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Inequalities for partial determinants of accretive block matrices

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Abstract

Let $A = [A_{i,j}]_{i,j=1}^{m} \in \mathbf{M}_m(\mathbf{M}_n)$ be an accretive block matrix. We write $\det_1$ and $\det_2$ for the first and second partial determinants, respectively. In this paper, we show that

$$\left\|\det_1(\operatorname{Re} A)\right\| \le \left\|\left(\frac{\operatorname{tr}(|A|)}{m}\right)^m I_n\right\|$$

and

$$\left\|\det_2(\operatorname{Re} A)\right\| \le \left\|\left(\frac{\operatorname{tr}(|A|)}{n}\right)^n I_m\right\|$$

hold for any unitarily invariant norm $\|\cdot\|$. The two inequalities generalize some known results related to partial determinants of positive-semidefinite block matrices.

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1 Introduction

The set of $n \times n$ complex matrices is denoted by $\mathbf{M}_n$. I_n is $n \times n$ identity matrix. Let $\mathbf{M}_m(\mathbf{M}_n)$ be the set of all $m \times m$ block matrices with each block in $\mathbf{M}_n$. If $A \in \mathbf{M}_n$ is positive-semidefinite (definite), then we write $A \ge 0$ ($A > 0$). For two Hermitian matrices A, B of the same size, $A \ge B$ ($A > B$) means that $A - B \ge 0$ ($A - B > 0$). For $A \in \mathbf{M}_n$, the singular values of A, denoted by $s_1(A), s_2(A), \dots, s_n(A)$, are the eigenvalues of the positive-semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in nonincreasing order and repeated according to multiplicity as $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A)$. If A is Hermitian, we enumerate eigenvalues of A in nonincreasing order $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$. We denote by A^T and A^* the transpose and conjugate transpose of A, respectively. Recall that a norm $\|\cdot\|$ is unitarily invariant if $\|UAV\| = \|A\|$ for any unitary matrices $U, V \in \mathbf{M}_n$ and any $A \in \mathbf{M}_n$. The Ky Fan k-norms, a special class of unitarily invariant norms, are defined as $\|\cdot\|_{(k)} = \sum_{j=1}^{k} s_j(A)$, $1 \le k \le n$.

The Schatten p-norms ($p \geq 1$) are defined as

$$\|A\|_p = \left(\operatorname{tr}\left(|A|^p\right)\right)^{\frac{1}{p}} = \left[\sum_{j=1}^{n} s_j^p(A)\right]^{\frac{1}{p}}.$$

The Schatten p-norms ($p \geq 1$) are also typical examples of unitarily invariant norms. We say that $A \in \mathbf{M}_m(\mathbf{M}_n)$ is an accretive block matrix if its real part $\operatorname{Re} A := \frac{A+A^*}{2}$ is positive-semidefinite.

In the following, two partial traces [6, p. 12] of $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ are defined by

$$\operatorname{tr}_1 A = \sum_{i=1}^{m} A_{i,i} \in \mathbb{M}_n \quad \text{and} \quad \operatorname{tr}_2 A = [\operatorname{tr} A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m.$$

Assume that $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$, where $A_{i,j} = [a_{l,k}^{i,j}]_{l,k=1}^n$. We introduce two partial determinants $\det_1 A \in \mathbf{M}_n$ and $\det_2 A \in \mathbf{M}_m$ analogous to the two partial traces as follows [2]:

$$\det_1 A = [\det G_{l,k}]_{l,k=1}^n,$$

where $G_{l,k} = [a_{l,k}^{i,j}]_{i,j=1}^m$, and

$$\det_2 A = [\det A_{i,j}]_{i,j=1}^m.$$

For $A = [[a_{l,k}^{i,j}]_{l,k=1}^n]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$, we will denote by $\widetilde{A} \in \mathbf{M}_n(\mathbf{M}_m)$ and $A^\tau \in \mathbf{M}_m(\mathbf{M}_n)$ the matrices

$$\widetilde{A} = [G_{l,k}]_{l,k=1}^n = \left[\left[a_{l,k}^{i,j}\right]_{i,j=1}^m\right]_{l,k=1}^n \quad \text{and} \quad A^\tau = [A_{j,i}]_{i,j=1}^m = \left[\left[a_{l,k}^{j,i}\right]_{l,k=1}^n\right]_{i,j=1}^m.$$

Note that $\tilde{\tilde{A}} = A$ and $\det_1 A = \det_2 \tilde{A}$ and therefore also $\det_2 A = \det_1 \tilde{A}$.

Recently, Xu et al. [8] presented the following unitarily invariant norm inequalities for two partial determinants of positive-semidefinite block matrices.

Theorem 1.1 *Let $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ be positive-semidefinite. Then, the inequalities*

$$\|\det_1 A\| \leq \left\| \left(\frac{\operatorname{tr} A}{m}\right)^m I_n \right\|$$

and

$$\|\det_2 A\| \leq \left\| \left(\frac{\operatorname{tr} A}{n}\right)^n I_m \right\|$$

hold for any unitarily invariant norm $\|\cdot\|$.

This theorem is inspired by a determinantal inequality for partial traces given by Lin [5, Theorem 1.2]. Actually, the two unitarily invariant norm inequalities for partial determinants of A^τ in Theorem 1.1 also hold; see [8, Theorem 2.12].

The main goal of this paper is to extend the above two inequalities to accretive block matrices that is a larger class of matrices than the class of positive-semidefinite block matrices. At the same time, some related results are obtained.

2 Partial determinant inequalities

We begin this section with some lemmas that are useful to present our main results. The following two results will be used in Theorem 2.6.

Lemma 2.1 [2, *Theorem* 7 *and Remark* 9] *For* $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$,

1. *A and $\tilde{A}$ are unitarily similar*;
2. *if A is positive-semidefinite, so is $\tilde{A}$.*

The next lemma is standard in matrix analysis.

Lemma 2.2 [4, *p.* 511] *Let $A, B \in \mathbf{M}_n$ be positive-semidefinite. Then,*

$$\det(A) + \det(B) \leq \det(A + B).$$

For the convenience of proofs, we also need to list some recent results as lemmas.

Lemma 2.3 [7] *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be positive-semidefinite. Then,*

$$\det_2 A \geq 0.$$

Lemma 2.4 [2, *Theorem* 6] *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be positive-semidefinite. Then,*

1. $\det_1 A \geq 0$,
2. $\det(\operatorname{tr}_2 A) \geq \operatorname{tr}(\det_1 A)$.

Lemma 2.5 [3, *Proposition* 2.1] *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be positive-semidefinite. Then,*

$$\det(\operatorname{tr}_1 A) \leq \left(\frac{\operatorname{tr} A}{n}\right)^n \quad \textit{and} \quad \det(\operatorname{tr}_2 A) \leq \left(\frac{\operatorname{tr} A}{m}\right)^m.$$

As an analog of Theorem 1.1, we prove the following inequalities for unitarily invariant norms.

Theorem 2.6 *Let $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ be a sector block matrix. Then, the inequalities*

$$\left\|\det_1(\operatorname{Re} A)\right\| \leq \left\|\left(\frac{\operatorname{tr}(|A|)}{m}\right)^m I_n\right\|$$

and

$$\left\|\det_2(\operatorname{Re} A)\right\| \leq \left\|\left(\frac{\operatorname{tr}(|A|)}{n}\right)^n I_m\right\|$$

hold for any unitarily invariant norm $\|\cdot\|$.

Proof To prove the desired results, by Ky Fan's dominance theorem [1, p. 93], we just need to show that for all $k = 1, \dots, n$,

$$\left\|\det_1(\operatorname{Re} A)\right\|_{(k)} \le \left\|\left(\frac{\operatorname{tr}(|A|)}{m}\right)^m I_n\right\|_{(k)}$$

and for all $k = 1, \dots, m$,

$$\left\|\det_2(\operatorname{Re} A)\right\|_{(k)} \le \left\|\left(\frac{\operatorname{tr}(|A|)}{n}\right)^n I_m\right\|_{(k)}.$$

Compute

$$\begin{aligned}
\left\|\det_1(\operatorname{Re} A)\right\|_{(k)} &= \sum_{j=1}^{k} s_j\big(\det_1(\operatorname{Re} A)\big) \\
&= \sum_{j=1}^{k} \lambda_j\big(\det_1(\operatorname{Re} A)\big) \quad \text{(by Lemma 2.4)} \\
&\le \operatorname{tr}\big(\det_1(\operatorname{Re} A)\big) \\
&\le \det\big(\operatorname{tr}_2(\operatorname{Re} A)\big) \quad \text{(by Lemma 2.4)} \\
&\le \left(\frac{\operatorname{tr}(\operatorname{Re} A)}{m}\right)^m \quad \text{(by Lemma 2.5)} \\
&= \frac{1}{k}\sum_{j=1}^{k} s_j\left(\left(\frac{\operatorname{tr}(\operatorname{Re} A)}{m}\right)^m I_n\right) \\
&= \frac{1}{k}\left\|\left(\frac{\operatorname{tr}(\operatorname{Re} A)}{m}\right)^m I_n\right\|_{(k)} \\
&\le \left\|\left(\frac{\operatorname{tr}(|A|)}{m}\right)^m I_n\right\|_{(k)}, \quad k = 1, \dots, n,
\end{aligned}$$

which means that

$$\left\|\det_1(\operatorname{Re} A)\right\| \le \left\|\left(\frac{\operatorname{tr}(|A|)}{m}\right)^m I_n\right\|.$$

By Lemma 2.1, we have $\operatorname{tr}(\operatorname{Re} A) = \operatorname{tr}(\widetilde{\operatorname{Re} A}) = \operatorname{tr}(\operatorname{Re} \widetilde{A})$. Therefore, by $\widetilde{\operatorname{Re} A} = \operatorname{Re} \widetilde{A}$,

$$\left\|\det_2(\operatorname{Re} A)\right\| = \left\|\det_1(\widetilde{\operatorname{Re} A})\right\| = \left\|\det_1(\operatorname{Re} \widetilde{A})\right\| \le \left\|\left(\frac{\operatorname{tr}(|\widetilde{A}|)}{n}\right)^n I_m\right\| = \left\|\left(\frac{\operatorname{tr}(|A|)}{n}\right)^n I_m\right\|. \quad \square$$

Remark 1 When $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ is positive-semidefinite in Theorem 2.6, our result is Theorem 1.1. Thus, our result is a generalization of Theorem 1.1.

Next, we will prove two determinantal inequalities for accretive block matrices involving partial determinants.

Theorem 2.7 *Let* $A = [A_{i,j}]_{i,j=1}^{m} \in \mathbf{M}_m(\mathbf{M}_n)$ *be an accretive block matrix. Then,*

$$\det\bigl(\det_1(\operatorname{Re} A)\bigr) \le \frac{(\operatorname{tr}(|A|))^{mn}}{m^{mn}n^n}$$

and

$$\det\bigl(\det_2(\operatorname{Re} A)\bigr) \le \frac{(\operatorname{tr}(|A|))^{mn}}{n^{mn}m^m}.$$

Proof Let $\lambda_j, j = 1, \dots, m$, be the eigenvalues of $\det_2(\operatorname{Re} A)$. Then, by the AM-GM inequality and Lemma 2.5, we have the following result:

$$\begin{aligned}
\det\bigl(\det_2(\operatorname{Re} A)\bigr) &= \lambda_1 \cdots \lambda_m \\
&\le \left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_m}{m}\right)^m \\
&= \left(\frac{\operatorname{tr}(\det_2(\operatorname{Re} A))}{m}\right)^m \\
&= \left(\frac{\sum_{i=1}^m \det(\operatorname{Re} A)_{ii}}{m}\right)^m \\
&\le \left(\frac{\det(\sum_{i=1}^m (\operatorname{Re} A)_{ii})}{m}\right)^m \quad \text{(by Lemma 2.2)} \\
&= \left(\frac{\det(\operatorname{tr}_1(\operatorname{Re} A))}{m}\right)^m \\
&\le \left(\frac{(\frac{\operatorname{tr}(\operatorname{Re} A)}{n})^n}{m}\right)^m \quad \text{(by Lemma 2.5)} \\
&\le \frac{(\operatorname{tr}(|A|))^{nm}}{n^{nm}m^m},
\end{aligned}$$

which means that

$$\det\bigl(\det_2(\operatorname{Re} A)\bigr) \le \frac{(\operatorname{tr}(|A|))^{nm}}{n^{nm}m^m}.$$

On the other hand, by $\det_1(\operatorname{Re} A) = \det_2(\widetilde{\operatorname{Re} A})$ and Lemma 2.1, we have

$$\det\bigl(\det_1(\operatorname{Re} A)\bigr) = \det(\det_2 \widetilde{\operatorname{Re} A}) = \det\bigl(\det_2(\operatorname{Re} \widetilde{A})\bigr) \le \frac{(\operatorname{tr}(|\tilde{A}|))^{mn}}{m^{mn}n^n} = \frac{(\operatorname{tr}(|A|))^{mn}}{m^{mn}n^n}. \qquad \square$$

We would like to know whether or not the inequalities above hold in the case of replacing A with A^τ. Now, we will present a result on sector block matrices that is the same as it was under the positive-semidefinite condition.

Theorem 2.8 *If* $A = [A_{i,j}]_{i,j=1}^{m} \in \mathbf{M}_m(\mathbf{M}_n)$ *is a sector block matrix, then*

$$\det_1\bigl(A^\tau\bigr) = \det_1 A \quad \textit{and} \quad \det_2\bigl(A^\tau\bigr) = (\det_2 A)^T = \det_2\bigl(A^T\bigr).$$

Proof Since $A^\tau = [A_{j,i}]_{i,j=1}^{m}$ and $\widetilde{A} = [G_{l,k}]_{l,k=1}^{n}$, we have $\widetilde{A^\tau} = [G_{l,k}^{\ T}]_{l,k=1}^{n}$.

Hence,

$$\begin{aligned}\det_1\left(A^\tau\right) &= \det_2\left(\widetilde{A^\tau}\right)\\ &= \left[\det G_{l,k}{}^T\right]_{l,k=1}^n\\ &= \left[\det\left[a_{l,k}^{j,i}\right]_{i,j=1}^m\right]_{l,k=1}^n\\ &= \left[\det\left[a_{l,k}^{i,j}\right]_{i,j=1}^m\right]_{l,k=1}^n\\ &= [\det G_{l,k}]_{l,k=1}^n\\ &= \det_1 A.\end{aligned}$$

On the other hand,

$$\begin{aligned}\det_2\left(A^T\right) &= \left[\det A^T{}_{i,j}\right]_{i,j=1}^m\\ &= \left[\det A_{j,i}{}^T\right]_{i,j=1}^m\\ &= \left[\det\left[a_{k,l}^{j,i}\right]_{l,k=1}^n\right]_{i,j=1}^m\\ &= \left[\det\left[a_{l,k}^{j,i}\right]_{l,k=1}^n\right]_{i,j=1}^m\\ &= \det_2\left(A^\tau\right)\\ &= [\det A_{j,i}]_{i,j=1}^m\\ &= \left([\det A_{i,j}]_{i,j=1}^m\right)^T\\ &= (\det_2 A)^T.\end{aligned}$$

□

The following result for sector block matrices involving partial determinants of A^τ can be regarded as a complement of Theorem 2.7.

Theorem 2.9 *Let* $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ *be a sector matrix. Then,*

$$\det\left(\det_1(\operatorname{Re} A)^\tau\right) \le \frac{(\operatorname{tr}(|A|))^{mn}}{m^{mn}n^n}$$

and

$$\det\left(\det_2(\operatorname{Re} A)^\tau\right) \le \frac{(\operatorname{tr}(|A|))^{mn}}{n^{nm}m^m}.$$

Proof The proof is similar to that of Theorem 2.7. □

Remark 2 In fact, the analogous inequalities below for partial traces are also valid using a similar idea to that of Lemma 2.5:

$$\det\left(\operatorname{tr}_1\left(\operatorname{Re} A^\tau\right)\right) \le \left(\frac{\operatorname{tr}(\operatorname{Re} A)}{n}\right)^n \le \left(\frac{\operatorname{tr}(|A|)}{n}\right)^n$$

and

$$\det\left(\operatorname{tr}_2\left(\operatorname{Re} A^\tau\right)\right) \le \left(\frac{\operatorname{tr}(\operatorname{Re} A)}{m}\right)^m \le \left(\frac{\operatorname{tr}(|A|)}{m}\right)^m.$$

Next, we give inequalities for partial determinants of A^τ involving unitarily invariant norms.

Theorem 2.10 *Let* $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ *be a sector matrix. Then, the inequalities*

$$\left\|\det_1\left(\operatorname{Re} A^\tau\right)\right\| \leq \left\|\left(\frac{\operatorname{tr}|A|}{m}\right)^m I_n\right\|$$

and

$$\left\|\det_2\left(\operatorname{Re} A^\tau\right)\right\| \leq \left\|\left(\frac{\operatorname{tr}|A|}{n}\right)^n I_m\right\|$$

hold for any unitarily invariant norm $\|\cdot\|$.

Proof Note that $\det_1((\operatorname{Re} A)^\tau) = \det_1(\operatorname{Re} A)$, $\det_2(\operatorname{Re} A^\tau) = \det_2((\operatorname{Re} A)^T)$ by Theorem 2.8 and $\operatorname{tr} A = \operatorname{tr}(A^T)$, hence the proof is similar to that of Theorem 2.6. □

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Competing interests
The authors declare no competing interests.

Author contributions
Xiaohui Fu wrote the main manuscript text and Lihong Hu, Abdul Haseeb Salarzay checked the proofs. All authors reviewed the manuscript

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References
1. Bhatia, R.: Matrix Analysis. Springer, New York (1997)
2. Choi, D.: Inequalities related to trace and determinant of positive semidefinite block matrices. Linear Algebra Appl. **532**, 1–7 (2017)
3. Fu, X., Lau, P.S., Tam, T.Y.: Inequalities on partial traces of positive semidefinite block matrices. Can. Math. Bull. **64**, 964–969 (2021)
4. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (2013)
5. Lin, M.: A determinantal inequality involving partial traces. Can. Math. Bull. **59**, 585–591 (2016)
6. Petz, D.: Quantum Information Theory and Quantum Statistics, in Theoretical and Mathematical Physics. Springer, Berlin (2008)
7. Thompson, R.: A determinantal inequality for positive definite matrices. Can. Math. Bull. **4**, 57–62 (1961)
8. Xu, H., Fu, X.: Inequalities for partial determinants of positive semidefinite block matrices. Linear Multilinear Algebra. https://doi.org/10.1080/03081087.2022.2158299

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