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Overdetermined problems in annular domains with a spherical-boundary component in space forms

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Abstract
We obtain a Serrin-type symmetry of the solutions to various overdetermined boundary value problems in annular domains with a spherical-boundary component in space forms by using the maximum principle for suitable subharmonic functions and integral identities.

MSC: 35N25; 35R01; 53C24

Keywords: Overdetermined problem; Annular domain; Space form

1 Introduction

In his pioneering work [18], Serrin proved that if there exists a solution of the following overdetermined boundary value problem for a smooth bounded open connected domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \text{const} = c & \text{on } \partial\Omega, \end{cases}$$

then the solution u is radially symmetric and Ω is a ball. Here, ν denotes the outward unit normal to $\partial\Omega$. His proof is based on the moving-plane method, which was initiated by Alexandrov [2]. Immediately, Weinberger [20] gave an alternative simple proof of Serrin's symmetry result, which was based on the maximum principle for a suitable subharmonic function and some integral identities. Thereafter, overdetermined boundary value problems have been actively studied. For instance, Serrin's symmetry result has been generalized into space forms (see [4–6, 9, 11, 15, 17] for example and references therein).

On the other hand, one may still expect the radial symmetry of the solutions to overdetermined problems in annular domains. To describe this precisely, let Ω_0 and Ω_1 be simply connected bounded C^2 domains in $\mathbb{R}^n$ ($n \geq 2$) such that $\overline{\Omega}_1 \subset \Omega_0$. For the annular domain

$\Omega := \Omega_0 \setminus \overline{\Omega}_1$, consider the following overdetermined boundary value problem

$$\begin{cases} \Delta u = -1 & \text{in } \overline{\Omega}, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = c_0 & \text{on } \partial\Omega_0, \\ u = a > 0, \quad \frac{\partial u}{\partial \nu} = c_1 & \text{on } \partial\Omega_1, \end{cases} \tag{1.1}$$

where ν is the outward unit normal to $\partial\Omega$ and c_0, c_1, and a are real constants. In 1990, Philippin [13] proved that if each domain Ω_i $(i = 0, 1)$ is star shaped then the solution to the overdetermined problem (1.1) is radially symmetric and the domain Ω is a standard annulus (see also [1, 12] for more general related results). Under the additional condition that $0 \le u \le a$ in $\overline{\Omega}$, Reichel [16] obtained the same result. Later, Sirakov [19] removed the extra condition (see [8, 21] for $n = 2$). Recently, Kamburov-Sciaraffia [7] constructed a bounded real analytic annular domain $\Omega \subset \mathbb{R}^n$, which is different from a standard annulus, satisfying that the overdetermined problem (1.1) admits a solution $u \in C^\infty(\overline{\Omega})$ with $a > 0$ and $c_0 = c_1 < 0$.

In this paper, we generalize the above Serrin-type result for annular domains in $\mathbb{R}^n$ to space forms by using the maximum principle for suitable subharmonic functions and some integral identities. This approach was also used to obtain the radial symmetry of solutions to partially overdetermined problems in domains inside a convex cone in space forms by the authors of [10]. In Sect. 2, we study overdetermined problems in annular domains with inner spherical boundary in the unit sphere $\mathbb{S}^n$. For the equations $\Delta u = -n \cos r$ and $\Delta u + nu = -n$, we obtain radial-symmetry results (see Theorem 2.1 and Theorem 2.2). Moreover, we prove rigidity theorems (Theorem 2.4 and Theorem 2.5) for annular domains in $\mathbb{S}^n$ under suitable conditions on the inner spherical boundary. In Sect. 3, we consider the equation $\Delta u - nu = -n$ on an annular domain with outer spherical boundary in the hyperbolic space $\mathbb{H}^n$. In Theorem 3.2, we are able to prove a Serrin-type symmetry result for such domains. Furthermore, under the additional assumption that the annular domain is weakly star shaped (see Definition 2.3), we obtain the same result with a weakened Dirichlet condition on the outer spherical boundary (see Theorem 3.3).

2 Annular domains with inner spherical boundary

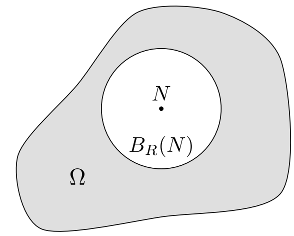
In this section, we study overdetermined boundary value problems in an annular domain whose inner boundary is spherical. Before we state our results, we start with some notations. Let M^n be an n-dimensional space form of constant sectional curvature $K = 0, 1$, and -1: the corresponding spaces are the Euclidean space $\mathbb{R}^n$, the unit sphere $\mathbb{S}^n$, and the hyperbolic space $\mathbb{H}^n$, respectively. These spaces can be regarded as the warped product space $M = I \times \mathbb{S}^{n-1}$ with the metric $g = dr^2 + h(r)^2 g_{\mathbb{S}^{n-1}}$, where r denotes the distance from the pole p of the model space and $g_{\mathbb{S}^{n-1}}$ denotes the round metric on $\mathbb{S}^{n-1}$. Moreover, the warping function $h(r)$ is given by

- $h(r) = r$ on $I = [0, \infty)$ in $\mathbb{R}^n$;
- $h(r) = \sin r$ on $I = [0, \pi)$ in $\mathbb{S}^n$;
- $h(r) = \sinh r$ on $I = [0, \infty)$ in $\mathbb{H}^n$.

Now we prove the radial symmetry of the solution to an overdetermined boundary value problem on annular domains in $\mathbb{S}^n$ with inner spherical boundary (see Fig. 1).

Theorem 2.1 *Let Ω be an annular domain in $\mathbb{S}^n \setminus B_R(N)$ such that $\partial B_R(N) \subset \partial\Omega$, where $B_R(N) \subset \mathbb{S}^n$ denotes the closed geodesic ball of radius $0 < R < \pi$ centered at the north pole*

Figure 1 An annular domain Ω with inner spherical boundary



$N \in \mathbb{S}^n$. *Suppose there is a solution* $u \in C^2(\overline{\Omega})$ *satisfying that*

$$\begin{cases} \Delta u = -n\cos r = -nh' & \textit{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = \text{const} = c_1 & \textit{on } \partial\Omega \setminus \partial B_R(N), \\ u = \text{const} = a > 0, \quad \frac{\partial u}{\partial \nu} = \sin R & \textit{on } \partial B_R(N), \end{cases}$$

where ν *is the outward unit normal to* $\partial\Omega$ *and* $r(x) = \operatorname{dist}(N, x)$. *Assume that either* Ω *is contained in the upper hemisphere* $\mathbb{S}^n_+$ *or* u *is positive. Then,* Ω *is the standard annulus* $\{x \in \mathbb{S}^n : R < r(x) < R_1\}$ *and the solution* u *is radial and is given by*

$$u(x) = \cos r(x) - \cos R_1,$$

where $R_1 = \sin^{-1}(-c_1)$.

Proof In the case where Ω is contained in $\mathbb{S}^n_+$, we see that u is positive in Ω by the maximum principle. Thus, we may assume that u is positive in Ω.

A straightforward computation yields

$$\operatorname{Hess} h' = -h' g \quad \text{and} \quad \Delta h' = -nh',$$

where $\operatorname{Hess} h'$ denotes the Hessian of h' and g denotes the metric of $\mathbb{S}^n$. Note that

$$(\Delta u)^2 \leq n \operatorname{tr}(\operatorname{Hess}^2 u), \tag{2.1}$$

where $\operatorname{Hess}^2 = \operatorname{Hess} \circ \operatorname{Hess}$. Moreover, equality holds if and only if $\operatorname{Hess} u$ is proportional to the metric g. By the polarized Bochner formula,

$$\begin{aligned} \Delta\langle \nabla(u - h'), \nabla u\rangle &= \langle \nabla(\Delta(u - h')), \nabla u\rangle + \langle \nabla(u - h'), \nabla(\Delta u)\rangle \\ &\quad + 2\operatorname{tr}(\operatorname{Hess}(u - h') \circ \operatorname{Hess} u) + 2\operatorname{Ric}(\nabla(u - h'), \nabla u). \end{aligned} \tag{2.2}$$

From (2.1), it follows that

$$\operatorname{tr}(\operatorname{Hess}(u - h') \circ \operatorname{Hess} u) = \operatorname{tr}(\operatorname{Hess}^2 u) + h' \Delta u = \operatorname{tr}(\operatorname{Hess}^2 u) - nh'^2 \geq 0.$$

Thus, (2.2) becomes

$$\Delta\langle \nabla(u - h'), \nabla u\rangle \geq -n\langle \nabla(u - h'), \nabla h'\rangle + 2(n-1)\langle \nabla(u - h'), \nabla u\rangle.$$

Since $u > 0$ in Ω, we obtain

$$\begin{aligned}&\int_\Omega u\Delta\langle\nabla(u-h'),\nabla u\rangle\,dV\\&\quad\geq -n\int_\Omega u\langle\nabla(u-h'),\nabla h'\rangle\,dV+2(n-1)\int_\Omega u\langle\nabla(u-h'),\nabla u\rangle\,dV.\end{aligned}\tag{2.3}$$

Since

$$\frac{\partial h'}{\partial\nu}=\sin R=\frac{\partial u}{\partial\nu}\quad\text{on }\partial B_R(N),$$

using the divergence theorem, we obtain

$$\begin{aligned}\int_\Omega\langle\nabla(u-h'),\nabla(u^2)\rangle\,dV&=\int_\Omega\operatorname{div}\bigl(u^2\nabla(u-h')\bigr)\,dV-\int_\Omega u^2\Delta(u-h')\,dV\\&=\int_{\partial\Omega}u^2\frac{\partial}{\partial\nu}(u-h')\,d\sigma=0,\end{aligned}$$

which yields

$$\int_\Omega u\langle\nabla(u-h'),\nabla u\rangle\,dV=0.\tag{2.4}$$

From (2.4),

$$\begin{aligned}\int_\Omega u\langle\nabla(u-h'),\nabla h'\rangle\,dV&=-\int_\Omega u\langle\nabla(u-h'),-\nabla h'\rangle\,dV\\&=-\int_\Omega u\bigl|\nabla(u-h')\bigr|^2\,dV.\end{aligned}\tag{2.5}$$

Combining (2.3), (2.4), and (2.5),

$$\int_\Omega u\Delta\langle\nabla(u-h'),\nabla u\rangle\,dV\geq n\int_\Omega u\bigl|\nabla(u-h')\bigr|^2\,dV\geq 0.\tag{2.6}$$

On the other hand, from Green's identity,

$$\begin{aligned}\int_\Omega u\Delta\langle\nabla(u-h'),\nabla u\rangle\,dV&=\int_\Omega\langle\nabla(u-h'),\nabla u\rangle\Delta u\,dV\\&\quad+\int_{\partial\Omega}u\frac{\partial}{\partial\nu}\langle\nabla(u-h'),\nabla u\rangle\,d\sigma\\&\quad-\int_{\partial\Omega}\langle\nabla(u-h'),\nabla u\rangle\frac{\partial u}{\partial\nu}\,d\sigma.\end{aligned}\tag{2.7}$$

Using the divergence theorem and the boundary conditions, we obtain

$$\begin{aligned}\int_\Omega\langle\nabla(u-h'),\nabla(uh')\rangle\,dV&=\int_\Omega\operatorname{div}\bigl(uh'\nabla(u-h')\bigr)\,dV-\int_\Omega uh'\Delta(u-h')\,dV\\&=\int_{\partial\Omega}uh'\frac{\partial}{\partial\nu}(u-h')\,d\sigma=0,\end{aligned}$$

which implies that

$$\begin{aligned}\int_{\Omega}\langle\nabla(u-h'),\nabla u\rangle\Delta u\,dV &= -n\int_{\Omega}h'\langle\nabla(u-h'),\nabla u\rangle\,dV \\ &= n\int_{\Omega}u\langle\nabla(u-h'),\nabla h'\rangle\,dV \\ &= -n\int_{\Omega}u\left|\nabla(u-h')\right|^2\,dV \le 0. \end{aligned}\tag{2.8}$$

Since $\partial\Omega$ is a level set of u, ∇u is parallel to ν on $\partial\Omega$. Thus,

$$\begin{aligned}\int_{\partial\Omega}\langle\nabla(u-h'),\nabla u\rangle\frac{\partial u}{\partial\nu}\,d\sigma &= {c_1}^2\int_{\partial\Omega\setminus\partial B}\langle\nabla(u-h'),\nu\rangle\,d\sigma \\ &= {c_1}^2\int_{\partial\Omega}\langle\nabla(u-h'),\nu\rangle\,d\sigma \\ &= {c_1}^2\int_{\Omega}\Delta(u-h')\,dV = 0. \end{aligned}\tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7),

$$\int_{\Omega}u\Delta\langle\nabla(u-h'),\nabla u\rangle\,dV \le a\int_{\partial B_R(N)}\frac{\partial}{\partial\nu}\langle\nabla(u-h'),\nabla u\rangle\,d\sigma.\tag{2.10}$$

To compute the right-hand side of (2.10), we choose a local orthonormal frame $\{e_i\}_{i=1}^n$ such that

$$e_n = \frac{\partial}{\partial r}.$$

Since u_r is constant on $\partial B_R(N)$ and each e_i is tangent to $\partial B_R(N)$ for all $i = 1,\dots,n-1$, we have

$$u_{ri} = 0 \quad \text{on } \partial B_R(N)\tag{2.11}$$

for all $i = 1,\dots,n-1$. Since $\partial B_R(N)$ is a level set of u,

$$u_i = 0 \quad\text{and}\quad u_{ij} = 0 \quad \text{on } \partial B_R(N)\tag{2.12}$$

for all $i,j = 1,\dots,n-1$. From (2.11) and (2.12),

$$\Delta u = (n-1)\frac{\cos r}{\sin r}u_r + u_{rr} \quad \text{on } \partial B_R(N),$$

which yields that

$$-nh' = (n-1)\frac{\cos R}{\sin R}(-\sin R) + u_{rr} \quad \text{on } \partial B_R(N).$$

That is,

$$u_{rr} = -nh' + (n-1)\cos R = -h' \quad \text{on } \partial B_R(N).$$

Note that

$$\begin{aligned}
\frac{\partial}{\partial \nu}\big\langle \nabla\big(u-h'\big), \nabla u\big\rangle &= \big\langle \nabla_\nu \nabla\big(u-h'\big), \nabla u\big\rangle + \big\langle \nabla\big(u-h'\big), \nabla_\nu \nabla u\big\rangle \\
&= 2\,\mathrm{Hess}\, u(\nabla u, \nu) - \mathrm{Hess}\, h'(\nabla u, \nu) - \mathrm{Hess}\, u\big(\nabla h', \nu\big) \\
&= 2\,\mathrm{Hess}\, u(\nabla u, \nu) + h'\frac{\partial u}{\partial \nu} - \mathrm{Hess}\, u\big(\nabla h', \nu\big) \\
&= 2u_{rr}\frac{\partial u}{\partial \nu} + h'\frac{\partial u}{\partial \nu} - u_{rr}\frac{\partial h'}{\partial \nu} \\
&= \sin R\big(u_{rr} + h'\big) = 0 \quad \text{on } \partial B_R(N).
\end{aligned}$$

Thus,

$$\int_\Omega u\Delta\big\langle \nabla\big(u-h'\big), \nabla u\big\rangle\, dV \leq 0. \tag{2.13}$$

Combining (2.6) and (2.13), we conclude that

$$\int_\Omega u\Delta\big\langle \nabla\big(u-h'\big), \nabla u\big\rangle\, dV = 0.$$

Note that the equality in (2.6) holds when $\nabla(u-h') \equiv 0$ in Ω. Thus,

$$u(x) = h' + c = \cos r + c$$

for some constant c. Since u vanishes on $\partial\Omega \setminus \partial B_R(N)$ and Ω is connected, the set $\partial\Omega \setminus \partial B_R(N)$ is the boundary of the geodesic ball $B_{R_1}(N)$ centered at N with radius $R_1 = \cos^{-1}(-c)$. Hence, Ω must be the standard annulus $\{x \in \mathbb{S}^n : R < r(x) < R_1\}$. Furthermore, the constant c can be expressed in terms of the constant c_1. To see this, observe that on $\partial\Omega \setminus \partial B_R(N)$

$$\frac{\partial u}{\partial \nu} = \langle -\sin r \nabla r, \nabla r\rangle = -\sin R_1 = c_1,$$

which implies that $R_1 = \sin^{-1}(-c_1)$. Therefore, the solution is given by

$$u(x) = \cos r + c = \cos r - \cos R_1,$$

where

$$R_1 = \sin^{-1}(-c_1). \qquad \square$$

We remark that the center N of the geodesic ball $B_R(N)$ can be replaced by any point $p \in \mathbb{S}^n$ in Theorem 2.1. In this case, the solution u is radially symmetric with respect to the point p. Note that consistency requires $-1 \leq c_1 < 0$ and $R < R_1$.

Theorem 2.2 *Let Ω be an annular domain in $\mathbb{S}^n_+ \setminus B_R(N)$ such that $\partial B_R(N) \subset \partial\Omega$, where $B_R(N) \subset \mathbb{S}^n_+$ denotes the closed geodesic ball of radius $0 < R < \frac{\pi}{2}$ centered at the north pole*

$N \in \mathbb{S}^n$. Given $R < R_1 < \frac{\pi}{2}$, suppose there is a solution $u \in C^2(\overline{\Omega})$ such that

$$\begin{cases} \Delta u + nu = -n & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = -\frac{\sin R_1}{\cos R_1} & \text{on } \partial\Omega \setminus \partial B_R(N), \\ u = \text{const} = a > 0, \quad \frac{\partial u}{\partial \nu} = \frac{\sin R}{\cos R_1} & \text{on } \partial B_R(N), \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$. Then, Ω is the standard annulus $\{x \in \mathbb{S}^n : R < r(x) < R_1\}$ and the solution u is radial and is given by

$$u(x) = \frac{1}{\cos R_1}\big(\cos r(x) - \cos R_1\big),$$

where $r(x) = \mathrm{dist}(N, x)$.

Proof Using the Bochner formula and (2.1), we have

$$\begin{aligned} \Delta|\nabla u|^2 &= 2\big\langle\nabla(\Delta u), \nabla u\big\rangle + 2\,\mathrm{tr}\big(\mathrm{Hess}^2\, u\big) + 2\,\mathrm{Ric}(\nabla u, \nabla u) \\ &\geq -2n|\nabla u|^2 + \frac{2}{n}(\Delta u)^2 + 2(n-1)|\nabla u|^2 \\ &= \frac{2}{n}(-n - nu)\Delta u - 2|\nabla u|^2 \\ &= -2\Delta u - \Delta u^2. \end{aligned} \tag{2.14}$$

Define

$$P(u) := |\nabla u|^2 + 2u + u^2.$$

Then, (2.14) shows that P is a subharmonic function in Ω. We claim that the function P is constant in Ω and the constant a satisfies that

$$a + 1 = \frac{\cos R}{\cos R_1}.$$

To see this, we first note that, on the boundary $\partial\Omega$, the function P is given by

$$P(u) = \begin{cases} \frac{\sin^2 R_1}{\cos^2 R_1} & \text{on } \partial\Omega \setminus \partial B_R(N), \\ \frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2 & \text{on } \partial B_R(N). \end{cases}$$

Suppose that

$$\frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2 > \frac{\sin^2 R_1}{\cos^2 R_1}.$$

Then,

$$(a+1)^2 > \frac{\sin^2 R_1 - \sin^2 R}{\cos^2 R_1} + 1 = \frac{1 - \cos^2 R_1 - (1 - \cos^2 R)}{\cos^2 R_1} + 1 = \frac{\cos^2 R}{\cos^2 R_1}.$$

Since a is positive,

$$a+1>\frac{\cos R}{\cos R_1}. \tag{2.15}$$

By the maximum principle,

$$\max_{\Omega} P(u)=\max_{\partial\Omega} P(u)=\frac{\sin^2 R}{\cos^2 R_1}+2a+a^2.$$

Since P is not a constant function under our assumption, we have

$$\frac{\partial P}{\partial \nu}>0 \quad \text{on } \partial B_R(N) \tag{2.16}$$

by the Hopf lemma. Choose a local orthonormal frame $\{e_i\}_{i=1}^n$ such that $e_n=\frac{\partial}{\partial r}$. Since u_r is constant on $\partial B_R(N)$ and each e_i is tangent to $\partial B_R(N)$ for $i=1,\dots,n-1$,

$$u_{ri}=0 \quad \text{on } \partial B_R(N) \tag{2.17}$$

for all $i=1,\dots,n-1$. Since $\partial B_R(N)$ is a level set of u,

$$u_i=0 \quad \text{and} \quad u_{ij}=0 \quad \text{on } \partial B_R(N) \tag{2.18}$$

for all $i,j=1,\dots,n-1$. From (2.17) and (2.18),

$$\Delta u=(n-1)\frac{\cos r}{\sin r}u_r+u_{rr} \quad \text{on } \partial B_R(N),$$

which yields that

$$-n-na=(n-1)\frac{\cos R}{\sin R}\left(-\frac{\sin R}{\cos R_1}\right)+u_{rr} \quad \text{on } \partial B_R(N).$$

That is,

$$u_{rr}=-n-na+(n-1)\frac{\cos R}{\cos R_1} \quad \text{on } \partial B_R(N).$$

Thus,

$$\begin{aligned}\frac{\partial P}{\partial \nu}&=2\operatorname{Hess} u(\nabla u,\nu)+2\frac{\partial u}{\partial \nu}+2u\frac{\partial u}{\partial \nu}\\&=2u_{rr}\frac{\partial u}{\partial \nu}+2\frac{\partial u}{\partial \nu}+2u\frac{\partial u}{\partial \nu}\\&=2(n-1)\frac{\sin R}{\cos R_1}\left(\frac{\cos R}{\cos R_1}-1-a\right) \quad \text{on } \partial B_R(N).\end{aligned}$$

By (2.15),

$$\frac{\partial P}{\partial \nu}<0 \quad \text{on } \partial B_R(N),$$

which is a contradiction to (2.16). Therefore, we see that

$$\frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2 \le \frac{\sin^2 R_1}{\cos^2 R_1}. \tag{2.19}$$

Then,

$$-\frac{\cos R}{\cos R_1} \le a + 1 \le \frac{\cos R}{\cos R_1}. \tag{2.20}$$

Suppose that P is not a constant function. Then, we obtain

$$P < \frac{\sin^2 R_1}{\cos^2 R_1} \quad \text{in } \Omega \tag{2.21}$$

by the maximum principle. Let

$$X = h\frac{\partial}{\partial r},$$

where $h(r) = \sin r$. Clearly,

$$h'(r) = \cos r > 0, \qquad h''(r) = -h(r), \quad \text{and} \quad \operatorname{div} X = nh' \quad \text{on } \mathbb{S}^n_+. \tag{2.22}$$

From (2.21),

$$\frac{\sin^2 R_1}{\cos^2 R_1}\int_\Omega h'\,dV > \int_\Omega Ph'\,dV = \int_\Omega \left(|\nabla u|^2 h' + u^2 h' + 2uh'\right) dV. \tag{2.23}$$

Note that

$$\langle \nabla h', \nabla u\rangle = \langle h''\nabla r, \nabla u\rangle = h'' u_r.$$

Applying the divergence theorem, we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2 h'\,dV &= \int_\Omega \langle \nabla(uh'), \nabla u\rangle\,dV - \int_\Omega u\langle \nabla h', \nabla u\rangle\,dV \\ &= \int_\Omega \operatorname{div}(uh'\nabla u)\,dV - \int_\Omega uh'\Delta u\,dV - \int_\Omega uh'' u_r\,dV \\ &= \int_{\partial\Omega} uh'\frac{\partial u}{\partial \nu}\,d\sigma - \int_\Omega uh'(-n-nu)\,dV + \int_\Omega uhu_r\,dV \\ &= \frac{a\cos R\sin R}{\cos R_1}\left|\partial B_R(N)\right| + n\int_\Omega uh'\,dV \\ &\quad + n\int_\Omega u^2 h'\,dV + \int_\Omega uhu_r\,dV.\end{aligned} \tag{2.24}$$

Using (2.22), we see that

$$\begin{aligned}\int_\Omega uhu_r\,dV &= \int_\Omega u\left\langle \nabla u, h\frac{\partial}{\partial r}\right\rangle dV = \frac{1}{2}\int_\Omega \langle \nabla u^2, X\rangle\,dV \\ &= \frac{1}{2}\int_\Omega \operatorname{div}(u^2 X)\,dV - \frac{1}{2}\int_\Omega u^2 \operatorname{div}(X)\,dV\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\partial\Omega} u^2 \langle X, \nu\rangle \, d\sigma - \frac{n}{2}\int_{\Omega} u^2 h' \, dV \\
&= -\frac{a^2}{2}\sin R\left|\partial B_R(N)\right| - \frac{n}{2}\int_{\Omega} u^2 h' \, dV,
\end{aligned}
$$

which is equivalent to

$$
\int_{\Omega} u^2 h' \, dV = -\frac{a^2}{n}\sin R\left|\partial B_R(N)\right| - \frac{2}{n}\int_{\Omega} uhu_r \, dV. \tag{2.25}
$$

Substituting (2.24) and (2.25) into (2.23), we obtain

$$
\begin{aligned}
\frac{\sin^2 R_1}{\cos^2 R_1}\int_{\Omega} h' \, dV &> a\sin R\left|\partial B_R(N)\right|\left(\frac{\cos R}{\cos R_1} - \frac{a(n+1)}{n}\right) \\
&\quad + (n+2)\int_{\Omega} uh' \, dV - \left(1 + \frac{2}{n}\right)\int_{\Omega} uhu_r \, dV.
\end{aligned} \tag{2.26}
$$

Using the following Pohozaev identity [3–5, 14, 20]

$$
\operatorname{div}\left(\frac{|\nabla u|^2}{2}X - hu_r\nabla u\right) = \frac{n-2}{2}|\nabla u|^2 h' - hu_r\Delta u,
$$

we have

$$
\begin{aligned}
&\int_{\Omega} \operatorname{div}\left(|\nabla u|^2 X - 2hu_r\nabla u\right) dV \\
&\quad = (n-2)\int_{\Omega} |\nabla u|^2 h' \, dV - 2\int_{\Omega} hu_r(-n - nu) \, dV.
\end{aligned}
$$

From (2.24) and (2.25),

$$
\begin{aligned}
&\int_{\Omega} \operatorname{div}\left(|\nabla u|^2 X - 2hu_r\nabla u\right) dV \\
&\quad = (n-2)\int_{\Omega} |\nabla u|^2 h' \, dV + 2n\int_{\Omega} hu_r \, dV + 2n\int_{\Omega} uhu_r \, dV \\
&\quad = (n-2)\left(\frac{a\cos R\sin R}{\cos R_1}\left|\partial B_R(N)\right| + n\int_{\Omega} uh' \, dV + n\int_{\Omega} u^2 h' \, dV + \int_{\Omega} uhu_r \, dV\right) \\
&\qquad + 2n\int_{\Omega} \langle X, \nabla u\rangle \, dV + 2n\int_{\Omega} uhu_r \, dV \\
&\quad = (n-2)\left(\frac{a\cos R\sin R}{\cos R_1}\left|\partial B_R(N)\right| + n\int_{\Omega} uh' \, dV + n\int_{\Omega} u^2 h' \, dV\right) \\
&\qquad + (3n-2)\int_{\Omega} uhu_r \, dV + 2n\int_{\Omega} \operatorname{div}(uX) \, dV - 2n\int_{\Omega} u \operatorname{div}(X) \, dV \\
&\quad = (n-2)a\sin R\left|\partial B_R(N)\right|\left(\frac{\cos R}{\cos R_1} - a\right) - 2an\sin R\left|\partial B_R(N)\right| \\
&\qquad + (n+2)\left(\int_{\Omega} uhu_r \, dV - n\int_{\Omega} uh' \, dV\right).
\end{aligned} \tag{2.27}
$$

On the other hand, since

$$\nabla u = \frac{\partial u}{\partial \nu}\nu \quad \text{on } \partial\Omega,$$

applying the divergence theorem, we have

$$\begin{aligned}
&\int_\Omega \operatorname{div}\bigl(|\nabla u|^2 X - 2hu_r\nabla u\bigr)\,dV \\
&\quad = \int_{\partial\Omega} |\nabla u|^2 \langle X,\nu\rangle\,d\sigma - 2\int_{\partial\Omega} \langle X,\nabla u\rangle\langle \nabla u,\nu\rangle\,d\sigma \\
&\quad = -\int_{\partial\Omega} |\nabla u|^2 \langle X,\nu\rangle\,d\sigma \\
&\quad = -\frac{\sin^2 R_1}{\cos^2 R_1}\int_{\partial\Omega\setminus\partial B_R(N)} \langle X,\nu\rangle\,d\sigma - \frac{\sin^2 R}{\cos^2 R_1}\int_{\partial B_R(N)} \langle X,\nu\rangle\,d\sigma \\
&\quad = -\frac{\sin^2 R_1}{\cos^2 R_1}\int_\Omega \operatorname{div}(X)\,dV + \frac{\sin^2 R_1 - \sin^2 R}{\cos^2 R_1}\bigl(-\sin R\bigl|\partial B_R(N)\bigr|\bigr) \\
&\quad = -n\frac{\sin^2 R_1}{\cos^2 R_1}\int_\Omega h'\,dV - \frac{\sin R(\sin^2 R_1 - \sin^2 R)}{\cos^2 R_1}\bigl|\partial B_R(N)\bigr|.
\end{aligned} \tag{2.28}$$

Combining (2.27) and (2.28),

$$\begin{aligned}
n\frac{\sin^2 R_1}{\cos^2 R_1}\int_\Omega h'\,dV = {}& -\frac{\sin R(\sin^2 R_1 - \sin^2 R)}{\cos^2 R_1}\bigl|\partial B_R(N)\bigr| \\
& - (n-2)a\sin R\bigl|\partial B_R(N)\bigr|\left(\frac{\cos R}{\cos R_1} - a\right) \\
& + 2na\sin R\bigl|\partial B_R(N)\bigr| \\
& + (n+2)\left(n\int_\Omega uh'\,dV - \int_\Omega uhu_r\,dV\right).
\end{aligned} \tag{2.29}$$

Combining (2.26) and (2.29),

$$0 < \frac{\sin^2 R - \sin^2 R_1}{\cos^2 R_1} - 2a(n-1)\frac{\cos R}{\cos R_1} + 2an + (2n-1)a^2.$$

Using (2.19), we can rewrite the above inequality as

$$\begin{aligned}
0 &< -a^2 - 2a - 2a(n-1)\frac{\cos R}{\cos R_1} + 2na + (2n-1)a^2 \\
&= 2a(n-1)\left(a + 1 - \frac{\cos R}{\cos R_1}\right).
\end{aligned}$$

However, this is a contradiction to (2.20). Therefore, we see that P is a constant function. It follows that the two boundary values of P must be equal:

$$\frac{\sin^2 R_1}{\cos^2 R_1} = \frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2,$$

which gives

$$a + 1 = \frac{\cos R}{\cos R_1}. \tag{2.30}$$

Since $\Delta P = 0$ in Ω, the equality in (2.14) holds. This implies that $\operatorname{Hess} u$ must be proportional to the metric g by (2.1). Thus,

$$\operatorname{Hess} u = \frac{\Delta u}{n} g = (-1 - u)g. \tag{2.31}$$

Let $\gamma : J \to \mathbb{S}^n_+$ be a unit-speed maximal geodesic satisfying that

$$\gamma(0) = N, \qquad \nabla_{\gamma'(s)}\gamma'(s) = 0 \quad \text{and} \quad \left|\gamma'(s)\right|^2 = 1.$$

Define $f(s) := u(\gamma(s))$ on J. Then,

$$f'(s) = \langle \nabla u, \gamma'(s) \rangle$$

and by (2.31)

$$\begin{aligned} f''(s) &= \langle \nabla_{\gamma'(s)} \nabla u, \gamma'(s) \rangle + \langle \nabla u, \nabla_{\gamma'(s)}\gamma'(s) \rangle \\ &= \operatorname{Hess} u\big(\gamma'(s), \gamma'(s)\big) \\ &= \big(-1 - u\big(\gamma(s)\big)\big)\left|\gamma'(s)\right|^2 \\ &= -1 - f(s). \end{aligned}$$

By the boundary conditions along $\partial B_R(N)$,

$$f(R) = a \quad \text{and} \quad f'(R) = -\frac{\sin R}{\cos R_1}.$$

Thus, we obtain the following initial value problem:

$$\begin{cases} f''(s) + f(s) = -1 & \text{on } J, \\ f'(R) = -\frac{\sin R}{\cos R_1}, \\ f(R) = a. \end{cases}$$

The general solution is given by

$$f(s) = c_1 \cos s + c_2 \sin s - 1$$

for some constants c_1 and c_2. From the initial conditions,

$$\begin{aligned} -\frac{\sin R}{\cos R_1} &= f'(R) = -c_1 \sin R + c_2 \cos R, \\ a &= f(R) = c_1 \cos R + c_2 \sin R - 1. \end{aligned}$$

Using (2.30), we obtain

$$c_1 = (a+1)\cos R + \frac{\sin^2 R}{\cos R_1} = \frac{1}{\cos R_1},$$
$$c_2 = (a+1)\sin R - \frac{\sin R}{\cos R_1}\cos R = 0.$$

Thus, the solution to the initial value problem is given by

$$f(s) = \frac{\cos s}{\cos R_1} - 1.$$

Since $f(s)$ depends only on the distance, we conclude that

$$u(x) = \frac{1}{\cos R_1}\big(\cos r(x) - \cos R_1\big),$$

where $r(x) = \mathrm{dist}(x, N)$. Since u vanishes on $\partial\Omega \setminus \partial B_R(N)$ by the boundary condition, $\partial\Omega \setminus \partial B_R(N)$ is the boundary of the geodesic ball centered at N with radius R_1. Therefore, Ω is the standard annulus $\{x \in \mathbb{S}^n : R < r(x) < R_1\}$. □

In Theorem 2.2, we assumed that the annular domain is contained in the upper hemisphere $\mathbb{S}^n_+$ and assumed that $u = a > 0$ and $\frac{\partial u}{\partial \nu} > 0$ on the inner spherical boundary. Instead, assuming each boundary component of the annular domain is geometrically rather simple, we are able to prove the same radial symmetry of the domain in the case where $u = a < -1$ and $\frac{\partial u}{\partial \nu} < 0$ on the spherical boundary.

Definition 2.3 An annular domain Ω is called *weakly star shaped with respect to p* if each component of the boundary $\partial\Omega$ can be written as a graph over a geodesic sphere with center p.

Theorem 2.4 *Let Ω be an annular domain in $\mathbb{S}^n \setminus B_R(N)$ such that $\partial B_R(N) \subset \partial\Omega$, where $B_R(N) \subset \mathbb{S}^n_+$ denotes the closed geodesic ball of radius R centered at the north pole $N \in \mathbb{S}^n$. Assume that Ω is weakly star shaped with respect to N. Given $R < R_1 < \pi$, suppose there is a solution $u \in C^2(\overline{\Omega})$ such that*

$$\begin{cases} \Delta u + nu = -n & \text{in } \Omega \\ u = 0, \quad \frac{\partial u}{\partial \nu} = -\frac{\sin R_1}{\cos R_1} & \text{on } \partial\Omega \setminus \partial B_R(N) \\ u = a < -1, \quad \frac{\partial u}{\partial \nu} = \frac{\sin R}{\cos R_1} < 0 & \text{on } \partial B_R(N), \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$. Then, Ω is the standard annulus $\{x \in \mathbb{S}^n : R < r(x) < R_1\}$ and the solution u is radial and is given by

$$u(x) = \frac{1}{\cos R_1}\big(\cos r(x) - \cos R_1\big),$$

where $r(x) = \mathrm{dist}(N, x)$.

Proof Define two P-functions as follows:

$$P(u) = |\nabla u|^2 + 2u + u^2 \quad \text{and} \quad \widetilde{P}(u) = \langle \nabla u, \nabla h' \rangle + uh' + h',$$

where $h(r) = \sin r$ and $r(x) = \operatorname{dist}(N, x)$. Then,

$$\Delta P \geq 0 \quad \text{and} \quad \Delta \widetilde{P} = 0.$$

The function P on the boundary is given by

$$P(u) = \begin{cases} \frac{\sin^2 R_1}{\cos^2 R_1} & \text{on } \partial\Omega \setminus \partial B_R(N), \\ \frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2 & \text{on } \partial B_R(N). \end{cases}$$

Suppose that

$$\frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2 > \frac{\sin^2 R_1}{\cos^2 R_1}.$$

In this case, we use the same argument as in the proof of Theorem 2.2 to obtain a contradiction. Therefore, we see that

$$\frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2 \leq \frac{\sin^2 R_1}{\cos^2 R_1},$$

which gives

$$\frac{\cos R}{\cos R_1} \leq a + 1 \leq -\frac{\cos R}{\cos R_1}. \tag{2.32}$$

Now suppose that neither P nor $\widetilde{P}$ is a constant function. Choose an orthonormal frame $\{e_1, \dots, e_n\}$ on $\partial\Omega \setminus \partial B_R(N)$ such that $e_n = \nu$. Using the same argument as above, we obtain

$$u_i = 0, \qquad u_{\nu i} = 0, \qquad u_{ij} = 0 \quad \text{for all } i, j = 1, \dots, n-1.$$

By the Hopf lemma,

$$\begin{aligned} 0 < \frac{\partial P}{\partial \nu} &= 2 \operatorname{Hess} u(\nabla u, \nu) + 2\frac{\partial u}{\partial \nu} + 2u\frac{\partial u}{\partial \nu} \\ &= 2u_{\nu\nu}\frac{\partial u}{\partial \nu} + 2\frac{\partial u}{\partial \nu} \\ &= -\frac{2\sin R_1}{\cos R_1}(u_{\nu\nu} + 1) \quad \text{on } \partial\Omega \setminus \partial B_R(N). \end{aligned}$$

Since $\frac{\sin R}{\cos R_1} < 0$ by our assumption,

$$u_{\nu\nu} + 1 > 0 \quad \text{on } \partial\Omega \setminus \partial B_R(N). \tag{2.33}$$

Note that both the maximum and minimum values of the function $\widetilde{P}$ are attained on the boundary of Ω because $\widetilde{P}$ is a harmonic function. We have the following two possibilities:

(a) $\widetilde{P}$ has the maximum value at $y_1 \in \partial\Omega \setminus \partial B_R(N)$;
(b) $\widetilde{P}$ has the maximum value at $y_2 \in \partial B_R(N)$.

In case (a), by the Hopf lemma,

$$\begin{aligned}
0 < \frac{\partial \widetilde{P}}{\partial \nu}(y_1) &= \operatorname{Hess} u\big(\nabla h', \nu\big) + \operatorname{Hess} h'(\nabla u, \nu) + u\frac{\partial h'}{\partial \nu} + h'\frac{\partial u}{\partial \nu} + \frac{\partial h'}{\partial \nu} \\
&= u_{\nu\nu}\frac{\partial h'}{\partial \nu} - h'\frac{\partial u}{\partial \nu} + u\frac{\partial h'}{\partial \nu} + h'\frac{\partial u}{\partial \nu} + \frac{\partial h'}{\partial \nu} \\
&= (u_{\nu\nu} + 1)\frac{\partial h'}{\partial \nu}.
\end{aligned}$$

Note that

$$\frac{\partial h'}{\partial \nu} = -\sin r\langle \nabla r, \nu\rangle < 0 \quad \text{on } \partial\Omega \setminus \partial B_R(N),$$

since $\Omega \cup B_R(N)$ is star shaped with respect to N. Thus,

$$u_{\nu\nu}(y_1) + 1 < 0,$$

which is a contradiction to (2.33). In case (b),

$$0 < \frac{\partial \widetilde{P}}{\partial \nu}(y_2) = \frac{\partial h'}{\partial \nu}(n-1)\left(\frac{\cos R}{\cos R_1} - a - 1\right).$$

Note that

$$\frac{\partial h'}{\partial \nu} = -\sin r\langle \nabla r, \nu\rangle = \sin R > 0 \quad \text{on } \partial B_R(N).$$

However, (2.32) shows that

$$\frac{\partial \widetilde{P}}{\partial \nu}(y_2) \leq 0,$$

which is a contradiction. Therefore, either P or $\widetilde{P}$ is a constant function.

Suppose $\widetilde{P}$ is a constant function. This implies that

$$\frac{\partial \widetilde{P}}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

and

$$u_{\nu\nu} + 1 = 0 \quad \text{on } \partial\Omega \setminus \partial B_R(N).$$

Then,

$$\frac{\partial P}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \partial B_R(N).$$

Since P has the maximum value on $\partial\Omega \setminus \partial B_R(N)$, P is a constant function in Ω by the Hopf lemma. Thus, we may assume that P is a constant function. Therefore, the two boundary values are equal, which implies that

$$\frac{\sin^2 R_1}{\cos^2 R_1} = \frac{\sin^2 R}{\cos^2 R_1} + 2a + a^2.$$

The remaining part of the proof is exactly the same as that of Theorem 2.2. Finally, we conclude that

$$u(x) = \frac{1}{\cos R_1}\bigl(\cos r(x) - \cos R_1\bigr),$$

where $r(x) = \operatorname{dist}(N, x)$. Moreover, Ω is the standard annulus $\{x \in \mathbb{S}^n : R < r(x) < R_1\}$. □

In Theorem 2.4, changing the boundary conditions on $\partial B_R(N)$ into

$$u = a > -1 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \frac{\sin R}{\cos R_1} > 0 \quad \text{on } \partial B_R(N)$$

gives the same conclusion. More precisely, applying the same argument as in the proof of Theorem 2.4, we have the following.

Theorem 2.5 *Let Ω be an annular domain in $\mathbb{S}^n \setminus B_R(N)$ such that $\partial B_R(N) \subset \partial\Omega$, where $B_R(N) \subset \mathbb{S}^n_+$ denotes the geodesic ball of radius R centered at the north pole $N \in \mathbb{S}^n$. Assume that Ω is a weakly star-shaped domain with respect to N. Given $R < R_1 < \pi$, suppose there is a solution $u \in C^2(\overline{\Omega})$ such that*

$$\begin{cases} \Delta u + nu = -n & \text{in } \Omega \\ u = 0, \quad \frac{\partial u}{\partial \nu} = -\frac{\sin R_1}{\cos R_1} & \text{on } \partial\Omega \setminus \partial B_R(N) \\ u = a > -1, \quad \frac{\partial u}{\partial \nu} = \frac{\sin R}{\cos R_1} > 0 & \text{on } \partial B_R(N), \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$. Then, Ω is the standard annulus $\{x \in \mathbb{S}^n : R < r(x) < R_1\}$ and the solution u is radial and is given by

$$u(x) = \frac{1}{\cos R_1}\bigl(\cos r(x) - \cos R_1\bigr),$$

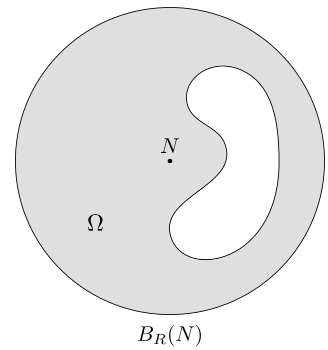
where $r = \operatorname{dist}(N, x)$.

3 Annular domains with outer spherical boundary

In the unit sphere $\mathbb{S}^n$, an overdetermined boundary value problem for an annular domain Ω with outer spherical boundary can be regarded as the problem for the domain with an inner spherical boundary. To be precise, for $p \in \mathbb{S}^n$, let Ω be a domain in $\mathbb{S}^n \setminus B_R(p)$ such that $\partial B_R(p) \subset \partial\Omega$, where $B_R(p)$ denotes the geodesic ball of radius R centered at p. Then, Ω is also a domain in $B_{\pi-R}(-p)$ such that $\partial B_{\pi-R}(-p) \subset \partial\Omega$. This observation gives the following result, which is basically the same as Theorem 2.1.

Theorem 3.1 *Let Ω be an annular domain in $B_R(N)$ such that $\partial B_R(N) \subset \partial\Omega$, where $B_R(N) \subset \mathbb{S}^n$ denotes the geodesic ball of radius $0 < R < \pi$ centered at the north pole $N \in \mathbb{S}^n$.*

Figure 2 An annular domain Ω with outer spherical boundary



Suppose there is a solution $u \in C^2(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = -n\cos r = -nh' & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = \text{const} = c_1 & \text{on } \partial\Omega \setminus \partial B_R(N), \\ u = \text{const} = a < 0, \quad \frac{\partial u}{\partial \nu} = -\sin R & \text{on } \partial B_R(N), \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$ and the function $h(r)$ is defined as before with $r(x) = \operatorname{dist}(N, x)$. Assume that either Ω is contained in $\mathbb{S}^n_+$ or u is negative. Then, Ω is an annulus $\{x \in \mathbb{S}^n : R_1 < r(x) < R\}$ and the radial solution u is given by

$$u(x) = \cos r(x) - \cos R_1.$$

In the following, we prove the radial symmetry of the solution to an overdetermined boundary value problem on annular domains in $\mathbb{H}^n$ with outer spherical boundary (see Fig. 2).

Theorem 3.2 *Let Ω be an annular domain in $B_R(p)$ such that $\partial B_R(p) \subset \partial\Omega$, where $B_R(p) \subset \mathbb{H}^n$ denotes the geodesic ball of radius R centered at $p \in \mathbb{H}^n$. Given $0 < R_1 < R$, suppose there is a solution $u \in C^2(\overline{\Omega})$ such that*

$$\begin{cases} \Delta u - nu = -n & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = \frac{\sinh R_1}{\cosh R_1} & \text{on } \partial\Omega \setminus \partial B_R(p), \\ u = \text{const} = a, \quad \frac{\partial u}{\partial \nu} = -\frac{\sinh R}{\cosh R_1} & \text{on } \partial B_R(p), \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$. If $1 - \frac{\cosh R}{\cosh R_1} \le a < 0$, then Ω is the standard annulus $\{x \in \mathbb{H}^n : R_1 < r(x) < R\}$. Moreover, the solution u is radial and is given by

$$u(x) = \frac{1}{\cosh R_1}\bigl(\cosh R_1 - \cosh r(x)\bigr),$$

where $r(x) = \operatorname{dist}(p, x)$.

Proof Let $P(u) := |\nabla u|^2 + 2u - u^2$. Then, from the Bochner formula,

$$\Delta P \ge 0.$$

Note that the maximum value of P is attained on the boundary of Ω by the maximum principle. Using the boundary conditions,

$$P(u) = \begin{cases} \frac{\sinh^2 R_1}{\cosh^2 R_1} & \text{on } \partial\Omega \setminus \partial B_R(p), \\ \frac{\sinh^2 R}{\cosh^2 R_1} + 2a - a^2 & \text{on } \partial B_R(p). \end{cases}$$

From the assumption that $1 - \frac{\cosh R}{\cosh R_1} \leq a < 0$, we have

$$(a-1)^2 \leq \frac{\cosh^2 R}{\cosh^2 R_1}.$$

Then,

$$a^2 - 2a \leq \frac{\cosh^2 R}{\cosh^2 R_1} - 1 = \frac{(\sinh^2 R + 1) - (\sinh^2 R_1 + 1)}{\cosh^2 R_1} = \frac{\sinh^2 R}{\cosh^2 R_1} - \frac{\sinh^2 R_1}{\cosh^2 R_1}.$$

Thus,

$$\frac{\sinh^2 R}{\cosh^2 R_1} + 2a - a^2 \geq \frac{\sinh^2 R_1}{\cosh^2 R_1},$$

which means that the function P defined on Ω attains its maximum value on $\partial B_R(p)$. Suppose P is not a constant function. From the Hopf lemma, it follows that

$$\frac{\partial P}{\partial \nu} > 0 \quad \text{on } \partial B_R(p). \tag{3.1}$$

Choose a local frame $\{e_i\}_{i=1}^n$ such that $e_n = \frac{\partial}{\partial r}$. Note that u_r is constant on $\partial B_R(p)$. Since each e_i is tangent to $\partial B_R(p)$ for $i = 1, \dots, n-1$, it follows that

$$u_{ri} = 0 \quad \text{on } \partial B_R(p) \tag{3.2}$$

for $i = 1, \dots, n-1$. Moreover, since $\partial B_R(p)$ is a level set of u,

$$u_i = 0 \quad \text{and} \quad u_{ij} = 0 \quad \text{on } \partial B_R(p) \tag{3.3}$$

for $i, j = 1, \dots, n-1$. From (3.2) and (3.3),

$$-n + na = \Delta u = (n-1)\frac{\cosh r}{\sinh r} u_r + u_{rr} = (n-1)\frac{\cosh R}{\sinh R}\left(-\frac{\sinh R}{\cosh R_1}\right) + u_{rr} \quad \text{on } \partial B_R(p),$$

which yields that

$$u_{rr} = -n + na + (n-1)\frac{\cosh R}{\cosh R_1} \quad \text{on } \partial B_R(p).$$

It follows that

$$\frac{\partial P}{\partial \nu} = -2(n-1)\frac{\sinh R}{\cosh R_1}\left(\frac{\cosh R}{\cosh R_1} - 1 + a\right) \leq 0 \quad \text{on } \partial B_R(p),$$

which is a contradiction to (3.1). Hence, P is a constant function and $\Delta P = 0$. As before, equality holds in (2.1), which implies that

$$\operatorname{Hess} u = \frac{\Delta u}{n} g = (u-1)g,$$

where g denotes the metric of $\mathbb{H}^n$. The same argument as in Theorem 2.2 shows that Ω is the standard annulus $\{x \in \mathbb{H}^n : R_1 < r(x) < R\}$ and

$$u(x) = \frac{1}{\cosh R_1}\big(\cosh R_1 - \cosh r(x)\big),$$

where $r(x) = \operatorname{dist}(p,x)$. $\square$

In the proof of Theorem 3.2, the assumption that $1 - \frac{\cosh R}{\cosh R_1} \leq a < 0$ is necessary. Instead, under the condition that the annular domain is weakly star shaped, the assumption on the constant a can be weakened as follows.

Theorem 3.3 *Let Ω be an annular domain in $B_R(p)$ such that $\partial B_R(p) \subset \partial\Omega$, where $B_R(p) \subset \mathbb{H}^n$ denotes the geodesic ball of radius R centered at $p \in \mathbb{H}^n$. Assume that Ω is weakly star shaped with respect to p. Given $0 < R_1 < R$, suppose there is a solution $u \in C^2(\overline{\Omega})$ such that*

$$\begin{cases} \Delta u - nu = -n & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = \frac{\sinh R_1}{\cosh R_1} & \text{on } \partial\Omega \setminus \partial B_R(p), \\ u = \text{const} = a < 0, \quad \frac{\partial u}{\partial \nu} = -\frac{\sinh R}{\cosh R_1} & \text{on } \partial B_R(p), \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$. Then, Ω is the standard annulus $\{x \in \mathbb{H}^n : R_1 < r(x) < R\}$. Moreover, the solution u is radial and is given by

$$u(x) = \frac{1}{\cosh R_1}\big(\cosh R_1 - \cosh r(x)\big),$$

where $r(x) = \operatorname{dist}(p,x)$.

Proof One can prove Theorem 3.3 by using the same argument as in the proof of Theorem 2.4. Here, we give the sketch of the proof. Define

$$P(u) := |\nabla u|^2 + 2u - u^2 \quad \text{and} \quad \widetilde{P} := \langle \nabla u, \nabla h' \rangle - uh' + h',$$

where $h(r) = \sinh r$ and $r(x) = \operatorname{dist}(p,x)$. Then,

$$\Delta P \geq 0 \quad \text{and} \quad \Delta \widetilde{P} = 0.$$

The boundary conditions show that

$$P(u) = \begin{cases} \frac{\sinh^2 R_1}{\cosh^2 R_1} & \text{on } \partial\Omega \setminus \partial B_R(p), \\ \frac{\sinh^2 R}{\cosh^2 R_1} + 2a - a^2 & \text{on } \partial B_R(p). \end{cases}$$

Suppose that

$$\frac{\sinh^2 R}{\cosh^2 R_1} + 2a - a^2 > \frac{\sinh^2 R_1}{\cosh^2 R_1}.$$

Then,

$$a - 1 > -\frac{\cosh R}{\cosh R_1}.$$

Applying the maximum principle,

$$\max_{\Omega} P(u) = \max_{\partial\Omega} P(u) = \frac{\sinh^2 R}{\cosh^2 R_1} + 2a - a^2.$$

By the Hopf lemma and the assumption,

$$0 < \frac{\partial P}{\partial \nu} = -2(n-1)\frac{\sinh R}{\cosh R_1}\left(\frac{\cosh R}{\cosh R_1} - 1 + a\right) < 0 \quad \text{on } \partial B_R(p),$$

which is a contradiction. Therefore, we see that

$$\frac{\sinh^2 R}{\cosh^2 R_1} + 2a - a^2 \le \frac{\sinh^2 R_1}{\cosh^2 R_1}. \tag{3.4}$$

Then,

$$a - 1 \le -\frac{\cosh R}{\cosh R_1}. \tag{3.5}$$

Suppose that neither P nor $\widetilde{P}$ is a constant function. Then,

$$0 < \frac{\partial P}{\partial \nu} = 2\frac{\sinh R_1}{\cosh R_1}(u_{\nu\nu} + 1) \quad \text{on } \partial\Omega \setminus \partial B_R(p),$$

which implies that

$$u_{\nu\nu} + 1 > 0 \quad \text{on } \partial\Omega \setminus \partial B_R(p). \tag{3.6}$$

Since $\widetilde{P}$ is harmonic, the maximum value of $\widetilde{P}$ is attained on $\partial\Omega$. Suppose that $\widetilde{P}$ has the maximum value at $y_1 \in \partial\Omega \setminus \partial B_R(p)$. By using (3.6), the Hopf lemma and the assumption that $B_R(p) \setminus \Omega$ is star shaped with respect to p,

$$0 < \frac{\partial \widetilde{P}}{\partial \nu}(y_1) = (u_{\nu\nu} + 1)\frac{\partial h'}{\partial \nu} = (u_{\nu\nu} + 1)\sinh r\langle \nabla r, \nu\rangle < 0,$$

which gives a contradiction. Now, suppose that $\widetilde{P}$ has the maximum value at $y_2 \in \partial B$. Similarly, by the Hopf lemma and (3.5),

$$\begin{aligned}0 &< \frac{\partial \widetilde{P}}{\partial \nu}(y_2) = \frac{\partial h'}{\partial \nu}(n-1)\left(\frac{\cosh R}{\cosh R_1} + a - 1\right) \\ &= \sinh r \langle \nabla r, \nu \rangle (n-1)\left(\frac{\cosh R}{\cosh R_1} + a - 1\right) \\ &\leq 0,\end{aligned}$$

which again gives a contradiction. Thus, either P or $\widetilde{P}$ is a constant function. Suppose $\widetilde{P}$ is a constant function. Then,

$$u_{\nu\nu} + 1 = 0 \quad \text{on } \partial\Omega \setminus \partial B_R(p),$$

which shows that

$$\frac{\partial P}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \partial B_R(p).$$

By (3.4), P attains its maximum value on $\partial\Omega \setminus \partial B_R(p)$. However, from the Hopf lemma, it follows that P is a constant function in Ω. Thus, we have $\Delta P = 0$ in Ω. Now, we can apply the same argument as in the proof of Theorem 2.2 to finish the proof. □

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Author contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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