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The stability of minimal solution sets for set optimization problems via improvement sets

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Abstract

In this paper we investigate the stability of solution sets for set optimization problems via improvement sets. Some sufficient conditions for the upper semicontinuity, lower semicontinuity, and compactness of E-minimal solution mappings are given for parametric set optimization under some suitable conditions. We also give some examples to illustrate our main results.

MSC: 90C

Keywords: Improvement set; Set optimization; Upper semicontinuity; Lower semicontinuity

1 Introduction

In recent years, set optimization problems have received increasing attention and have been intensively discussed due to extensive applications in many areas such as vector optimization, vector variational inequalities, mathematical economics, game theory, engineering management, control system field, and many others; for details, see [1–6] and the references therein.

It is well known that the stability of solutions is a very interesting topic in the study of set optimization. Xu and Li [7] established the semicontinuity of minimal solution mappings and weak minimal solution mappings to a parametric set optimization problem by making use of the converse u-property of objective mappings. Han and Huang [8] studied the upper semicontinuity, lower semicontinuity, and convexity of solution mappings of parametric set optimization problems. Khoshkhabar-amiranloo [9] studied the semicontinuity and compactness of minimal solutions of parametric set optimization problems. Karuna and Lalitha [10] discussed the continuity of approximate weak efficient solution mappings of parametric set optimization problems under the strict quasiconvexity of the objective map. Under some suitable conditions Zhang et al. [11] discussed the semicontinuity and compactness of minimal solution mappings to a parametric set optimization with the general pre-order relations. Mao and Han [12] discussed the semicontinuity of solution mappings for parametric set optimization problems via improvement sets under several suitable conditions. Since the compactness of objective values is so strong, it limits the application of stability of set optimization problems. The main purpose of this paper is to investigate the upper semicontinuity, lower semicontinuity, and closedness of solution maps to the parametric set optimization problem via improvement sets under appropriate assumptions. In our results we have no compactness assumption on the objective maps. Moreover, the continuity of the objective map is replaced by (weak, converse) $\leq_E^l$-continuity assumptions, which is of great interest and importance. Our results extend and improve the corresponding ones of [7, 8, 10–14].

The rest of this paper is organized as follows. The next section presents some necessary notations, concepts, and results to be used throughout the paper. In Sect. 3 we obtain the upper semicontinuity, lower semicontinuity, and compactness of E-minimal solution mappings for a parametric set optimization problem under weaker and simpler assumptions.

2 Preliminaries

Throughout the paper, unless otherwise specified, we assume that X, Y, and Λ are real normed vector spaces. We assume that $K \subseteq Y$ is a convex, closed, and pointed cone with nonempty interior. We denote the family of nonempty subsets of Y by $P(Y)$. We denote by $\operatorname{int} A$, $\operatorname{cl} A$, and A^c the topological interior, the closure of A, and the complementary set of A, respectively.

A set $A \in P(Y)$ is called K-closed if $A + K$ is closed; K-proper if $A + K \neq Y$.

The sets of all minimal solutions and weak minimal solutions of $A \in P(Y)$ are defined as follows:

$$minA := \{a \in A : (A - a) \cap (-K) = \{0\}\};$$
$$WminA := \{a \in A : (A - a) \cap (-\operatorname{int} K) = \emptyset\}.$$

In [15], for any $A, B \in P(Y)$, the lower set less order relation $\leq_K^l$ and the strict lower set less order relation $\ll_K^l$ on $P(Y)$ are defined by

$$A \leq_K^l B \quad \Leftrightarrow \quad B \subseteq A + K;$$
$$A \ll_K^l B \quad \Leftrightarrow \quad B \subseteq A + \operatorname{int} K.$$

In [16], Chicco et al. defined the upper comprehensive set of a set $E \subseteq Y$ by

$$u - compr(E) := E + K.$$

Definition 2.1 ([16]) Let E be a nonempty subset in Y. E is called an improvement set with respect to K iff $0 \notin E$ and $E + K = E$.

In the sequel, we assume that $E \subseteq Y$ is an improvement set with $E \subseteq K \backslash \{0\}$. Dhigra and Lalitha [17] defined a set relation as follows:

$$A \leq_E^l B \quad \Leftrightarrow \quad B \subseteq A + E;$$
$$A \ll_E^l B \quad \Leftrightarrow \quad B \subseteq A + \operatorname{int} E.$$

Let $F: X \to 2^Y$ be a set-valued mapping with nonempty values and $G \subseteq X$ with $G \neq \emptyset$. We consider the following set optimization problem:

$$\begin{aligned} &(\text{SOP}) \quad \min \quad F(x) \\ &\qquad\qquad \text{s.t.} \quad x \in G. \end{aligned}$$

Definition 2.2 ([17]) An element $\bar{x} \in G$ is said to be

(i) a K-l-minimal solution of (SOP) if $x \in G$ such that $F(x) \preceq_K^l F(\bar{x})$ imply $F(\bar{x}) \preceq_K^l F(x)$;

(ii) an E-l-minimal solution of (SOP) if $x \in G$ such that $F(x) \preceq_E^l F(\bar{x})$ imply $F(\bar{x}) \preceq_E^l F(x)$;

(iii) a weak E-l-minimal solution of (SOP) if $x \in G$ such that $F(x) \ll_E^l F(\bar{x})$ imply $F(\bar{x}) \ll_E^l F(x)$.

Let $K_l(G)$, $E_l(G)$, $W_l(G)$ denote the K-l-minimal solution, E-l-minimal solution, and the weak E-l-minimal solution set of (SOP), respectively.

Remark 2.1 ([12]) $K_l(G) \subseteq E_l(G) \subseteq W_l(G)$.

We say that F is K-closed-valued (K-proper-valued) on G if $F(x)$ is K-closed (K-proper) for each $x \in G$.

Definition 2.3 Let G be a nonempty convex subset of X and E be an improvement set. A set-valued mapping $F: X \to 2^Y$ is said to be l-E-strictly quasiconvex at $\bar{x} \in G$ if, for any $x \in G$ with $x \neq \bar{x}$, $t \in (0,1)$

$$F(x) \preceq_E^l F(\bar{x}) \quad \Rightarrow \quad F\bigl(tx + (1-t)\bar{x}\bigr) \ll_E^l F(\bar{x}).$$

The map $F: X \to 2^Y$ is said to be l-E-strictly quasiconvex on G if it is l-E-strictly quasiconvex at every $x \in G$.

We use the following example to illustrate the existence of l-E-strictly quasiconvex for set-valued mappings.

Example 2.1 Let $X = Y = \mathbb{R}^2$, $G = [0,2] \times [0,2]$, and $K = \mathbb{R}^2_+$, $E = [0.1, +\infty) \times [0.1, +\infty)$. Consider $F: X \to 2^Y$ defined as

$$F(x_1, x_2) = \begin{cases} \{(3,3)\}, & \text{if } x_1 = x_2 = 0, \\ (0,1) + \mathbb{R}^2_+, & \text{if } x_1 = 0, 0 < x_2 \leq 1, \\ (1,0) + \mathbb{R}^2_+, & \text{if } 0 < x_1 \leq 1, x_2 = 0, \\ \{(5,5)\}, & \text{otherwise.} \end{cases}$$

We can verify that F is l-E-strictly quasiconvex at $(0,0)$.

Definition 2.4 ([4]) Let $F: X \to 2^Y$ be a set-valued mapping. Then F is said to be

(i) K-lower semicontinuous (K-l.s.c.) at $\bar{x}$ if, for any open subset U of Y with $F(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that $F(x) \cap (U - K) \neq \emptyset$ for all $x \in N(\bar{x})$.

(ii) K-upper semicontinuous (K-u.s.c.) at $\bar{x}$ if, for any open neighborhood U of $F(\bar{x})$, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for every $x \in N(\bar{x})$, $F(x) \subseteq U + K$.

(iii) K-Hausdorff lower semicontinuous (K-H-l.s.c.) at $\bar{x}$ if, for any open neighborhood U of zero, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for every $x \in N(\bar{x})$, $F(\bar{x}) \subseteq F(x) + U + K$.

(iv) K-Hausdorff upper semicontinuous (K-H-u.s.c.) at $\bar{x}$ if, for any open neighborhood U of zero, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for every $x \in N(\bar{x})$, $F(x) \subseteq F(\bar{x}) + U + K$.

We say that F is K-l.s.c., K-u.s.c., K-H-l.s.c., and K-H-u.s.c. on X if it is K-l.s.c., K-u.s.c., K-H-l.s.c., and K-H-u.s.c. at each point $x \in X$, respectively. Taking $K = \{0\}$, we get the definition of lower semicontinuity (l.s.c.), upper semicontinuity (u.s.c.), Hausdorff lower semicontinuity (H-l.s.c.), and Hausdorff upper semicontinuity (H-u.s.c.), respectively.

Remark 2.2 ([4, 6, 9]) u.s.c. $\Rightarrow$ H-u.s.c. $\Rightarrow$ K-H-u.s.c.; u.s.c. $\Rightarrow$ K-u.s.c. $\Rightarrow$ K-H-u.s.c.; and H-l.s.c. $\Rightarrow$ l.s.c. $\Rightarrow$ K-l.s.c.; H-l.s.c. $\Rightarrow$ K-H-l.s.c. $\Rightarrow$ K-l.s.c.

As pointed out in [4], F is H-l.s.c. (resp. K-H-l.s.c.) at x if F is l.s.c. (resp. K-l.s.c.) at x and $F(x)$ is compact. Moreover, F is K-u.s.c. (resp.u.s.c.) at x if F is K-H-u.s.c. (resp. H-u.s.c.) at x and $F(x)$ is compact.

Lemma 2.1 ([3, 10]) *A set-valued mapping $F : X \to 2^Y$ is l.s.c. at $\bar{x} \in X$ if and only if, for any sequence $\{x_n\} \subseteq X$ with $x_n \to \bar{x}$ and for any $\bar{y} \in F(\bar{x})$, there exists $y_n \in F(x_n)$ such that $y_n \to \bar{y}$.*

Lemma 2.2 ([3, 10]) *Let $F : X \to 2^Y$ be a set-valued mapping. For any given $\bar{x} \in X$, if $F(\bar{x})$ is compact, then F is u.s.c. at $\bar{x} \in X$. If and only if for any sequence $\{x_n\} \subseteq X$ with $x_n \to \bar{x}$ and for any $y_n \in F(x_n)$ there exist $\bar{y} \in F(\bar{x})$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \bar{y}$.*

Lemma 2.3 ([18]) *If E is an improvement set, then $E + \operatorname{int} K = \operatorname{int} E$ and $\operatorname{int} E + K = \operatorname{int} E$.*

Lemma 2.4 ([9]) *Let $A \in P(Y)$. Then $WminA \neq \emptyset$ if A is K-closed and K-proper.*

Lemma 2.5 ([12, 17]) *Let E be an improvement set. For any given $\bar{x} \in G$, if $F(\bar{x})$ is compact, then $\bar{x}$ is an E-l-minimal solution if and only if there is no $x \in G$ such that $F(x) \preceq_E^l F(\bar{x})$.*

Theorem 2.1 *Let E be an improvement set. For any given $\bar{x} \in G$, if $F(\bar{x})$ is K-closed and K-proper, then $\bar{x}$ is a weak E-l-minimal solution if and only if there is no $x \in G$ such that $F(x) \ll_E^l F(\bar{x})$.*

Proof In view of Lemma 2.4, we have $WminF(\bar{x}) \neq \emptyset$. Suppose that there exists $x \in G$ such that $F(x) \ll_E^l F(\bar{x})$. Since $\bar{x} \in W_l(G)$, we have $F(\bar{x}) \ll_E^l F(x)$. So, we get

$$F(\bar{x}) \subseteq F(x) + \operatorname{int} E \subseteq F(\bar{x}) + \operatorname{int} E + \operatorname{int} E \subseteq F(\bar{x}) + \operatorname{int} E + K \subseteq F(\bar{x}) + \operatorname{int} K. \tag{2.1}$$

Let $\bar{y} \in WminF(\bar{x})$. Consequently,

$$\big(F(\bar{x}) - \bar{y}\big) \cap (-\operatorname{int} K) = \emptyset. \tag{2.2}$$

It follows from (2.1) that there exist $y_0 \in F(\bar{x})$ and $k \in \operatorname{int} K$ such that $\bar{y} = y_0 + k$. Thus,

$$-k = y_0 - \bar{y} \in \big(F(\bar{x}) - \bar{y}\big) \cap (-\operatorname{int} K),$$

which contradicts (2.2). The converse is obvious. □

Lemma 2.6 ([12]) *Let A be a nonempty and compact subset of Y. Then $Min(A) \neq \emptyset$ and $Max(A) \neq \emptyset$. Furthermore, $A \nsubseteq A + K\backslash\{0\}$ and $A \nsubseteq A - K\backslash\{0\}$.*

Lemma 2.7 *Let G be a nonempty convex subset of X and E be an improvement set. $F : X \to 2^Y$ is l-E-strictly quasiconvex on G with nonempty compact values. Then $E_l(G) = W_l(G)$.*

Proof It suffices to prove that $W_l(G) \subseteq E_l(G)$. Let $\bar{x} \in W_l(G)$. If there exists $x_0 \in G$ such that $F(x_0) \leq_E^l F(\bar{x})$, then

$$F(\bar{x}) \subseteq F(x_0) + E.$$

Since $E \subseteq K\backslash\{0\}$, we get $F(\bar{x}) \subseteq F(x_0) + K\backslash\{0\}$. This together with Lemma 2.6 implies that $x_0 \neq \bar{x}$. Since F is *l*-*E*-strictly quasiconvex on G, one has

$$F\big(tx_0 + (1-t)\bar{x}\big) \ll_E^l F(\bar{x}).$$

By Theorem 2.1, we can see that $\bar{x} \notin W_l(G)$, which contradicts $\bar{x} \in W_l(G)$. Therefore, $\bar{x} \in E_l(G)$. □

Definition 2.5 Let G be a nonempty subset of X and $F : D \to 2^Y$ be a set-valued mapping. We say that

(i) F is $\leq_E^l$-continuous at $x_0 \in G$ with respect to $y_0 \in G$ if for all sequences $\{x_n\}$, $\{y_n\} \subseteq G$ satisfy $x_n \to x_0$, $y_n \to y_0$ such that $F(x_n) \leq_E^l F(y_n)$ for sufficiently large n, then $F(x_0) \leq_E^l F(y_0)$;

(ii) F is weak $\ll_E^l$-continuous at $x_0 \in G$ with respect to $y_0 \in G$ if for all sequences $\{x_n\}$, $\{y_n\} \subseteq G$ satisfy $x_n \to x_0$, $y_n \to y_0$ such that $F(x_n) \ll_E^l F(y_n)$ for sufficiently large n, then $F(x_0) \ll_E^l F(y_0)$;

(iii) F is converse $\leq_E^l$-continuous at $x_0 \in G$ with respect to $y_0 \in G$ if $F(x_0) \leq_E^l F(y_0)$ and for any sequences $\{x_n\}$, $\{y_n\} \subseteq G$ satisfy $x_n \to x_0$, $y_n \to y_0$, we have $F(x_n) \leq_E^l F(y_n)$ for sufficiently large n;

(iv) F is weak converse $\ll_E^l$-continuous at $x_0 \in G$ with respect to $y_0 \in G$ if $F(x_0) \ll_E^l F(y_0)$ and for any sequences $\{x_n\}$, $\{y_n\} \subseteq G$ satisfy $x_n \to x_0$, $y_n \to y_0$, we have $F(x_n) \ll_E^l F(y_n)$ for sufficiently large n.

F is $\leq_E^l$-continuous (resp., weak $\ll_E^l$-continuous) on G if F is $\leq_E^l$-continuous (resp., weak $\ll_E^l$-continuous) at each $x_0 \in G$ with respect to each $y_0 \in G$; F is converse $\leq_E^l$-continuous (resp., weak converse $\ll_E^l$-continuous) on G if F is converse $\leq_E^l$-continuous (resp., weak converse $\ll_E^l$-continuous) at each $x_0 \in G$ with respect to each $y_0 \in G$.

Proposition 2.1 *Let G be a nonempty subset of X. Assume that a set-valued mapping $F : G \to 2^Y$ is K-H-u.s.c. at $x_0 \in G$ and K-l.s.c. at $y_0 \in G$. If $F(x_0) + E$ is closed, then F is $\leq_E^l$-continuous at $x_0 \in G$ with respect to $y_0 \in G$.*

Proof Since $F: G \to 2^Y$ is K-H-u.s.c. at $x_0 \in G$, take two sequences $\{x_n\}$ and $\{y_n\}$ that satisfy $x_n \to x_0$, $y_n \to y_0$ such that $F(x_n) \preceq_E^l F(y_n)$ for sufficiently large n. Let $e \in \operatorname{int} K$ and $\epsilon > 0$. Then $-\epsilon e + \operatorname{int} K$ is an open neighborhood of 0 in Y. Since F is K-H-u.s.c. at x_0, we have $F(x_n) \subseteq F(x_0) - \epsilon e + \operatorname{int} K + K$ for sufficiently large n. Then we have

$$F(y_n) \subseteq F(x_0) - \epsilon e + \operatorname{int} K + K + E \subseteq F(x_0) - \epsilon e + E.$$

Since $F(x_0) + E$ is closed, letting $\epsilon \to 0$, we have

$$F(y_n) \subseteq F(x_0) + E. \tag{2.3}$$

We now claim that $F(x_0) \preceq_E^l F(y_0)$; otherwise, $F(y_0) \cap (F(x_0) + E)^C \neq \emptyset$. Since F is K-l.s.c. at y_0, we have $F(y_n) \cap ((F(x_0) + E)^C - K) \neq \emptyset$ for n sufficiently large. That is, $F(x_0) \npreceq_E^l F(y_n)$, (indeed, if $F(x_0) \preceq_E^l F(y_n)$, then we have $(F(y_n) + K) \subseteq (F(x_0) + E + K) = F(x_0) + E$, so we get $(F(y_n) + K)) \cap (F(x_0) + E)^C = \emptyset$. We conclude that $F(y_n) \cap ((F(x_0) + E)^C - K) = \emptyset$. This is a contradiction.), which contradicts (2.3). □

Proposition 2.2 *Let G be a nonempty subset of X. Assume that a set-valued mapping $F: G \to 2^Y$ is H-K-u.s.c. at $x_0 \in G$ and H-K-l.s.c. at $y_0 \in G$. If $F(x) + K$ is closed for any $x \in G$, then F is weak $\ll_E^l$-continuous at $x_0 \in G$ with respect to $y_0 \in G$.*

Proof Since $F: G \to 2^Y$ is H-K-u.s.c. at $x_0 \in G$ and H-K-l.s.c. at $y_0 \in G$. Take two sequences $\{x_n\}$ and $\{y_n\}$ that satisfy $x_n \to x_0$, $y_n \to y_0$ such that $F(x_n) \ll_E^l F(y_n)$ for n sufficiently large. Let $e \in \operatorname{int} K$ and $\epsilon > 0$. Then $-\epsilon e + \operatorname{int} K$ is a neighborhood of 0 in Y. We have $F(x_n) \subseteq F(x_0) - \epsilon e + \operatorname{int} K + K$ and $F(y_0) \subseteq F(y_n) - \epsilon e + \operatorname{int} K + K$ for n sufficiently large. Since $F(x_0) + K$ and $F(y_n) + K$ is closed, letting $\epsilon \to 0$, we have $F(x_n) \subseteq F(x_0) + K$ and $F(y_0) \subseteq F(y_n) + K$. Then we have

$$F(y_0) \subseteq F(y_n) + K \subseteq F(x_n) + \operatorname{int} E + K \subseteq F(x_0) + K + \operatorname{int} E + K \subseteq F(x_0) + \operatorname{int} E.$$

That is, $F(x_0) \ll_E^l F(y_0)$. □

Proposition 2.3 *Let G be a nonempty subset of X. Assume that a set-valued mapping $F: G \to 2^Y$ is H-K-u.s.c. at $y_0 \in G$ and H-K-l.s.c. at $x_0 \in G$. If $F(x) + K$ is closed for any $x \in G$, then F is converse $\preceq_E^l$-continuous and weak converse $\ll_E^l$-continuous at $x_0 \in G$ with respect to $y_0 \in G$.*

Proof Let $e \in \operatorname{int} K$ and $\epsilon > 0$, then $-\epsilon e + \operatorname{int} K$ is an open neighborhood of 0 in Y. Take two sequences $\{x_n\}$ and $\{y_n\}$ satisfy $x_n \to x_0$, $y_n \to y_0$ such that $F(x_0) \ll_E^l F(y_0)$. Since $F: G \to 2^Y$ is H-K-u.s.c. at $x_0 \in G$ and H-K-l.s.c. at $y_0 \in G$, we have $F(x_0) \subseteq F(x_n) - \epsilon e + \operatorname{int} K + K$ and $F(y_n) \subseteq F(y_0) - \epsilon e + \operatorname{int} K + K$ for n sufficiently large. Since $F(x_n) + K$ and $F(y_0) + K$ is closed, letting $\epsilon \to 0$, we have $F(x_0) \subseteq F(x_n) + K$ and $F(y_n) \subseteq F(y_0) + K$ for n sufficiently large. Then we have

$$F(y_n) \subseteq F(y_0) + K \subseteq F(x_0) + \operatorname{int} E + K \subseteq F(x_n) + K + \operatorname{int} E + K \subseteq F(x_n) + \operatorname{int} E,$$

for n sufficiently large. That is, $F(x_n) \ll_E^l F(y_n)$ for n sufficiently large.

The proof of converse $\preceq_E^l$-continuous property is similar to the proof of weak converse $\ll_E^l$-continuous property. □

Assume that $G:\Lambda \to 2^X$ and $F:X\to 2^Y$ are two set-valued mappings with nonempty values. For any $\lambda \in \Lambda$, now we consider the following parametric set optimization problem (for short, PSOP):

$$\text{(PSOP)}\quad \min \quad F(x)$$
$$\text{s.t.}\quad x\in G(\lambda).$$

We denote the solution mappings $E_l:\Lambda\to 2^X$, $W_l:\Lambda\to 2^X$ for (PSOP) as follows: $E_l(\lambda) = E_l(G(\lambda))$, $W_l(\lambda) = W_l(G(\lambda))$.

3 Main results

In this section, we make an investigation of the continuity of solution mappings for (PSOP). Firstly, we give the upper semicontinuity and compactness of the weak E-minimal solution mapping for (PSOP).

Theorem 3.1 *Let $\lambda_0 \in \Lambda$. Suppose that*

(i) *G is continuous and compact-valued at λ_0;*

(ii) *F is weak converse $\ll_E^l$-continuous on $G(\lambda_0)$, K-closed-valued, and K-proper-valued.*

Then $W_l(\lambda)$ is u.s.c. at λ_0 and $W_l(\lambda_0)$ is compact.

Proof We first assert that $W_l(\lambda)$ is u.s.c. at λ_0. Suppose on the contrary that $W_l(\lambda)$ is not u.s.c. at λ_0, hence there exist an open neighborhood W_0 in X with $W_l(\lambda_0)\subseteq W_0$ and a sequence $\{\lambda_n\}$ with $\lambda_n\to\lambda_0$ such that $W_l(\lambda_n)\nsubseteq W_0$. Therefore, there is a sequence $\{x_n\}$ with $x_n\in W_l(\lambda_n)$ and

$$x_n \notin W_0, \quad \forall n\in \mathbf{N}. \tag{3.1}$$

Since G is upper semicontinuous and compact-valued at λ_0, by Lemma 2.2, there exist $x_0\in G(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to x_0$. Without loss of generality, let $x_n \to x_0$.

We now claim that $x_0\in W_l(\lambda_0)$. If $x_0\notin W_l(\lambda_0)$, then by Theorem 2.1 there exists $y_0\in G(\lambda_0)$ such that $F(y_0)\ll_E^l F(x_0)$. By the lower semicontinuity of G at λ_0 and Lemma 2.1, there exists a sequence $\{y_n\}$ with $y_n\in G(\lambda_n)$ such that $y_n\to y_0$. F is weak converse $\ll_E^l$-continuous on $G(\lambda_0)$. By Proposition 2.3 we have $F(y_n)\ll_E^l F(x_n)$ for n sufficiently large, which is a contradiction to $x_n\in W_l(\lambda_n)$, and so $x_0\in W_l(\lambda_0)$. Therefore, from the assumption that $x_n\to x_0$, we have $x_n\in W_0$ for n large enough, which contradicts (3.1). Therefore, $W_l(\lambda)$ is u.s.c. at λ_0.

Next, we state that $W_l(\lambda_0)$ is compact. In fact, since $W_l(\lambda_0)\subseteq G(\lambda_0)$ and $G(\lambda_0)$ is compact, it is only sufficient to prove that $W_l(\lambda_0)$ is closed. Let $\{z_n\}\subseteq W_l(\lambda_0)$ be a sequence with $z_n\to z_0$. If $z_0\notin W_l(\lambda_0)$, then there exists $z^*\in G(\lambda_0)$ such that $F(z^*)\ll_E^l F(z_0)$. It yields the same proof as above that we have $F(z^*)\ll_E^l F(z_n)$ for n large enough, which contradicts $\{z_n\}\subseteq W_l(\lambda_0)$, and so $\{z_0\}\subseteq W_l(\lambda_0)$. Therefore $W_l(\lambda_0)$ is compact. □

Now, we give the following example to illustrate Theorem 3.1.

Example 3.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = [0,1]$, and $K = \mathbb{R}^2_+$, $E = [0.1,+\infty) \times [0.1,+\infty)$. Assume that $G(\lambda) = [0,\lambda]$ for all $\lambda \in \Lambda$. Let $F : X \to 2^Y$ be a set-valued mapping defined as $F(x) = [-1,x) \times [0,1)$ for all $x \in \mathbb{R}$. Let $\lambda_0 = 1$. Then it is easy to check that all conditions of Theorem 3.1 are satisfied. By a simple computation, we know that $W_l(\lambda) = [0,\lambda]$ for all $\lambda \in \Lambda$. Clearly, we can see that $W_l(\lambda)$ is u.s.c. at 1 and $W_l(1) = [0,1]$ is compact.

Theorem 3.2 *Let $\lambda_0 \in \Lambda$. Suppose that*

(i) *G is continuous and compact-valued at λ_0;*

(ii) *F is weak converse $\ll^l_E$-continuous on $G(\lambda_0)$, K-closed-valued, and K-proper-valued;*

(iii) *F is l-E-strictly quasiconvex on G.*

Then $E_l(\lambda)$ is u.s.c. at λ_0 and $E_l(\lambda_0)$ is compact.

Proof We show that $E_l(\lambda)$ is u.s.c. at λ_0. Suppose on the contrary that $E_l(\lambda)$ is not u.s.c. at λ_0, then there exist an open neighborhood W in X with $E_l(\lambda_0) \subseteq W$ and a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$E_l(\lambda_n) \nsubseteq W. \tag{3.2}$$

Since F is l-E-strictly quasiconvex on G, then by Lemma 2.7 we get $E_l(\lambda_0) = W_l(\lambda_0) \subseteq W$. By employing Theorem 3.1, it follows that $W_l(\lambda)$ is u.s.c. at λ_0; therefore, there exists a neighborhood V of λ_0 such that $W_l(\lambda) \subseteq W$ for all $\lambda \in V$. Since $\lambda_n \to \lambda_0$, it is clear that $\lambda_n \in V$ for n large enough. Thus, $E_l(\lambda_n) \subseteq W_l(\lambda_n) \subseteq W$ for n large enough, which is a contradiction to (3.2). Hence, $E_l(\lambda)$ is u.s.c. at λ_0. $E_l(\lambda_0) = W_l(\lambda_0)$ together with Theorem 3.1 indicates that $E_l(\lambda_0)$ is compact. □

Remark 3.1 By Remark 2.2 and Proposition 2.3, it is easy to see that Theorem 3.1 and Theorem 3.2 are the generalizations of Theorem 4.1 and Theorem 4.2 in [12], Theorem 3.1 and Theorem 3.7 in [10]. In Example 3.1, F is K-closed-valued but not compact-valued. Therefore, Theorem 4.1 and Theorem 4.2 in [12], as well as Theorem 3.1 and Theorem 3.7 in [10], cannot be used.

The following theorem establishes the lower semicontinuity of the E-minimal solution mapping for (PSOP).

Theorem 3.3 *Let $\lambda_0 \in \Lambda$. Assume that*

(i) *G is continuous and compact-valued at λ_0;*

(ii) *F is $\leq^l_E$-continuous on $G(\lambda_0)$ with nonempty and compact values and $F(x) + E$ is closed on $G(\lambda_0)$.*

Then $E_l(\lambda)$ is l.s.c. at λ_0.

Proof Assume on the contrary that $E_l(\lambda)$ is not l.s.c. at λ_0, then there exist $y \in E_l(\lambda_0)$, an open neighborhood W_0 of 0 in X, and a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$(y + W_0) \cap E_l(\lambda_n) = \emptyset, \quad \forall n \in N. \tag{3.3}$$

We conclude from $y \in E_l(\lambda_0)$ that $y \in G(\lambda_0)$. Since G is l.s.c. at λ_0, by Lemma 2.1 there exists a sequence $\{y_n\}$ with $y_n \in G(\lambda_n)$ such that $y_n \to y$. We now claim that $y_n \in E_l(\lambda_n)$

for n large enough. Indeed, if not, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that $y_{n_k} \notin E_l(\lambda_{n_k})$ for $k = 1, 2, \ldots$. Without loss of generality, we suppose $y_n \notin E_l(\lambda_n)$ for $n = 1, 2, \ldots$. Then by Lemma 2.5 there exists $x_n \in G(\lambda_n)$ such that $F(x_n) \preceq_E^l F(y_n)$. Since G is upper semicontinuous and compact-valued at λ_0, by Lemma 2.2, there exist $x \in G(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$. Without loss of generality, let $x_n \to x$. It follows from $F(x_n) \preceq_E^l F(y_n)$ and F is $\preceq_E^l$-continuous on $G(\lambda_0)$. By Proposition 2.1 we get $F(x) \preceq_E^l F(y)$, which contradicts $y \in E_l(\lambda_0)$. Hence $y_n \in E_l(\lambda_n)$ for n large enough. Therefore, it is obvious to find that $y_n \in (y + W_0) \cap E_l(\lambda_n)$ for n large enough, which contradicts (3.3). Hence, $E_l(\lambda)$ is l.s.c. at λ_0. □

Now, we give the following example to illustrate Theorem 3.3.

Example 3.2 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = [0, 1]$, and $K = \mathbb{R}^2_+$, $E = [0.1, +\infty) \times [0.1, +\infty)$. Assume that $G(\lambda) = [0, 1]$ for all $\lambda \in \Lambda$. Let $F : X \to 2^Y$ be a set-valued mapping defined as

$$F(x) = [0, x] \times [0, x], \quad \forall x \in G.$$

Let $\lambda_0 = 0$. Then it is easy to check that all conditions of Theorem 3.3 are satisfied. By a simple computation, we know that $E_l(\lambda) = [0, 1]$ for all $\lambda \in \Lambda$. Clearly, we can see that $E_l(\lambda)$ is l.s.c. at 0.

We show the lower semicontinuity of the weak E-minimal solution mapping for (PSOP) as follows.

Theorem 3.4 *Let $\lambda_0 \in \Lambda$. Assume that*

(i) *G is continuous and compact-valued at λ_0;*

(ii) *F is $\ll_E^l$-continuous on $G(\lambda_0)$ with nonempty and compact values and $F(x) + E$ is closed on $G(\lambda_0)$;*

(iii) *F is l-E-strictly quasiconvex on G.*

Then $W_l(\lambda)$ is l.s.c. at λ_0.

Proof Applying Theorem 3.3, we know that $E_l(\lambda)$ is l.s.c. at λ_0. By Lemma 2.7, we have $E_l(\lambda_0) = W_l(\lambda_0)$. For any open set V with $V \cap W_l(\lambda_0) \neq \emptyset$, so we have that $V \cap E_l(\lambda_0) \neq \emptyset$. Since $E_l(\lambda)$ is l.s.c. at λ_0, there exists a neighbourhood U of λ_0 such that, for all $\lambda \in U$,

$$V \cap E_l(\lambda) \neq \emptyset.$$

By Remark 2.1, we have $E_l(\lambda) \subseteq W_l(\lambda)$, then

$$V \cap W_l(\lambda) \neq \emptyset, \quad \forall \lambda \in U.$$

Hence, $W_l(\lambda)$ is l.s.c. at λ_0. □

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Declarations

Competing interests
The authors declare no competing interests.

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