{p}}{\int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{p} \diamond_{\alpha} \lambda} \le \left(\frac{M}{m}\right)^{p} \tag{3.10}$$

and

$$\left(\frac{n}{N}\right)^{q} \le \frac{|t(\lambda)|^{q}}{\int_{\xi}^{\omega} |z(\lambda)| |t(\lambda)|^{q} \diamond_{\alpha} \lambda} \le \left(\frac{N}{n}\right)^{q}. \tag{3.11}$$

Therefore,

$$\begin{aligned}
\left[\left(\frac{M}{m}\right)^{p} \left(\frac{N}{n}\right)^{q}\right]^{-1} &\le \left(\frac{|z(\lambda)| |s(\lambda)|^{p}}{\int_{\xi}^{\omega} |z(\lambda)| |s(\lambda)|^{p} \diamond_{\alpha} \lambda}\right) \left(\frac{\int_{\xi}^{\omega} |z(\lambda)| |t(\lambda)|^{q} \diamond_{\alpha} \lambda}{|z(\lambda)| |t(\lambda)|^{q}}\right) \\
&\le \left(\frac{M}{m}\right)^{p} \left(\frac{N}{n}\right)^{q}.
\end{aligned} \tag{3.12}$$

Using the inequality (2.3) with $\nu = \frac{1}{q}$, $L = (\frac{M}{m})^p(\frac{N}{n})^q$, $\Phi(\lambda) = \frac{|z(\lambda)||s(\lambda)|^p}{\int_\xi^\omega |z(\lambda)||s(\lambda)|^p \diamond_\alpha \lambda}$, and $\Psi(\lambda) = \frac{|z(\lambda)||t(\lambda)|^q}{\int_\xi^\omega |z(\lambda)||t(\lambda)|^q \diamond_\alpha \lambda}$, we get

$$\begin{aligned}
&\frac{1}{p}\frac{|z(\lambda)||s(\lambda)|^p}{\int_\xi^\omega |z(\lambda)||s(\lambda)|^p \diamond_\alpha \lambda} + \frac{1}{q}\frac{|z(\lambda)||t(\lambda)|^q}{\int_\xi^\omega |z(\lambda)||t(\lambda)|^q \diamond_\alpha \lambda} \\
&\quad \le S(L)\frac{|z(\lambda)||s(\lambda)t(\lambda)|}{(\int_\xi^\omega |z(\lambda)||s(\lambda)|^p \diamond_\alpha \lambda)^{\frac{1}{p}}(\int_\xi^\omega |z(\lambda)||t(\lambda)|^q \diamond_\alpha \lambda)^{\frac{1}{q}}}.
\end{aligned} \tag{3.13}$$

Integrating (3.13) with respect to λ from ξ to ω, we obtain

$$1 \le S(L)\frac{\int_\xi^\omega |z(\lambda)||s(\lambda)t(\lambda)| \diamond_\alpha \lambda}{(\int_\xi^\omega |z(\lambda)||s(\lambda)|^p \diamond_\alpha \lambda)^{\frac{1}{p}}(\int_\xi^\omega |z(\lambda)||t(\lambda)|^q \diamond_\alpha \lambda)^{\frac{1}{q}}}. \tag{3.14}$$

This completes the proof of Theorem 3.2. □

Next, we give an extension of reverse Cauchy–Schwarz inequality on time scales.

Corollary 3.4 *Let $z, s, t \in C([\xi,\omega]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions satisfying $\int_\xi^\omega |z(\lambda)| \diamond_\alpha \lambda = 1$. Assume further that $0 < m \le |s(\lambda)| \le M < \infty$ and $0 < n \le |t(\lambda)| \le N < \infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Then the following inequality holds true*:

$$\begin{aligned}
&\left(\int_\xi^\omega |z(\lambda)|\left|s(\lambda)\right|^2 \diamond_\alpha \lambda\right)^{\frac{1}{2}}\left(\int_\xi^\omega |z(\lambda)|\left|t(\lambda)\right|^2 \diamond_\alpha \lambda\right)^{\frac{1}{2}} \\
&\quad \le S\left(\left(\frac{MN}{mn}\right)^2\right)\int_\xi^\omega |z(\lambda)|\left|s(\lambda)t(\lambda)\right| \diamond_\alpha \lambda.
\end{aligned} \tag{3.15}$$

Proof Take $p = q = 2$ in Theorem 3.2, and the result follows. □

Remark 3.1 We have the following:

(i) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = \eta + 1$, $s(k) = x_k > 0$, $t(k) = y_k > 0$, and $z(k) = w_k \ge 0$ for any $k \in \{1, 2, \dots, \eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. Then inequality (3.1) reduces to inequality (1.1).

(ii) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = \eta + 1$, $s(k) = x_k > 0$, $t(k) = y_k > 0$, and $z(k) = w_k \ge 0$ for any $k \in \{1, 2, \dots, \eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. Then inequality (3.6) reduces to inequality (1.2).

(iii) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = \eta + 1$, $s(k) = x_k > 0$, $t(k) = y_k > 0$, and $z(k) = w_k \ge 0$ for any $k \in \{1, 2, \dots, \eta\}$. Then inequality (3.9) reduces to inequality (1.3).

(iv) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = \eta + 1$, $s(k) = x_k > 0$, $t(k) = y_k > 0$, and $z(k) = w_k \ge 0$ for any $k \in \{1, 2, \dots, \eta\}$. Then inequality (3.15) reduces to inequality (1.4).

Finally, we give another extension of reverse Rogers–Hölder dynamic inequality.

Theorem 3.3 *Let $z, u_1, u_2, s, t \in C([\xi,\omega]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions. Assume further that $0 < m \le |s(\lambda)| \le M < \infty$ and $0 < n \le |t(\lambda)| \le N < \infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $\frac{1}{p} + \frac{1}{q} = 1$*

with $p > 1$. Then the following inequalities hold true:

$$\begin{aligned}
&\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)s(\lambda)|\diamond_\alpha\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||u_2(\lambda)t(\lambda)|\diamond_\alpha\lambda\right)\\
&\quad\le\frac{1}{p}\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)||s(\lambda)|^p\diamond_\alpha\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||u_2(\lambda)|\diamond_\alpha\lambda\right)\\
&\qquad+\frac{1}{q}\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)|\diamond_\alpha\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||u_2(\lambda)||t(\lambda)|^q\diamond_\alpha\lambda\right)\\
&\quad\le\max\left\{S\left(\frac{N^q}{m^p}\right),S\left(\frac{M^p}{n^q}\right)\right\}\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)s(\lambda)|\diamond_\alpha\lambda\right)\\
&\qquad\times\left(\int_{\xi}^{\omega}|z(\lambda)||u_2(\lambda)t(\lambda)|\diamond_\alpha\lambda\right).
\end{aligned}\tag{3.16}$$

Proof For $\lambda,\zeta\in[\xi,\omega]_{\mathbb{T}}$, it is clear that

$$\frac{m^p}{N^q}\le\frac{|s(\lambda)|^p}{|t(\zeta)|^q}\le\frac{M^p}{n^q}.\tag{3.17}$$

Let $l=\frac{N^q}{m^p}$, $L=\frac{M^p}{n^q}$, $\Phi(\lambda)=|s(\lambda)|^p$, $\Psi(\zeta)=|t(\zeta)|^q$, and $\nu=\frac{1}{q}$. Then using the inequalities (2.1) and (2.4), respectively, we have

$$|s(\lambda)||t(\zeta)|\le\frac{1}{p}|s(\lambda)|^p+\frac{1}{q}|t(\zeta)|^q\le\max\left\{S\left(\frac{N^q}{m^p}\right),S\left(\frac{M^p}{n^q}\right)\right\}|s(\lambda)||t(\zeta)|.\tag{3.18}$$

Multiplying by $|z(\lambda)||u_1(\lambda)|$ and integrating (3.18) with respect to λ from ξ to ω, we obtain

$$\begin{aligned}
&\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)s(\lambda)|\diamond_\alpha\lambda\right)|t(\zeta)|\\
&\quad\le\frac{1}{p}\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)||s(\lambda)|^p\diamond_\alpha\lambda\right)+\frac{1}{q}\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)|\diamond_\alpha\lambda\right)|t(\zeta)|^q\\
&\quad\le\max\left\{S\left(\frac{N^q}{m^p}\right),S\left(\frac{M^p}{n^q}\right)\right\}\left(\int_{\xi}^{\omega}|z(\lambda)||u_1(\lambda)s(\lambda)|\diamond_\alpha\lambda\right)|t(\zeta)|.
\end{aligned}\tag{3.19}$$

Multiplying by $|z(\zeta)||u_2(\zeta)|$ and integrating (3.19) with respect to ζ from ξ to ω, we obtain the desired inequality (3.16). □

Next, we give an extension of reverse Rogers–Hölder inequality on time scales.

Corollary 3.5 *Let $z,s,t\in C([\xi,\omega]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions. Assume further that $0<m\le|s(\lambda)|\le M<\infty$ and $0<n\le|t(\lambda)|\le N<\infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$.*

Then the following inequalities hold true:

$$\begin{aligned}
&\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)t(\lambda)|\diamond_{\alpha}\lambda\right)^{2} \\
&\leq \frac{1}{p}\left(\int_{\xi}^{\omega}|z(\lambda)||t(\lambda)||s(\lambda)|^{p}\diamond_{\alpha}\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)|\diamond_{\alpha}\lambda\right) \\
&\quad+\frac{1}{q}\left(\int_{\xi}^{\omega}|z(\lambda)||t(\lambda)|\diamond_{\alpha}\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)||t(\lambda)|^{q}\diamond_{\alpha}\lambda\right) \\
&\leq \max\left\{S\left(\frac{N^{q}}{m^{p}}\right),S\left(\frac{M^{p}}{n^{q}}\right)\right\}\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)t(\lambda)|\diamond_{\alpha}\lambda\right)^{2}.
\end{aligned}\tag{3.20}$$

Proof Put $|u_1(\lambda)| = |t(\lambda)|$ and $|u_2(\lambda)| = |s(\lambda)|$ on $[\xi,\omega]_{\mathbb{T}}$ in Theorem 3.3, and then the result follows. □

Now, we give another extension of reverse Rogers–Hölder inequality on time scales.

Corollary 3.6 *Let $z,s,t \in C([\xi,\omega]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions. Assume further that $0 < m \leq |s(\lambda)| \leq M < \infty$ and $0 < n \leq |t(\lambda)| \leq N < \infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Then the following inequalities hold true*:

$$\begin{aligned}
&\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)|^{2}\diamond_{\alpha}\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||t(\lambda)|^{2}\diamond_{\alpha}\lambda\right) \\
&\leq \frac{1}{p}\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)|^{p+1}\diamond_{\alpha}\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||t(\lambda)|\diamond_{\alpha}\lambda\right) \\
&\quad+\frac{1}{q}\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)|\diamond_{\alpha}\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||t(\lambda)|^{q+1}\diamond_{\alpha}\lambda\right) \\
&\leq \max\left\{S\left(\frac{N^{q}}{m^{p}}\right),S\left(\frac{M^{p}}{n^{q}}\right)\right\}\left(\int_{\xi}^{\omega}|z(\lambda)||s(\lambda)|^{2}\diamond_{\alpha}\lambda\right)\left(\int_{\xi}^{\omega}|z(\lambda)||t(\lambda)|^{2}\diamond_{\alpha}\lambda\right).
\end{aligned}\tag{3.21}$$

Proof Put $|u_1(\lambda)| = |s(\lambda)|$ and $|u_2(\lambda)| = |t(\lambda)|$ on $[\xi,\omega]_{\mathbb{T}}$ in Theorem 3.3, and then the result follows. □

Next, we give another extension of reverse Rogers–Hölder inequality on time scales.

Corollary 3.7 *Let $z,f_1,f_2 \in C([\xi,\omega]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions, with neither $f_1 \equiv 0$ nor $f_2 \equiv 0$. Assume further that $0 < m \leq \frac{|f_1(\lambda)|}{|f_2(\lambda)|} \leq M < \infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Then the following inequalities hold true*:

$$\begin{aligned}
&\left(\int_{\xi}^{\omega}|z(\lambda)||f_1(\lambda)f_2(\lambda)|\diamond_{\alpha}\lambda\right)^{2} \\
&\leq \left[\frac{1}{p}\int_{\xi}^{\omega}|z(\lambda)||f_1(\lambda)|^{p}|f_2(\lambda)|^{2-p}\diamond_{\alpha}\lambda\right. \\
&\quad\left.+\frac{1}{q}\int_{\xi}^{\omega}|z(\lambda)||f_1(\lambda)|^{q}|f_2(\lambda)|^{2-q}\diamond_{\alpha}\lambda\right]\int_{\xi}^{\omega}|z(\lambda)||f_2(\lambda)|^{2}\diamond_{\alpha}\lambda \\
&\leq \max\left\{S\left(\frac{M^{q}}{m^{p}}\right),S\left(\frac{M^{p}}{m^{q}}\right)\right\}\left(\int_{\xi}^{\omega}|z(\lambda)||f_1(\lambda)f_2(\lambda)|\diamond_{\alpha}\lambda\right)^{2}.
\end{aligned}\tag{3.22}$$

Proof Put $|s(\lambda)| = |t(\lambda)| = \frac{|f_1(\lambda)|}{|f_2(\lambda)|}$, $|u_1(\lambda)| = |u_2(\lambda)| = |f_2(\lambda)|^2$ on $[\xi,\omega]_{\mathbb{T}}$, $M = N$, and $m = n$ in Theorem 3.3, and then the result follows. □

Remark 3.2 We have the following:

(i) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = \eta + 1$, $s(k) = x_k > 0$, $t(k) = y_k > 0$, and $z(k) = w_k \geq 0$ for any $k \in \{1,2,\dots,\eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. Then inequality (3.20) reduces to inequality [5]

$$\begin{aligned}\left(\sum_{k=1}^{\eta} w_k x_k y_k\right)^2 &\leq \frac{1}{p}\sum_{k=1}^{\eta} w_k y_k x_k^p \sum_{k=1}^{\eta} w_k x_k + \frac{1}{q}\sum_{k=1}^{\eta} w_k y_k \sum_{k=1}^{\eta} w_k x_k y_k^q \\ &\leq \max\left\{S\left(\frac{N^q}{m^p}\right), S\left(\frac{M^p}{n^q}\right)\right\}\left(\sum_{k=1}^{\eta} w_k x_k y_k\right)^2.\end{aligned} \tag{3.23}$$

(ii) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = \eta + 1$, $s(k) = x_k > 0$, $t(k) = y_k > 0$, and $z(k) = w_k \geq 0$ for any $k \in \{1,2,\dots,\eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. Then inequality (3.21) reduces to inequality [5]

$$\begin{aligned}\sum_{k=1}^{\eta} w_k x_k^2 \sum_{k=1}^{\eta} w_k y_k^2 &\leq \frac{1}{p}\sum_{k=1}^{\eta} w_k x_k^{p+1} \sum_{k=1}^{\eta} w_k y_k + \frac{1}{q}\sum_{k=1}^{\eta} w_k x_k \sum_{k=1}^{\eta} w_k y_k^{q+1} \\ &\leq \max\left\{S\left(\frac{N^q}{m^p}\right), S\left(\frac{M^p}{n^q}\right)\right\}\sum_{k=1}^{\eta} w_k x_k^2 \sum_{k=1}^{\eta} w_k y_k^2.\end{aligned} \tag{3.24}$$

(iii) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = \eta + 1$, $f_1(k) = x_k > 0$, $f_2(k) = y_k > 0$, and $z(k) = w_k \geq 0$ for any $k \in \{1,2,\dots,\eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. Then inequality (3.22) reduces to inequality [5]

$$\begin{aligned}\left(\sum_{k=1}^{\eta} w_k x_k y_k\right)^2 &\leq \left(\frac{1}{p}\sum_{k=1}^{\eta} w_k x_k^p y_k^{2-p} + \frac{1}{q}\sum_{k=1}^{\eta} w_k x_k^q y_k^{2-q}\right)\sum_{k=1}^{\eta} w_k y_k^2 \\ &\leq \max\left\{S\left(\frac{M^q}{m^p}\right), S\left(\frac{M^p}{m^q}\right)\right\}\left(\sum_{k=1}^{\eta} w_k x_k y_k\right)^2.\end{aligned} \tag{3.25}$$

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Author contributions
MJSS and DA carried out preparatory investigation. Formal analysis was done by MI and ASA. The manuscript was prepared by MJSS and DA. ASA was responsible for the main funding. MJSS and MI worked together to conclude the result. All authors read and approved the final manuscript.

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