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Formation of singularity of solution to a nonlinear shallow water equation

Sen Ming[1], Jiayi Du[1*], Yaxian Ma[1] and Yeqin Su[2]

*Correspondence:
dujy_0603@163.com
[1]Department of Mathematics, North University of China, College Road, 030051, Taiyuan, China
Full list of author information is available at the end of the article

Abstract

This paper is mainly concerned with behaviors of solution to the Cauchy problem for a generalized shallow water equation with dispersive term and dissipative term in the Besov space. It is shown that the problem of nonlinear shallow water equation is locally well posed. The $H^1(\mathbb{R})$ norm of solution to the problem is bounded under certain assumption on the initial value. Several blow-up criteria of solution are presented. The solution has compact support provided that the initial value has compact support. More specifically, the solution exponentially decays at infinity if the initial value exponentially decays at infinity. Our main new contribution is that the effects of coefficients λ and β on solution are illustrated. To the best of our knowledge, the results in Theorems 1.1–1.7 are new.

Keywords: Generalized shallow water equation; Locally well posed; Blow-up; Exponential decay

1 Introduction

The main aim of the present work is to consider the following Cauchy problem of a generalized shallow water equation:

$$\begin{cases} v_t - v_{xxt} + \beta(v_x - v_{xxx}) + \lambda(v - v_{xx}) \\ \quad = (1+\partial_x)(v^2 v_{xx} + v v_x^2 - 2v^2 v_x), \quad t > 0, x \in \mathbb{R}, \\ v(0,x) = v_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Here, $v(t,x)$ is the velocity of shallow water wave, $\beta(v_x - v_{xxx})(\beta \in \mathbb{R})$ is the dispersive term, $\lambda(v - v_{xx})(\lambda > 0)$ is the dissipative term. The initial value satisfies $v_0 \in B^s_{p,r}(\mathbb{R})(s > \max(\frac{3}{2}, 1 + \frac{1}{p}))$.

It is worthwhile pointing out that problem (1.1) and the Camassa–Holm (CH) equation

$$v_t - v_{xxt} + \beta v_x + 3vv_x = 2v_x v_{xx} + vv_{xxx}$$

are special equations of the following shallow water model:

$$(1 - \partial_x^2)v_t = F(v, v_x, v_{xx}, v_{xxx}),$$

which is investigated in [1]. In recent decades, local well-posedness for the Cauchy problem of the CH equation in $H^s(\mathbb{R})$ and $B^s_{p,r}(\mathbb{R})$ has attracted a lot of attention (see detailed instructions in [2–14]). Zhou and Chen [14] discovered that the solution v to problem (1.1) could be regarded as a perturbation around β by investigating asymptotic behavior of solution. The Cauchy problem of the CH equation with dissipative term and dispersion term was considered (see [9]). Finite time blow-up result and existence of global solution were obtained. Cui and Han [2] studied the asymptotic behavior of solution to a generalized CH equation. Gao et al. [3] established the dispersive regularization for a modified CH equation in one space dimension. Huang and Yu [4] derived the soliton and peakon solution to a generalized CH equation. Gui et al. [15] considered the nonlocal shallow water equation in two space dimensions by employing the asymptotic perturbation method. Local well-posedness and blow-up dynamics of solution to the Cauchy problem were demonstrated in the Sobolev space. Li and Zhang [7] discovered the generic regularity of conservative solution to a CH type equation by making use of the Thom transversality lemma. Silva and Freire [10] studied local well-posedness and traveling waves for the Cauchy problem of CH equation. Meanwhile, existence and uniqueness of solution were derived by applying the Kato approach. Mi et al. [8] demonstrated that the generalized CH equation is locally well posed in the Sobolev spaces $H^s(s > \frac{3}{2})$ in periodic and nonperiodic cases. Local well-posedness for the Cauchy problem of the periodic shallow water equation was proved in $H^s(\mathbb{T})(s > -n + \frac{3}{2}, n \geq 2)$ (see [12]). Zhang [13] investigated local well-posedness for the Cauchy problem of rotation CH equation on the torus $\mathbb{T}$. Peakon solutions to the μ-CH equation in $H^s(\mathbb{S})(s > \frac{7}{2})$ were discovered (see [11]). In terms of other dynamic properties of the generalized CH models, the readers are referred to [16–25] for more details.

There has been increasing interest in the other two shallow water models, namely, the Degasperis–Procesi (DP) equation

$$v_t - v_{xxt} + 4vv_x = 3v_x v_{xx} + vv_{xxx}$$

and the Novikov equation

$$v_t - v_{xxt} + 4v^2 v_x = 3vv_x v_{xx} + v^2 v_{xxx}.$$

The wave breaking phenomena of solution to the DP equation were considered (see [26]). Constantin and Ivanov [27] investigated soliton solution to the DP equation by utilizing the dressing method. Molinet [28] established asymptotic stability of the DP peakon. Cai et al. [29] considered the Lipschitz metric for the Novikov equation. Himonas et al. [30] obtained the construction of two-peakon solution for the Novikov equation. Zheng and Yin [31] established the wave breaking and solitary wave solution for a generalized Novikov equation. Blow-up mechanisms of solution to a degenerated Novikov equation in $H^s(\mathbb{R})(s > \frac{5}{2})$ were analyzed (see [32]).

Motivated by the previous works [2, 9, 21, 33, 34], we are devoted to investigating local well-posedness and several blow-up results of solution to the Cauchy problem of generalized shallow water Eq. (1.1). We observe that Li and Yin [21] have obtained local well-posedness and blow-up dynamics of solution to the Cauchy problem of the Camassa–Holm equation, which is a special case of problem (1.1) in the case $\lambda = \beta = 0$. The approaches were based on the transport equation and Littlewood–Paley theory in the Besov space. Constantin [35] investigated finite propagation speed for the Camassa–Holm equation. It was shown that classical solution to the Camassa–Holm equation has compact support if its initial data has compact support. Henry [36] considered infinite propagation speed for the Degasperis–Procesi equation. Himonas et al. [37] demonstrated persistence properties of solution to the Camassa–Holm equation. The results indicated that a solution to the Camassa–Holm equation decays exponentially when the initial value decays exponentially. The asymptotic behaviors of solution to a generalized CH equation were discussed (see [2]). More precisely, the solution does not have compact support in the framework of compactly supported initial value. In this work, it is shown that the Cauchy problem of generalized shallow water equation with dispersive term and dissipative term is locally well posed in the Besov space. The $H^1(\mathbb{R})$ norm of solution to the problem is bounded under certain assumption on the initial value. Meanwhile, we establish several blow-up criteria of solution to problem (1.1). We recognize that the solution has compact support provided that the initial value has compact support. The solution exponentially decays at infinity if the initial value exponentially decays at infinity. The advantage of the present paper is to derive the effects of dispersive term $\beta(v - v_{xxx})$ and dissipative term $\lambda(v - v_{xx})$ on behaviors of solution to problem (1.1). In addition, we extend parts of results in [2, 21]. To the best of authors' knowledge, the results in Theorems 1.1–1.7 are new.

Let $s \in \mathbb{R}$, $T > 0, p \in [1,\infty]$, $r \in [1,\infty]$. Here, we set the framework of space

$$E^s_{p,r}(T) = \begin{cases} C([0,T]; B^s_{p,r}(\mathbb{R})) \cap C^1([0,T]; B^{s-1}_{p,r}(\mathbb{R})), & 1 \leq r < \infty, \\ L^\infty([0,T]; B^s_{p,\infty}(\mathbb{R})) \cap \mathrm{Lip}([0,T]; B^{s-1}_{p,\infty}(\mathbb{R})), & r = \infty. \end{cases}$$

Assume $w_0(x) = (1 - \partial_x)v_0(x)$ and $w(t,x) = (1 - \partial_x)v(t,x)$. Therefore, problem (1.1) is reformulated as

$$\begin{cases} w_t + (v^2 + \beta)w_x = vw^2 - v^2 w - \lambda w, & t > 0, x \in \mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

or

$$\begin{cases} w_t + (v^2 + \beta)w_x = -vv_x w - \lambda w, & t > 0, x \in \mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

We summarize our results as follows.

Theorem 1.1 *Assume* $1 \leq p, r \leq \infty$, $v_0 \in B^s_{p,r}(\mathbb{R})(s > \max(\frac{3}{2}, 1 + \frac{1}{p}))$. *Then there exists a unique solution* $v \in E^s_{p,r}(T)$ *to problem* (1.1) *for a suitable positive constant* T.

Theorem 1.2 *Let* $1 \leq p, r \leq \infty$, $v_0 \in B^s_{p,r}(\mathbb{R})(s > \max(\frac{3}{2}, 1 + \frac{1}{p}))$. *Then a solution* v *to problem* (1.1) *satisfies*

$$\|v(t)\|_{H^1} \leq \|v_0\|_{H^1}, \quad t \in [0,T].$$

Theorem 1.3 *Let* $1 \le p, r \le \infty$, $v_0 \in B^s_{p,r}(\mathbb{R})(\max(\frac{3}{2}, 1 + \frac{1}{p}) < s < 2)$, $t \in [0, T]$. *Then a solution v to problem* (1.1) *blows up if and only if*

$$\int_0^t \left(\|w(\tau)\|^2_{L^\infty} - \lambda\right) d\tau = \infty.$$

Theorem 1.4 *Assume* $v_0 \in H^s(\mathbb{R})(s > \frac{3}{2})$ *and* $t \in [0, T]$. *Then a solution v to problem* (1.1) *blows up if and only if*

$$\lim_{t \to T^-} \left[\sup_{x \in \mathbb{R}} \left(vw(t, x) - \lambda\right)\right] = \infty.$$

Theorem 1.5 *Let* $1 \le p, r \le \infty$, $v_0 \in B^s_{p,r}(\mathbb{R})(s > \max(\frac{3}{2}, 1 + \frac{1}{p}))$, v_0 *is compactly supported in the interval* $[a_{u_0}, b_{u_0}]$. *Assume* $w_0 = v_0 - v_{0,x} \ge 0$. $T > 0$ *is the maximal existence time of the corresponding solution v to problem* (1.1). *Then the solution v is compactly supported in* $[p(t, a_{u_0}), p(t, b_{u_0})]$ *for all* $t \in [0, T)$.

Theorem 1.6 *Let* $v_0 \in H^s(\mathbb{R})(s > \frac{3}{2})$, $w_0 = v_0 - v_{0,x} \ge 0$. *Assume that* $v_0(v_0 - v_{0,x})(x_0) > 2\lambda + \|v_0\|^2_{H^1}$, *where* x_0 *is defined as* $v_0(v_0 - v_{0,x})(x_0) = \sup_{x \in \mathbb{R}}[v_0(v_0 - v_{0,x})]$. *Then a solution v to problem* (1.1) *blows up in finite time.*

Theorem 1.7 *Suppose that* $v_0 \in H^s(\mathbb{R})(s > \frac{5}{2})$ *and v is the corresponding solution to problem* (1.1). *Let* $t \in [0, T]$ *and* $\theta \in (0, 1)$. *Assume that* v_0 *satisfies*

$$|v_0(x)| \sim O(e^{-\theta x}) \quad \text{as } x \to \infty.$$

Then it holds that

$$|v(t, x)| \sim O(e^{-\theta x}) \quad \text{as } x \to \infty$$

uniformly on $[0, T]$.

Remark 1.1 Problem (1.1) is locally well posed in $B^s_{p,r}(\mathbb{R})(s > \max(\frac{3}{2}, 1 + \frac{1}{p}))$. We obtain that $\|v(t)\|_{H^1(\mathbb{R})}$ is bounded. Blow-up criterion of solution in the Besov space is shown in Theorem 1.3. It is illustrated in Theorem 1.4 that the wave breaking of solution u occurs in the case that vw is unbounded in finite time. Theorems 1.3, 1.4, and 1.6 indicate that the dissipative coefficient λ is related to the blow-up of solution. From Theorem 1.5, we observe that the solution has compact support provided that the initial value has compact support. From Theorem 1.7, we deduce that the solution exponentially decays at infinity if the initial value exponentially decays at infinity. Parts of the results in [2, 21] are extended.

2 Proof of Theorem 1.1

First of all, we show the proof in five steps. We note that $w_0 \in B^s_{p,r}(s > \max(\frac{1}{p}, \frac{1}{2}))$ in (1.2).

Step one: Let $w^0 = 0$ for all $t > 0, x \in \mathbb{R}$. We assume that a sequence of smooth functions $(w^i)_{i \in \mathbb{N}} \in C(\mathbb{R}^+; B^\infty_{p,r})$ satisfies

$$\begin{cases} (\partial_t + ((v^i)^2 + \beta)\partial_x)w^{i+1} = F(t, x), \\ w^{i+1}(0, x) = w^{i+1}_0(x) = S_{i+1}w_0 \end{cases} \tag{2.1}$$

and

$$F(t,x) = v^i\big(w^i\big)^2 - \big(v^i\big)^2 w^i - \lambda w^i. \tag{2.2}$$

We observe that $S_{i+1}w_0 \in B^{\infty}_{p,r}$. Taking advantage of Lemma 2.5 in [38], we derive that $w^i \in C(\mathbb{R}^+; B^{\infty}_{p,r})$ is global for all $i \in \mathbb{N}$.

Step two: It is deduced from Lemma 2.4 in [38] that

$$\begin{aligned}\big\|w^{i+1}(t)\big\|_{B^s_{p,r}} &\le e^{C_1\int_0^t \|\partial_x (v^i)^2(\tau)\|_{B^s_{p,r}}\,d\tau} \\ &\quad \times \bigg[\|w_0\|_{B^s_{p,r}} + \int_0^t e^{-C_1\int_0^\tau \|\partial_x (v^i)^2(\xi)\|_{B^s_{p,r}}\,d\xi} \big\|F(\tau,\cdot)\big\|_{B^s_{p,r}}\,d\tau\bigg].\end{aligned} \tag{2.3}$$

Making use of $v^i = (1-\partial_x^2)^{-1}(1+\partial_x)w^i$, we obtain

$$\big\|\partial_x\big(v^i\big)^2\big\|_{B^s_{p,r}} \lesssim \big\|w^i\big\|^2_{B^s_{p,r}}, \tag{2.4}$$

which results in

$$\big\|F(t,\cdot)\big\|_{B^s_{p,r}} \lesssim \big(1+\lambda+\big\|w^i(t)\big\|_{B^s_{p,r}}\big)^2 \big\|w^i(t)\big\|_{B^s_{p,r}}. \tag{2.5}$$

Applying (2.3), (2.4), and (2.5), we have

$$\begin{aligned}\big\|w^{i+1}(t)\big\|_{B^s_{p,r}} &\le C_2 e^{C_2\int_0^t (1+\lambda+\|w^i(\tau)\|_{B^s_{p,r}})^2\,d\tau}\bigg[\|w_0\|_{B^s_{p,r}} \\ &\quad + \int_0^t e^{-C_2\int_0^\tau (1+\lambda+\|w^i(\xi)\|_{B^s_{p,r}})^2\,d\xi} \\ &\quad \times \big(1+\lambda+\big\|w^i(\tau)\big\|_{B^s_{p,r}}\big)^2\big\|w^i(\tau)\big\|_{B^s_{p,r}}\,d\tau\bigg].\end{aligned} \tag{2.6}$$

Let the positive constant T satisfy $4C_2^3(1+\lambda+\|w_0\|_{B^s_{p,r}})^2 T < 1$ and

$$\big(1+\lambda+\big\|w^i(t)\big\|_{B^s_{p,r}}\big)^2 \le \frac{C_2^2(1+\lambda+\|w_0\|_{B^s_{p,r}})^2}{1-4C_2^3(1+\lambda+\|w_0\|_{B^s_{p,r}})^2 t}. \tag{2.7}$$

Thus, we calculate from (2.6) and (2.7) that

$$\big(1+\lambda+\big\|w^{i+1}(t)\big\|_{B^s_{p,r}}\big)^2 \le \frac{C_2^2(1+\lambda+\|w_0\|_{B^s_{p,r}})^2}{1-4C_2^3(1+\lambda+\|w_0\|_{B^s_{p,r}})^2 t}.$$

This implies that $(w^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$.

Step three: We set $i,j \in \mathbb{N}$. From (2.1), we observe

$$\begin{aligned}&\big(\partial_t + \big(\big(v^{i+j}\big)^2+\beta\big)\partial_x\big)\big(w^{i+j+1}-w^{i+1}\big) \\ &\quad = -\big[\big(v^{i+j}\big)^2-\big(v^i\big)^2\big]w_x^{i+1} + \big(v^{i+j}-v^i\big)\big(w^{i+j}\big)^2 \\ &\qquad + v^i\big(\big(w^{i+j}\big)^2-\big(w^i\big)^2\big) - \big(v^{i+j}-v^i\big)\big(v^{i+j}+v^i\big)w^{i+j}\end{aligned}$$

$$- \left(v^i\right)^2\left(w^{i+j} - w^i\right) - \lambda\left(w^{i+j} - w^i\right). \tag{2.8}$$

Employing Lemma 2.14 in [39] yields

$$\begin{aligned} \left\|w^{i+j+1} - w^{i+1}\right\|_{B^{s-1}_{p,r}} \le e^{C\int_0^t \|w^{i+j}\|^2_{B^s_{p,r}}\,d\tau}\Bigg[&\left\|w_0^{i+j+1} - w_0^{i+1}\right\|_{B^{s-1}_{p,r}} \\ &+ C\int_0^t e^{-C\int_0^\tau \|w^{i+j}\|^2_{B^s_{p,r}}\,d\xi}\left\|w^{i+j} - w^i\right\|_{B^{s-1}_{p,r}} \\ &\times \left(\left\|w^i\right\|_{B^s_{p,r}} + \left\|w^{i+j}\right\|_{B^s_{p,r}} + \left\|w^{i+1}\right\|_{B^s_{p,r}} + 1\right)^2 d\tau\Bigg]. \end{aligned} \tag{2.9}$$

We note that the initial values satisfy

$$w_0^{i+j+1} - w_0^{i+1} = \sum_{q=i+1}^{i+j} \Delta_q w_0.$$

We recognize that there exists a positive constant C_1 independent of i such that

$$\left\|w^{i+j+1} - w^{i+1}\right\|_{L^\infty([0,T];B^{s-1}_{p,r})} \le C_1 2^{-i}.$$

We arrive at the desired result.

Step four: Similar to the discussions in Step 4 in Sect. 3 in [38], we derive that the solution $w \in E^s_{p,r}(T)$ is continuous.

Step five: We are in the position to present the uniqueness of solution.

Let $(p,r) \in [1,\infty]^2, s > \max(\frac{1}{p}, \frac{1}{2}), w_0^1, w_0^2 \in B^s_{p,r}$. w^1 and w^2 satisfy (1.2). $w^1, w^2 \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; B^{s-1}_{p,r})$. We assume $w^{12} = w^1 - w^2$ and

$$w^{12} \in L^\infty\left([0,T]; B^s_{p,r}\right) \cap C\left([0,T]; B^{s-1}_{p,r}\right),$$

which satisfies

$$\begin{cases} (\partial_t + ((v^1)^2 + \beta)\partial_x)w^{12} = -v^{12}(v^1 + v^2)(w^2)_x + F_1(t,x), \\ w^{12}(0,x) = w_0^{12} = w_0^1 - w_0^2 \end{cases} \tag{2.10}$$

and

$$F_1(t,x) = v^{12}\left(w^1\right)^2 + v^2 w^{12}\left(w^1 + w^2\right) - v^{12}\left(v^1 + v^2\right)w^1 - \left(v^2\right)^2 w^{12} - \lambda w^{12}.$$

Utilizing Lemma 2.14 in [39], we achieve

$$\begin{aligned} &e^{-C\int_0^t \|w^1\|^2_{B^s_{p,r}}\,d\tau}\left\|w^{12}\right\|_{B^s_{p,r}} \\ &\quad \le \left\|w_0^{12}\right\|_{B^{s-1}_{p,r}} \\ &\qquad + C\int_0^t e^{-C\int_0^\tau \|w^1\|^2_{B^s_{p,r}}\,d\xi}\left\|w^{12}\right\|_{B^s_{p,r}}\left(1 + \lambda + \left\|w^1\right\|_{B^s_{p,r}} + \left\|w^2\right\|_{B^s_{p,r}}\right)^2 d\tau. \end{aligned}$$

It holds that

$$\|w^{12}\|_{B^{s-1}_{p,r}} \le \|w^{12}_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t (1+\lambda+\|w^1\|_{B^s_{p,r}}+\|w^2\|_{B^s_{p,r}})^2\,d\tau}. \tag{2.11}$$

This completes the proof of Theorem 1.1.

Remark 2.1 We note that $(1-\partial_x)^{-1} = (1-\partial_x^2)^{-1}(1+\partial_x)$ is S^{-1} multiplier. That is

$$\begin{aligned}
&\|v\|_{B^s_{p,r}} \lesssim \|w\|_{B^{s-1}_{p,r}},\\
&v(x) = (1-\partial_x)^{-1}w(x)\\
&\quad = (1-\partial_x^2)^{-1}(1+\partial_x)w(x)\\
&\quad = \int_x^\infty e^{x-\xi}w(\xi)\,d\xi.
\end{aligned} \tag{2.12}$$

3 Proofs of Theorems 1.2, 1.3, 1.4, and 1.5

3.1 Proof of Theorem 1.2

We assume that $f(x)\in C_c^\infty(\mathbb{R})$ is the nonnegative mollifier. It holds that $\int_{\mathbb{R}} f(x)\,dx = 1$, $f^j(x) = jf(jx)$, $v_0^j = f^j * v_0$, and $\|f^j\|_{L^1} = 1$. v^j is the solution to problem (1.1) with initial value v_0^j. $\Delta_q(f^j * v_0) = f^j * \Delta_q v_0$. That is,

$$\|v_0^j\|_{B^s_{p,r}} \le \|v_0\|_{B^s_{p,r}}, \qquad \|v_0^j\|_{H^1} \le \|v_0\|_{H^1}. \tag{3.1}$$

It follows that $v_0^j \in H^3$. From (1.1), we conclude

$$v^j(t,x) \in C([0,T];H^3\cap B^s_{p,r}) \cap C^1([0,T];H^2\cap B^{s-1}_{p,r}) \quad \text{for all } [0,T]\times\mathbb{R},$$

where $T = \frac{C_3}{\|v\|_{L^\infty([0,T];B^s_{p,r})}}$. The sequence $(v^j)_{j\in\mathbb{N}} \in E^s_{p,r}(T)$ is bounded.

It is deduced from (1.2) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^j\|_{L^2}^2\\
&\quad = \int_{\mathbb{R}} w^j w^j_t\,dx\\
&\quad = \int_{\mathbb{R}} w^j\bigl[v^j(w^j)^2 - (v^j)^2w^j - \lambda w^j - \beta w^j_x - (v^j)^2 w^j_x\bigr]\,dx\\
&\quad = \int_{\mathbb{R}} w^j\bigl[v^j w^j(-v^j_x) - \lambda w^j - \beta w^j_x - (v^j)^2 w^j_x\bigr]\,dx\\
&\quad \le -\lambda\|w^j\|_{L^2}^2.
\end{aligned}$$

Equivalently, we have

$$\|w^j\|_{L^2}^2 = \int_{\mathbb{R}} (v^j - v^j_x)^2\,dx = \int_{\mathbb{R}} \bigl[(v^j)^2 - 2v^jv^j_x + (v^j_x)^2\bigr]\,dx = \|v^j\|_{H^1}^2.$$

It is equal to check that

$$\|v^j(t)\|_{H^1} \le \|v_0^j\|_{H^1} \le \|v_0\|_{H^1} \quad \text{for all } t\in[0,T]. \tag{3.2}$$

Sending $j \to \infty$ in (3.2), we come to the estimate

$$\|v(t)\|_{H^1} \le \|v_0\|_{H^1} \quad \text{for all } t \in [0,T]. \tag{3.3}$$

The proof of Theorem 1.2 is finished.

3.2 Proof of Theorem 1.3

We present a lemma that is applied in the proof.

Lemma 3.1 ([40]) *Let $1 \le p, r \le \infty$, and $0 < s < 1$. There exists a positive constant C, which is independent of w, g such that*

$$\|[\Delta_j, w \cdot \nabla]g\|_{B^s_{p,r}} \le C\|\nabla w\|_{L^\infty}\|g\|_{B^s_{p,r}}.$$

Applying the operator Δ_q to (1.2) yields

$$\left(\partial_t + \left(v^2 + \beta\right)\partial_x\right)\Delta_q w = \left[v^2, \Delta_q\right]\partial_x w + F_2(t,x) - \lambda\Delta_q w, \tag{3.4}$$

where

$$F_2(t,x) = \Delta_q\left[vw^2 - v^2 w\right].$$

If $\frac{1}{2} < s < 1$, then taking advantage of Lemma 3.1, we acquire

$$\begin{aligned}\left\|\left[v^2, \Delta_q\right]\partial_x w\right\|_{B^s_{p,r}} &\lesssim \left\|\partial_x\left(v^2\right)\right\|_{L^\infty}\|w\|_{B^s_{p,r}} \\ &\le \|v\|_{L^\infty}\|v_x\|_{L^\infty}\|w\|_{B^s_{p,r}}.\end{aligned} \tag{3.5}$$

A straightforward computation gives rise to

$$\left\|vw^2\right\|_{B^s_{p,r}} \lesssim \left(\|v\|_{L^\infty}\|w\|_{L^\infty} + \|w\|^2_{L^\infty}\right)\|w\|_{B^s_{p,r}} \tag{3.6}$$

and

$$\left\|v^2 w\right\|_{B^s_{p,r}} \lesssim \left(\|v\|^2_{L^\infty} + \|v\|_{L^\infty}\|w\|_{L^\infty}\right)\|w\|_{B^s_{p,r}}. \tag{3.7}$$

A simple calculation shows

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\Delta_q w\|^p_{L^p} \\ &\quad\lesssim \left\|\partial_x\left(v^2 + \beta\right)\right\|_{L^\infty}\|\Delta_q w\|^p_{L^p} + \left\|\left[v^2, \Delta_q\right]\partial_x w\right\|_{L^p}\|\Delta_q w\|^{p-1}_{L^p} \\ &\qquad + \left\|F_2(t,x)\right\|_{L^p}\|\Delta_q w\|^{p-1}_{L^p} - \lambda\|\Delta_q w\|^p_{L^p},\end{aligned} \tag{3.8}$$

which results in

$$\begin{aligned}\frac{d}{dt}\|\Delta_q w\|_{L^p} &\lesssim \left\|\partial_x\left(v^2\right)\right\|_{L^\infty}\|\Delta_q w\|_{L^p} + \left\|\left[v^2, \Delta_q\right]\partial_x w\right\|_{L^p} \\ &\quad + \left\|F_2(t,x)\right\|_{L^p} - \lambda\|\Delta_q w\|_{L^p}.\end{aligned} \tag{3.9}$$

That is,

$$\begin{aligned}
&\|w(t)\|_{B^s_{p,r}} \\
&\quad\lesssim \|w_0\|_{B^s_{p,r}} + \int_0^t \big(\|v\|_{L^\infty}\|v_x\|_{L^\infty} + \|v\|_{L^\infty}\|w\|_{L^\infty} \\
&\qquad + \|w\|^2_{L^\infty} + \|v\|^2_{L^\infty} - \lambda\big)\|w(\tau)\|_{B^s_{p,r}}\,d\tau \\
&\quad\lesssim \|w_0\|_{B^s_{p,r}} + \int_0^t \big(\|w\|^2_{L^\infty} - \lambda\big)\|w(\tau)\|_{B^s_{p,r}}\,d\tau.
\end{aligned} \tag{3.10}$$

Employing (3.10) leads to

$$\|w(t)\|_{B^s_{p,r}} \lesssim \|w_0\|_{B^s_{p,r}} e^{\int_0^t (\|w\|^2_{L^\infty} - \lambda)\,d\tau}. \tag{3.11}$$

If

$$\int_0^t \big(\|w\|^2_{L^\infty} - \lambda\big)\,d\tau < \infty, \tag{3.12}$$

then we derive that $\|w(T^*)\|_{B^s_{p,r}}$ is bounded, where $T^* < \infty$ is the maximal existence time. This yields a contradiction. This finishes the proof of Theorem 1.3.

3.3 Proof of Theorem 1.4

Based on the density argument, we need to illustrate the proof of Theorem 1.4 with $s \geq 2$. Therefore, we calculate $\|w\|_{H^1}$ for simplicity.

According to (1.2), we come to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \big[w^2 + w_x^2\big]\,dx \\
&\quad= \int_{\mathbb{R}} w_x\big[\big(vw^2 - v^2w - v^2w_x\big)_x - \lambda w_x + \beta w_{xx}\big]\,dx \\
&\quad\lesssim \int_{\mathbb{R}} \big(vww_x^2 - \lambda w_x^2\big)\,dx + C_1\|w\|^2_{H^1}.
\end{aligned} \tag{3.13}$$

We suppose that the positive constant M satisfies $vw \leq M$, $t \in [0, T]$, $T < \infty$. We then conclude that

$$\|w(t)\|^2_{H^1} \lesssim \|w_0\|^2_{H^1} e^{(M-\lambda)t},$$

which contradicts that $T < \infty$ is the maximal existence time. The proof of Theorem 1.4 is completed.

3.4 Proof of Theorem 1.5

We consider the problem

$$\begin{cases} \frac{d}{dt}p(t,x) = (v^2 + \beta)(t, p(t,x)), & t \in (0,T), x \in \mathbb{R}, \\ p(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{3.14}$$

The solution $p \in C^1([0,T],\mathbb{R})$ to problem (3.14) is unique. We recognize that

$$p_x(t,x) = e^{\int_0^t (2vv_x)(\tau,p(\tau,x))\,d\tau} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}. \tag{3.15}$$

We deduce

$$\begin{aligned}
&\frac{d}{dt}\bigl[t, w\bigl(p(t,x)\bigr)p_x^{\frac{1}{2}}(t,x)\bigr] \\
&\quad = (w_t + w_x p_t)p_x^{\frac{1}{2}} + \frac{1}{2} w p_x^{-\frac{1}{2}} p_{xt} \\
&\quad = \bigl[w_t + w_x\bigl(v^2 + \beta\bigr) + vv_x w\bigr]p_x^{\frac{1}{2}} \\
&\quad = -\lambda w p_x^{\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

Thus, we achieve

$$w\bigl(p(t,x),t\bigr)p_x^{\frac{1}{2}}(t,x) = w_0(x)e^{-\lambda t}.$$

An application of (3.15) gives rise to

$$w\bigl(t,p(t,x)\bigr) = w_0(x)e^{\int_0^t (-vv_x - \lambda)\,d\tau}. \tag{3.17}$$

If u_0 is compactly supported in $[a_{u_0}, b_{u_0}]$, then w_0 is compactly supported in $[p(t,a_{u_0}), p(t,b_{u_0})]$. We deduce from (3.17) that $w(t,x)$ has its support in the interval $[p(t,a_{u_0}), p(t,b_{u_0})]$. From (2.12), we acquire that $u(t,x)$ is compactly supported in $[p(t,a_{u_0}), p(t,b_{u_0})]$. This completes the proof of Theorem 1.5.

3.5 Proof of Theorem 1.6

First of all, we illustrate a useful lemma.

Lemma 3.2 ([41]) *Let $v \in C^1([0,T); H^3(\mathbb{R}))$ and $n = v(v - v_x)$. Then, for all $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$n(t) = \sup_{x\in\mathbb{R}} n(t,x) = n\bigl(t,\xi(t)\bigr). \tag{3.18}$$

The function $n(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{d}{dt}n(t) = n_t\bigl(t,\xi(t)\bigr).$$

Here, we set $s = 2$ in view of the density argument. We observe

$$\begin{aligned}
&\frac{d}{dt}\bigl[vw\bigl(t,p(t,x)\bigr)\bigr] + \bigl(v^2 + \beta\bigr)\partial_x(vw) \\
&\quad = v_t w + v w_t + v^2 v_x w + v^3 w_x + \beta(vw)_x
\end{aligned}$$

$$
\begin{aligned}
&= -2\lambda vw + v^2w^2 - w\left(1-\partial_x^2\right)^{-1}(1+\partial_x)\left[vw^2\right] \\
&\geq (vw)^2 - 2\lambda vw - w\int_x^{\infty} e^{x-\xi} v(t,\xi)w^2(t,\xi)\,d\xi \\
&\geq (vw)^2 - 2\lambda vw - we^x\Big[\sup_{\xi\geq x} e^{-\xi} v(t,\xi)\Big]\int_x^{\infty} w^2(t,\xi)\,d\xi.
\end{aligned}
\tag{3.19}
$$

It is deduced by a straightforward computation that

$$
\frac{d}{d\xi}\left(e^{-\xi}v(t,\xi)\right) = e^{-\xi}\left(v_x(t,\xi) - v(t,\xi)\right) = -e^{-\xi}w(t,\xi) \leq 0. \tag{3.20}
$$

Eventually, we derive $\sup_{\xi\geq x}[e^{-\xi}v(t,\xi)] = e^{-x}v(t,x)$. It follows from Theorem 1.2 that

$$
\int_x^{\infty} w^2(t,\xi)\,d\xi \leq \|w\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 = \|v_0\|_{H^1}^2. \tag{3.21}
$$

Utilizing (3.17), (3.18), and (3.19), we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\left[vw\big(t,p(t,x)\big)\right] + \left[\left(v^2+\beta\right)\partial_x(vw)\big(t,p(t,x)\big)\right] \\
&\quad\geq (vw)^2 - 2\lambda vw - we^x\sup_{\xi\geq x} e^{-\xi}v(\xi)\int_x^{\infty} w^2(\xi)\,d\xi \\
&\quad\geq (vw)^2 - 2\lambda vw - vw\|v_0\|_{H^1}^2 \\
&\quad\geq (vw)^2 - \lambda_1 vw,
\end{aligned}
\tag{3.22}
$$

where $\lambda_1 = 2\lambda + \|v_0\|_{H^1}^2$.

Let $n(t) = \sup_{x\in\mathbb{R}}[vw(t,p(t,x))]$. Making use of Lemma 3.2, we acquire that there exists $\xi(t)$ with $t\in[0,T)$ such that

$$
n_1(t) = \sup_{x\in\mathbb{R}} n(t,x) = n\big(t,\xi(t)\big) \quad \text{for all } t\in[0,T).
$$

This in turn implies that $n_x(t,\xi(t)) = 0$ for all $t\in[0,T)$.

On the other hand, since $p(t,\cdot):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism for all $t\in[0,T)$, there exists $x_1(t)\in\mathbb{R}$ such that $p(t,x_1(t)) = \xi(t)$ for all $t\in[0,T)$.

Applying (3.22) gives rise to

$$
\frac{d}{dt}n(t) \geq \big(n(t)\big)^2 - \lambda_1 n(t). \tag{3.23}
$$

Setting $n_2(t) = -(n(t) - \frac{\lambda_1}{2})$, we have

$$
\frac{d}{dt}n_2(t) \leq \big(n_2(t)\big)^2 + K, \tag{3.24}
$$

where $K = \frac{\lambda_1^2}{4}$.

Employing the assumption $n_0(x_0) > \lambda_1$ with the point x_0 defined by $n(x_0) = \sup_{x\in\mathbb{R}} n_0(x)$ in Theorem 1.6 and letting $\xi(0) = x_0$, we deduce

$$n_2(0) = -\left(n_1(0) - \frac{\lambda_1}{2}\right) = -\left(n_0\big(\xi(0)\big) - \frac{\lambda_1}{2}\right) = -\left(n_0(x_0) - \frac{\lambda_1}{2}\right) < -\sqrt{K}.$$

We choose $\delta \in (0,1)$ to satisfy $-\sqrt{\delta} n_2(0) = \sqrt{K}$.

We observe that $n_2(0) = -(n(0) - \frac{\lambda_1}{2}) < -\sqrt{K}$ and $n_2(t)$ decreases on $[0,T)$. We set $-\sqrt{\delta} n_2(0) = \sqrt{K}$, where $\delta \in (0,1)$. Direct calculations show

$$-\frac{1}{n_2(t)} + \frac{1}{n_2(0)} \le -(1-\delta)t.$$

We arrive at $n_2(t) < 0$, $t \in [0,T)$, $T \le \frac{-1}{(1-\delta)n_2(0)} < \infty$, and $n_2(0) = -(n(0) - \frac{\lambda_1}{2}) < 0$. As a consequence, we obtain

$$\begin{aligned}
-\left[n\big(t,\xi(t)\big) - \frac{\lambda_1}{2}\right] &= -\left[\big(v(v-v_x)\big)\big(t,\xi(t)\big) - \frac{\lambda_1}{2}\right] \\
&\le \frac{n(0) - \frac{\lambda_1}{2}}{-1 + t(1-\delta)(n(0) - \frac{\lambda_1}{2})} \to -\infty \\
&\quad \text{as } t \to \frac{1}{(1-\delta)(n(0) - \frac{\lambda_1}{2})}.
\end{aligned} \tag{3.25}$$

The proof of Theorem 1.6 is finished.

4 Proof of Theorem 1.7

Let $M = \sup_{t\in[0,T]} \|v(t)\|_{H^s} > 0$ with $s > \frac{5}{2}$. We derive

$$\big\|v(t)\big\|_{L^\infty} \le \big\|v_x(t)\big\|_{L^\infty} \le \big\|v_{xx}(t)\big\|_{L^\infty} \le \big\|v(t)\big\|_{H^s} \le M.$$

We observe that the weight function

$$\varphi_N(x) = \begin{cases} 1, & x \le 0, \\ e^{\theta x}, & x \in (0,N), \\ e^{\theta N}, & x \ge N, \end{cases}$$

satisfies $0 \le (\varphi_N(x))_x \le \varphi_N(x)$, where $N \in \mathbb{N}^*$ and $\theta \in (0,1)$. There exists a positive constant M_0, which depends on θ such that

$$\varphi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_N(y)}\, dy \le M_0.$$

We rewrite the first equation in problem (1.1) in the form

$$v_t = G - \lambda v - \beta v_x, \tag{4.1}$$

where

$$G(t,x) = (1-\partial_x)^{-1}\big[v^2 v_{xx} + v v_x^2 - 2v^2 v_x\big]$$

$$= \int_x^\infty e^{x-y}\big(v^2 v_{xx} + v v_x^2 - 2v^2 v_x\big)(y)\,dy. \tag{4.2}$$

Multiplying both sides of (4.2) by $v^{2n-1}\varphi_N^{2n}$ with respect to the variable x over $\mathbb{R}$ leads to

$$\begin{aligned}
&\frac{1}{2n}\frac{d}{dt}\|v\varphi_N\|_{L^{2n}}^{2n} \\
&\quad= \lambda\int_{\mathbb{R}}(v\varphi_N)^{2n}\,dx - \beta\int_{\mathbb{R}}\big[\partial_x(v\varphi_N) - v(\varphi_N)_x\big](v\varphi_N)^{2n-1}\,dx \\
&\qquad+ \int_{\mathbb{R}}(v\varphi_N)^{2n-1}G\varphi_N\,dx \\
&\quad\le \lambda\int_{\mathbb{R}}(v\varphi_N)^{2n}\,dx + \beta\int_{\mathbb{R}}(v\varphi_N)(v\varphi_N)^{2n-1}\,dx \\
&\qquad+ \int_{\mathbb{R}}(v\varphi_N)^{2n-1}G\varphi_N\,dx \\
&\quad\le (\lambda+\beta)\|v\varphi_N\|_{L^{2n}} + \|v\varphi_N\|_{L^{2n}}^{2n-1}\|G\varphi_N\|_{L^{2n}}.
\end{aligned} \tag{4.3}$$

We then conclude that

$$\frac{d}{dt}\|v\varphi_N\|_{L^{2n}} \le (\lambda+\beta)\|v\varphi_N\|_{L^{2n}} + \|G\varphi_N\|_{L^{2n}}. \tag{4.4}$$

Sending $n\to\infty$ in (4.4) gives rise to the estimate

$$\frac{d}{dt}\|v\varphi_N\|_{L^\infty} \le (\lambda+\beta)\|v\varphi_N\|_{L^\infty} + \|G\varphi_N\|_{L^\infty}. \tag{4.5}$$

A direct computation shows

$$\begin{aligned}
&|G\varphi_N| \\
&\quad= \left|\phi_N(x)\int_x^\infty e^{x-y}\big[v^2 v_{xx} + v v_x^2 - 2v^2 v_x\big](y)\,dy\right| \\
&\quad\le \phi_N(x)\int_x^\infty e^{x-y}\big|\big[v^2 v_{xx} + v v_x^2 - 2v^2 v_x\big](y)\big|\,dy \\
&\quad\le \phi_N(x)\int_{\mathbb{R}} e^{-|x-y|}\frac{1}{\phi_N(y)}\phi_N(y)\big|\big[v^2 v_{xx} + v v_x^2 - 2v^2 v_x\big](y)\big|\,dy \\
&\quad\lesssim \big(\|v\|_{L^\infty}\|v_{xx}\|_{L^\infty} + \|v_x\|_{L^\infty}^2 + 2\|v\|_{L^\infty}\|v_x\|_{L^\infty}\big)\|v\varphi_N\|_{L^\infty} \\
&\quad\lesssim M\|v\varphi_N\|_{L^\infty}.
\end{aligned} \tag{4.6}$$

Taking advantage of the Gronwall inequality yields

$$\|v\varphi_N\|_{L^\infty} \lesssim e^{(M+\lambda+\beta)t}\|v_0\varphi_N\|_{L^\infty}. \tag{4.7}$$

We achieve

$$\sup_{t\in[0,T]}\big\|e^{\theta x}v\big\|_{L^\infty} \lesssim \big\|e^{\theta x}v_0\big\|_{L^\infty}.$$

Therefore, we deduce

$$|v| \sim O(e^{-\theta x}) \quad \text{as } x \to \infty$$

uniformly on $[0, T]$. The proof of Theorem 1.7 is completed.

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Author contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, North University of China, College Road, 030051, Taiyuan, China. [2]Department of Securities and Futures, Southwestern University of Finance and Economics, Liutai Road, 611130, Chengdu, China.

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