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Newton–Simpson-type inequalities via majorization

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Abstract

In this article, the main objective is construction of fractional Newton–Simpson-type inequalities with the concept of majorization. We established a new identity on estimates of definite integrals utilizing majorization and this identity will lead us to develop new generalized forms of prior estimates. Some basic inequalities such as Hölder's, power-mean, and Young's along with the Niezgoda–Jensen–Mercer inequality have been used to obtain new bounds and it has been determined that the main findings are generalizations of many results that exist in the literature. Applications to the quadrature rule are given as well. We make links between our findings and a number of well-known discoveries in the literature.

MSC: 26A33; 51M16

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1 Introduction

The notable character of inequalities in the growth and enlargement of mathematics is well known. In several areas of science and technology, mathematical inequalities have effectively spread their impact and they are now recognized and imparted as some of the most useful mathematical disciplines. Information theory, economics, engineering, and other fields have benefited from their use (see [1, 2]). Inequalities and their affiliated theory have rapidly expanded as a result of this applicability, leading to the establishment of various new and generalized forms of inequalities. For illustration, the Hermite–Hadamard inequality, Newton's inequality, Simpson's inequality, Jensen's inequality, and the Jensen–Mercer inequality (see [2–5]) are some of the most well-known identities among scientists. Currently, scientists are specifically interested in generalized inequalities that include many of the previously stated variants in one version or another. We selected Newton's and Simpson's inequalities for convex functions in generalized form.

The relationship between inequalities and the theory of convex functions was discovered to be extremely strong. Convex functions play a notable role in the fields of both theoretical and applied studies. The study of convex functions always presents stunning and magnificent sights of the beauty in advanced mathematics. Mathematicians always put potential in this direction as a result, discover and survey a large variety of results that are beneficial and remarkable for applications. This method is strong in relationship with countless issues, most of which are found in both the basic and applied sciences. Convexity also has a major effect on our daily life through countless implementations in medicine, industry, business, and art. Due to the wide range of implementations, it is among the most advanced branches of mathematical modeling. Convex functions are the topic of research in a number of disciplines due to their applicability in inequality theory and are defined as:

$$\psi\big(\kappa\varkappa + (1-\kappa)\varkappa_1\big) \le \kappa\psi(\varkappa) + (1-\kappa)\psi(\varkappa_1), \tag{1}$$

where $\psi : [\mathfrak{a},\mathfrak{b}] \subseteq \Re \to \Re$ is a convex function that holds for all $\varkappa, \varkappa_1 \in [\mathfrak{a},\mathfrak{b}]$ and $\kappa \in [0,1]$.

Additional information of different types of convexity and their contribution to inequalities can be found in [1, 2].

Newton's inequality is defined as:

Theorem 1.1 ([4]) *Consider* $\psi : [\mathfrak{a},\mathfrak{b}] \to \mathbb{R}$ *is a four-times continuously differentiable mapping, and*

$$\big\|\psi^{(4)}\big\|_\infty = \sup_{\varkappa\in(\mathfrak{a},\mathfrak{b})} \big|\psi^{(4)}(\varkappa)\big| < \infty, \tag{2}$$

then, the following estimation holds:

$$\left|\frac{1}{8}\left[\psi(\mathfrak{a}) + 3\psi\left(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right) + 3\psi\left(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right) + \psi(\mathfrak{b})\right] - \frac{1}{\mathfrak{b}-\mathfrak{a}}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi(\varkappa)\,d\varkappa\right| \le \frac{1}{6480}\big\|\psi^{(4)}\big\|_\infty (\mathfrak{b}-\mathfrak{a})^4.$$

The results of Newton-type inequalities involving convex mappings have been looked at by various authors because convex theory is an excellent technique to deal with a sizable number of issues from various mathematical disciplines. In the papers [4, 6], based on differentiable convex mapping, new modifications of Newton-type inequalities were discovered. Furthermore, the authors presented Newton's inequality for convex functions in quantum calculus [7, 8]. Iftikhar et al. in [9], presented a novel Newton-type inequality for functions with the local fractional derivative, which is generalized and convex.

Simpson's inequality is explained as:

Theorem 1.2 ([3]) *Suppose that* $\psi : [\mathfrak{a},\mathfrak{b}] \to \mathbb{R}$ *is a four-times continuously differentiable mapping on* $(\mathfrak{a},\mathfrak{b})$, *and let* $\|\psi^{(4)}\|_\infty = \sup_{\varkappa\in(\mathfrak{a},\mathfrak{b})} |\psi^{(4)}(\varkappa)| < \infty$, *then the following inequality holds*:

$$\left|\frac{1}{3}\left[\frac{\psi(\mathfrak{a})+\psi(\mathfrak{b})}{2} + 2\psi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)\right] - \frac{1}{\mathfrak{b}-\mathfrak{a}}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi(\varkappa)\,d\varkappa\right| \le \frac{1}{2880}\big\|\psi^{(4)}\big\|_\infty (\mathfrak{b}-\mathfrak{a})^4.$$

The results of Simpson-type inequalities for convex mappings have been looked at by various authors because convex theory is an excellent technique to deal with a sizable number of issues from various mathematical disciplines. In the papers [10, 11], based on differentiable convex mapping, new modifications of Simpson-type inequalities are discovered. Furthermore, these inequalities were also the subject of some publications [12, 13].

The introduction of new approaches and the generalization of integral inequalities using fractional integral operators resulted in a revolution in inequality theory. For some recent results see [14, 15]. Among the many fractional integrals that have emerged, the Riemann–Liouville fractional integral has been widely considered as a result of its uses in numerous fields of science.

Definition 1.1 ([16]) For an integrable function ψ on $[\mathfrak{a},\mathfrak{b}]$, the left and right Riemann–Liouville fractional integrals of order $\varpi > 0$ are defined as:

$$J_{\mathfrak{a}^+}^{\varpi}\psi(\varkappa) = \frac{1}{\Gamma(\varpi)}\int_{\mathfrak{a}}^{\varkappa}(\varkappa-\kappa)^{\varpi-1}\psi(\kappa)\,d\kappa, \quad \varkappa > \mathfrak{a},$$

$$J_{\mathfrak{b}^-}^{\varpi}\psi(\varkappa) = \frac{1}{\Gamma(\varpi)}\int_{\varkappa}^{\mathfrak{b}}(\kappa-\varkappa)^{\varpi-1}\psi(\kappa)\,d\kappa, \quad \varkappa < \mathfrak{b},$$

where $\Gamma(\varpi)$ is the Gamma function.

Let $0 < \zeta_1 \le \zeta_2 \le \cdots \le \zeta_n$ and let $\rho = (\rho_1, \rho_2, \dots, \rho_\epsilon)$ be nonnegative weights such that $\sum_{\lambda=1}^{\epsilon}\rho_\lambda = 1$. The famous Jensen inequality [1] in the literature states that if ψ is a convex function on the interval $[\mathfrak{a},\mathfrak{b}]$, then

$$\psi\left(\sum_{\lambda=1}^{\epsilon}\rho_\lambda\zeta_\lambda\right) \le \left(\sum_{\lambda=1}^{\epsilon}\rho_\lambda\psi(\zeta_\lambda)\right), \tag{3}$$

for all $\zeta_\lambda \in [\mathfrak{a},\mathfrak{b}]$, $\rho_\lambda \in [0,1]$ and $(\lambda = 1,2,\dots,\epsilon)$.

In 2003, a new variant of Jensen's inequality was introduced by Mercer [17] as:

If ψ is a convex function on $[\mathfrak{a},\mathfrak{b}]$, then

$$\psi\left(\mathfrak{a}+\mathfrak{b}-\sum_{\lambda=1}^{\epsilon}\rho_\lambda\zeta_\lambda\right) \le \psi(\mathfrak{a})+\psi(\mathfrak{b})-\sum_{\lambda=1}^{\epsilon}\rho_\lambda\psi(\zeta_\lambda), \tag{4}$$

holds for all $\zeta_\lambda \in [\mathfrak{a},\mathfrak{b}]$, $\rho_\lambda \in [0,1]$ and $(\lambda = 1,2,\dots,\epsilon)$. It is one of the key inequalities that helps to extract bounds for useful distances in information theory [5, 18, 19].

Definition 1.2 ([20]) Let $\varrho = (\varrho_1, \varrho_2, \dots, \varrho_\lambda)$ and $\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_\lambda)$ be two λ-tuples of real numbers with their arrangements $\varrho_\lambda \le \varrho_{\lambda-1} \le \cdots \le \varrho_1$, $\mathfrak{y}_\lambda \le \mathfrak{y}_{\lambda-1} \le \cdots \le \mathfrak{y}_1$, then ϱ is said to majorize $\mathfrak{y}$ (or $\mathfrak{y}$ is said to be majorized by ϱ, symbolically $\mathfrak{y} \prec \varrho$), if:

$$\sum_{\Theta=1}^{s}\mathfrak{y}_{[\Theta]} \le \sum_{\Theta=1}^{s}\varrho_{[\Theta]} \quad \text{for } s = 1,2,\dots,\lambda-1,$$

and:

$$\sum_{\Theta=1}^{\lambda}\mathfrak{y}_{[\Theta]} = \sum_{\Theta=1}^{\lambda}\varrho_{[\Theta]}. \tag{5}$$

It is an inequality in elementary algebra that generalizes Jensen's inequality for convex real-valued functions defined on an interval of the real line. Niezgoda [21] utilized the idea of majorization and extended the Jensen–Mercer inequality given as follows:

Theorem 1.3 *Let* $(\varkappa_{\mathrm{i}\Theta})$ *be an* $\epsilon \times \lambda$ *real matrix and* $\theta = (\theta_1, \dots, \theta_\lambda)$ *be a* λ*-tuple such that* $\theta_\Theta, \varkappa_{\mathrm{i}\Theta} \in I$ *for all* $\mathrm{i} = 1, 2, \dots, \epsilon$, $\Theta \in \{1, \dots, \lambda\}$ *and* ψ *be a continuous convex function defined on an interval* $I \subset \mathbb{R}$. *Furthermore, let* $\sigma_{\mathrm{i}} \geq 0$ *for* $\mathrm{i} = 1, 2, \dots, \epsilon$ *with* $\sum_{\mathrm{i}=1}^{\epsilon} \sigma_{\mathrm{i}} = 1$. *If* θ *majorizes every row of* $(\varkappa_{\mathrm{i}\Theta})$, *then*:

$$\psi\left(\sum_{\Theta=1}^{\lambda} \theta_\Theta - \sum_{\Theta=1}^{\lambda-1} \sum_{\mathrm{i}=1}^{\epsilon} \sigma_{\mathrm{i}} \varkappa_{\mathrm{i}\Theta}\right) \leq \sum_{\Theta=1}^{\lambda} \psi(\theta_\Theta) - \sum_{\Theta=1}^{\lambda-1} \sum_{\mathrm{i}=1}^{\epsilon} \sigma_{\mathrm{i}} \psi(\varkappa_{\mathrm{i}\Theta}). \tag{6}$$

For researchers working with various integrals or convex functions, the theory of majorization provides a unique opportunity. Researchers from a variety of disciplines have been paying close attention to it. Numerous majorization ideas have been recreated and applied to various fields of study, including economics, graph theory, and optimization. The theory of majorization is a very important topic in mathematics; Olkin and Marshall's book [22] is a remarkable and comprehensive reference on the subject. The concept of majorization, for instance, is a powerful component for converting nonconvex complicated constrained optimization problems with matrix-valued variables into simple problems with scalar variables that can be quickly resolved [23, 24]. Majorization theory can be traced back to some modern applications in signal processing and communication [25]. One can see some recent results related to majorization in [26, 27].

In this paper, the main focus is on majorization-type results for Newton–Simpson-type inequalities involving convex functions. To extend majorization the Riemann–Liouville fractional integral is used for both Simpson- and Newton-type inequalities. Inequalities of the classical Mercer type and their different versions are created in the case when we have $\varpi = 1$ and $\lambda = 2$ in the obtained results. Finally, the obtained outcomes were backed up by diminished outcomes and implementations.

Here, $\mathbf{\Omega} = (\Omega_1, \Omega_2, \dots, \Omega_\lambda)$, $\mathbf{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_\lambda)$ and $\mathbf{b} = (\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_\lambda)$ are three λ-tuples that will be used throughout the paper.

2 Main results

New Newton–Simpson-type Lemma's via majorization are presented in this section.

Lemma 2.1 *Let* $\Omega_\Theta, \mathfrak{a}_\Theta, \mathfrak{b}_\Theta \in I$ *for all* $\Theta \in \{1, \dots, \lambda\}$ *be three* λ*-tuples such that* $\mathfrak{a}_\lambda > \mathfrak{b}_\lambda$, $\varpi > 0$ *and* ψ *be a differentiable function on an interval* $I \subset \mathbb{R}$. *If* $\psi' \in L(I)$ *and* $\mathbf{\Omega}$ *majorizes both* $\mathbf{a}$ *and* $\mathbf{b}$, *then*:

$$\frac{1}{8}\left[\psi\left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{a}_\Theta\right) + 3\psi\left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \frac{2\mathfrak{a}_\Theta + \mathfrak{b}_\Theta}{3}\right)\right.$$
$$\left. + 3\psi\left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \frac{\mathfrak{a}_\Theta + 2\mathfrak{b}_\Theta}{3}\right) + \psi\left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{b}_\Theta\right)\right]$$

$$-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^\varpi}J^{(\varpi)}_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)$$
$$=\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_0^1\Phi^\varpi(\kappa)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa,$$

where $\Phi^\varpi(\kappa)$ *is defined by*

$$\Phi^\varpi(\kappa)=\begin{cases}\kappa^\varpi-\frac{1}{8}, & if\, t\in[0,\frac{1}{3}),\\ \kappa^\varpi-\frac{1}{2}, & if\, t\in[\frac{1}{3},\frac{2}{3}),\\ \kappa^\varpi-\frac{7}{8}, & if\, t\in[\frac{2}{3},1].\end{cases}$$

Proof Let us denote

$$\mathbb{I}=\int_0^1\Phi^\varpi(\kappa)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa.$$

Now, splitting the integral, we have

$$\mathbb{I}=\int_0^{\frac{1}{3}}\left(\kappa^\varpi-\frac{1}{8}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\kappa^\varpi-\frac{1}{2}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa$$
$$+\int_{\frac{2}{3}}^{1}\left(\kappa^\varpi-\frac{7}{8}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa$$
$$=\mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3.$$

Using integration by parts

$$\mathbb{I}_1=\int_0^{\frac{1}{3}}\left(\kappa^\varpi-\frac{1}{8}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa$$
$$=\left(\left(\frac{1}{3}\right)^\varpi-\frac{1}{8}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}))}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}+\left(\frac{1}{8}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}$$
$$-\frac{\varpi}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\int_0^{\frac{1}{3}}\kappa^{\varpi-1}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa,$$

$$\mathbb{I}_2=\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\kappa^\varpi-\frac{1}{2}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa$$
$$=\left(\left(\frac{2}{3}\right)^\varpi-\frac{1}{2}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}))}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}$$
$$-\left(\left(\frac{1}{3}\right)^\varpi-\frac{1}{2}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}))}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}$$
$$-\frac{\varpi}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\int_{\frac{1}{3}}^{\frac{2}{3}}\kappa^{\varpi-1}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)d\kappa,$$

$$\begin{aligned}\mathbb{I}_3 &= \int_{\frac{2}{3}}^{1}\left(\kappa^{\varpi}-\frac{7}{8}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa \\ &= -\left(\left(\frac{2}{3}\right)^{\varpi}-\frac{7}{8}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}))}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}+\left(\frac{1}{8}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)} \\ &\quad -\frac{\varpi}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\int_{\frac{2}{3}}^{1}\kappa^{\varpi-1}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa\end{aligned}$$

and adding $\mathbb{I}_1$, $\mathbb{I}_2$, and $\mathbb{I}_3$ we have

$$\begin{aligned}\mathbb{I} &= \frac{1}{8\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\right. \\ &\quad \left.+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right] \\ &\quad -\frac{\varpi}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\int_{0}^{1}\kappa^{\varpi-1}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa. \qquad (7)\end{aligned}$$

Now, substituting $\varkappa=\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta)$ we have

$$\begin{aligned}\mathbb{I} &= \frac{1}{8\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\right. \\ &\quad \left.+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right] \\ &\quad -\frac{\varpi}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi+1}}\int_{\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta}^{\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta}\psi(\varkappa)\left[\varkappa-\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]^{\varpi-1}d\varkappa. \qquad (8)\end{aligned}$$

To apply the definition of a fractional integral in (8), we need to show that

$$\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\le\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta.$$

As

$$\begin{aligned}&\mathfrak{a}_\lambda>\mathfrak{b}_\lambda,\\ &\mathfrak{a}_\lambda-\mathfrak{b}_\lambda>0. \qquad (9)\end{aligned}$$

Furthermore, $\mathbf{a}\prec\mathbf{\Omega}$ and $\mathbf{b}\prec\mathbf{\Omega}$. Then,

$$\begin{aligned}&\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta+\mathfrak{b}_\lambda=\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta+\mathfrak{a}_\lambda,\\ &\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta=\mathfrak{a}_\lambda-\mathfrak{b}_\lambda \qquad (10)\end{aligned}$$

and using (9) in (10), we have

$$\begin{aligned}&\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta>0,\\&-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta<-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\end{aligned}\tag{11}$$

and adding $\sum_{\Theta=1}^{\lambda-1}\Omega_\Theta$ to both sides of (11) we have

$$\sum_{\Theta=1}^{\lambda-1}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta<\sum_{\Theta=1}^{\lambda-1}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta.\tag{12}$$

Now, (8) implies

$$\begin{aligned}\mathbb{I}={}&\frac{1}{8\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\Bigg[\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\Bigg)+3\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\Bigg)\\&+3\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\Bigg)+\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\Bigg)\Bigg]\\&-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi+1}}J^{(\varpi)}_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\Bigg).\end{aligned}\tag{13}$$

Multiplying (13) by $\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)$, we obtain the required equality. □

Throughout the paper we take $\mathfrak{a}_1=\mathfrak{a}$ and $\mathfrak{b}_1=\mathfrak{b}$ to enhance the beauty of the paper.

Remark 1 If we take $\lambda=2$ in Lemma 2.1, then we have

$$\begin{aligned}&\frac{1}{8}\bigg[\psi(\Omega_1+\Omega_2-\mathfrak{a})+3\psi\bigg(\Omega_1+\Omega_2-\frac{2\mathfrak{a}+\mathfrak{b}}{3}\bigg)+3\psi\bigg(\Omega_1+\Omega_2-\frac{\mathfrak{a}+2\mathfrak{b}}{3}\bigg)\\&\quad+\psi(\Omega_1+\Omega_2-\mathfrak{b})\bigg]-\frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^\varpi}J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\\&\quad=(\mathfrak{b}-\mathfrak{a})\int_0^1\Phi^\varpi(\kappa)\psi'\big(\Omega_1+\Omega_2-\big(\kappa\mathfrak{a}+(1-\kappa)\mathfrak{b}\big)\big)\,d\kappa,\end{aligned}\tag{14}$$

which is a new equality in the literature.

- If we take $\varpi=1$ in (14), then we have

$$\begin{aligned}&\frac{1}{8}\bigg\{\psi(\Omega_1+\Omega_2-\mathfrak{a})+3\psi\bigg(\Omega_1+\Omega_2-\bigg(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\bigg)\bigg)\\&\quad+3\psi\bigg(\Omega_1+\Omega_2-\bigg(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\bigg)\bigg)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\bigg\}\\&\quad-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{(\Omega_1+\Omega_2-\mathfrak{b})}^{(\Omega_1+\Omega_2-\mathfrak{a})}\psi(\varkappa)\,d\varkappa\\&=(\mathfrak{b}-\mathfrak{a})\int_0^1\Phi(\kappa)\psi'\big(\Omega_1+\Omega_2-\big(\kappa\mathfrak{a}+(1-\kappa)\mathfrak{b}\big)\big)\,d\kappa,\end{aligned}$$

which is a new equality in the literature.

- If we take $\Omega_1 = \mathfrak{a}$, $\Omega_2 = \mathfrak{b}$ and $\varpi = 1$ in (14)

$$\frac{1}{8}\left[\psi(\mathfrak{a}) + 3\psi\left(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right) + 3\psi\left(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right) + \psi(\mathfrak{b})\right] - \frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi(\varkappa)\,d\varkappa$$
$$= (\mathfrak{b}-\mathfrak{a})\int_0^1 \Phi(\kappa)\psi'\big(\kappa\mathfrak{b} + (1-\kappa)\mathfrak{a}\big)\,d\kappa,$$

which is a result for generalized convexity and classical convex functions in the literature, respectively.

Lemma 2.2 *Let $\Omega_\Theta, \mathfrak{a}_\Theta, \mathfrak{b}_\Theta \in I$ for all $\Theta \in \{1,\ldots,\lambda\}$, $\mathfrak{a}_\lambda > \mathfrak{b}_\lambda$, $\varpi > 0$ and ψ be a differentiable function on an interval $I \subset \mathbb{R}$. If $\psi' \in L(I)$ and $\mathbf{\Omega}$ majorizes both $\mathbf{a}$ and $\mathbf{b}$, then*:

$$\frac{1}{6}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right) + 4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right) + \psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]$$
$$- \frac{\Gamma(1+\varpi)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)^{\varpi}} J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)$$
$$= \sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_0^1 p^{\varpi}(\kappa)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa,$$

where $p^{\varpi}(\kappa)$ is defined by

$$p^{\varpi}(\kappa) = \begin{cases} \kappa^{\varpi} - \frac{1}{6}, & \text{if } \kappa \in [0, \frac{1}{2}),\\ \kappa^{\varpi} - \frac{5}{6}, & \text{if } \kappa \in [\frac{1}{2}, 1].\end{cases}$$

Proof We can write

$$\mathbb{I} = \int_0^1 p^{\varpi}(\kappa)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa.$$

Now, using the definition of our kernel, we obtain

$$\mathbb{I} = \int_0^{\frac{1}{2}}\left(\kappa^{\varpi} - \frac{1}{6}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa$$
$$+ \int_{\frac{1}{2}}^1\left(\kappa^{\varpi} - \frac{5}{6}\right)\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa$$
$$= \mathbb{I}_1 + \mathbb{I}_2.$$

Where we have, by using integration by parts,

$$\mathbb{I}_1 = \left(\frac{1}{2}^{\varpi} - \frac{1}{6}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2})}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)} + \left(\frac{1}{6}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}$$
$$- \frac{\varpi}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\left(\int_0^{\frac{1}{2}}\kappa^{\varpi-1}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\big(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta\big)\right)d\kappa\right)$$

and

$$\mathbb{I}_2 = \left(\frac{1}{6}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta)} - \left(\frac{1}{2}^{\varpi} - \frac{5}{6}\right)\frac{\psi(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2})}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta)}$$
$$- \frac{\varpi}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta)}\left(\int_{\frac{1}{2}}^{1}\kappa^{\varpi-1}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\left(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta\right)\right)d\kappa\right)$$

and adding $\mathbb{I}_1$ and $\mathbb{I}_2$, we have

$$\begin{aligned}\mathbb{I} = {} & \frac{1}{6\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta)}\Bigg[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right) \\ & + 4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right) + \psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg] \\ & - \frac{\varpi}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta)}\left(\int_{0}^{1}\kappa^{\varpi-1}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\left(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta\right)\right)d\kappa\right). \end{aligned} \tag{15}$$

Now, Substituting $\varkappa = \sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}(\kappa\mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta)$ we have

$$\begin{aligned}\mathbb{I} = {} & \frac{1}{6\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta)}\Bigg[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right) \\ & + 4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right) + \psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg] - \frac{\varpi}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta))^{\varpi+1}} \\ & \times \int_{\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta}^{\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta}\psi(\varkappa)\left[\varkappa - \left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]^{\varpi-1}d\varkappa. \end{aligned} \tag{16}$$

To apply the definition we have to show that

$$\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta \leq \sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta.$$

As

$$\begin{aligned} & \mathfrak{a}_\lambda > \mathfrak{b}_\lambda, \\ & \mathfrak{a}_\lambda - \mathfrak{b}_\lambda > 0. \end{aligned} \tag{17}$$

Furthermore, $\mathbf{a} \prec \boldsymbol{\Omega}$ and $\mathbf{b} \prec \boldsymbol{\Omega}$. Then, we may write:

$$\begin{aligned} & \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta + \mathfrak{a}_\lambda = \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta + \mathfrak{a}_\lambda, \\ & \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta = \mathfrak{a}_\lambda - \mathfrak{b}_\lambda \end{aligned} \tag{18}$$

and using (17) in (18), we have

$$\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta>0,$$
$$-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta<-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta \tag{19}$$

and adding $\sum_{\Theta=1}^{\lambda-1}\Omega_\Theta$ to both sides of (19), we have

$$\sum_{\Theta=1}^{\lambda-1}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta<\sum_{\Theta=1}^{\lambda-1}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta. \tag{20}$$

Now, (16) implies

$$\begin{aligned}\mathbb{I}&=\frac{1}{6\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)}\Bigg[\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\Bigg)\\&\quad+4\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\Bigg)+\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\Bigg)\Bigg]\\&\quad-\frac{\varpi}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi+1}}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\Bigg(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\Bigg).\end{aligned} \tag{21}$$

Multiplying (21) by $\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)$, we obtain the required equality. □

Remark 2 If we take $\lambda=2$ in Lemma 2.2, we have

$$\begin{aligned}&\frac{1}{6}\bigg[\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\bigg(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\bigg)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\bigg]\\&\quad-\frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^\varpi}J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\\&=(\mathfrak{b}-\mathfrak{a})\int_0^1p^\varpi(\kappa)\psi'\big(\Omega_1+\Omega_2-\big(\kappa\mathfrak{a}+(1-\kappa)\mathfrak{b}\big)\big)\,d\kappa,\end{aligned} \tag{22}$$

which is a new inequality in the literature.

Here, we have different scenarios given as:

- If we take $\varpi=1$ in (22), we have

$$\begin{aligned}&\frac{1}{6}\bigg[\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\bigg(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\bigg)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\bigg]\\&\quad-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\Omega_1+\Omega_2-\mathfrak{b}}^{\Omega_1+\Omega_2-\mathfrak{a}}\psi(\varkappa)\,d\varkappa\\&=(\mathfrak{b}-\mathfrak{a})\int_0^1p(\kappa)\psi'\big(\Omega_1+\Omega_2-\big(\kappa\mathfrak{a}+(1-\kappa)\mathfrak{b}\big)\big)\,d\kappa,\end{aligned} \tag{23}$$

which is a new equality in the literature.

- If we take $\Omega_1 = \mathfrak{a}$, $\Omega_2 = \mathfrak{b}$ and $\varpi = 1$ in (22), we have

$$\frac{1}{6}\left\{\psi(\mathfrak{a}) + 4\psi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) + \psi(\mathfrak{b})\right\} - \frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi(\varkappa)\,d\varkappa$$
$$= (\mathfrak{b}-\mathfrak{a})\int_0^1 p(\kappa)\psi'\big(\kappa\mathfrak{b} + (1-\kappa)\mathfrak{a}\big)\,d\kappa, \tag{24}$$

which was proved by Alomari in [10].

3 Newton–Simpson-type inequalities via majorization

Numerous Newton–Mercer-type inequalities via majorization for convex function are presented in this section.

Theorem 3.1 *Under the assumptions of Lemma 2.1, if the mapping $|\psi'|$ is continuous convex on the interval I, then we have the following inequality*:

$$\begin{aligned}
&\Bigg|\frac{1}{8}\Bigg[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right) + 3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right) \\
&\quad + 3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right) + \psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg] \\
&\quad - \frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi}} J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg| \\
&\leq \sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\Bigg[\left(\Theta_1\sum_{\Theta=1}^{\lambda}\big|\psi'(\Omega_\Theta)\big| - \left(\Upsilon_1\sum_{\Theta=1}^{\lambda-1}\big|\psi'(\mathfrak{a}_\Theta)\big| + \Upsilon_2\sum_{\Theta=1}^{\lambda-1}\big|\psi'(\mathfrak{b}_\Theta)\big|\right)\right) \\
&\quad + \left(\Theta_2\sum_{\Theta=1}^{\lambda}\big|\psi'(\Omega_\Theta)\big| - \left(\Upsilon_3\sum_{\Theta=1}^{\lambda-1}\big|\psi'(\mathfrak{a}_\Theta)\big| + \Upsilon_4\sum_{\Theta=1}^{\lambda-1}\big|\psi'(\mathfrak{b}_\Theta)\big|\right)\right) \\
&\quad + \left(\Theta_3\sum_{\Theta=1}^{\lambda}\big|\psi'(\Omega_\Theta)\big| - \left(\Upsilon_5\sum_{\Theta=1}^{\lambda-1}\big|\psi'(\mathfrak{a}_\Theta)\big| + \Upsilon_6\sum_{\Theta=1}^{\lambda-1}\big|\psi'(\mathfrak{b}_\Theta)\big|\right)\right)\Bigg],
\end{aligned}$$

where notions Θ_1–Θ_3 and Υ_1–Υ_6 are explained below.

Proof By taking the modulus on both sides in Lemma 2.1 and utilizing the Niezgoda–Jensen–Mercer inequality (6) for $\epsilon = 2$, $\sigma_1 = \kappa$ and $\sigma_2 = 1-\kappa$, we have

$$\begin{aligned}
&\Bigg|\frac{1}{8}\Bigg[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right) + 3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right) \\
&\quad + 3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right) + \psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg] \\
&\quad - \frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi}} J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta)\int_0^1 \left|\Phi^{\varpi}(\kappa)\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\left(\kappa\mathfrak{b}_\Theta + (1-\kappa)\mathfrak{a}_\Theta\right)\right)\right| d\kappa \\
&\leq \int_0^{\frac{1}{3}}\left|\left(\kappa^{\varpi} - \frac{1}{8}\right)\right|\left[\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right| - \left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right| + (1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|\right)\right] d\kappa \\
&\quad + \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\kappa^{\varpi} - \frac{1}{2}\right)\right|\left[\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right| - \left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right| + (1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|\right)\right] d\kappa \\
&\quad + \int_{\frac{2}{3}}^{1}\left|\left(\kappa^{\varpi} - \frac{7}{8}\right)\right|\left[\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right| - \left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right| + (1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|\right)\right] d\kappa
\end{aligned} \tag{25}
$$

and we will utilize the following computations throughout the paper to develop new Newton–Mercer-type inequalities;

$$
\begin{aligned}
\Theta_1 &= \int_0^{\frac{1}{3}}\left|\left(\kappa^{\varpi} - \frac{1}{8}\right)\right| d\kappa \\
&= -\frac{1}{96} - \frac{1}{(1+\varpi)}\left[8^{-\varpi-1} - \left(3^{-\varpi-1} - 8^{-\varpi-1}\right)\right], \\
\Theta_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\kappa^{\varpi} - \frac{1}{2}\right)\right| d\kappa \\
&= \frac{1}{\varpi+1}\left[\frac{2^{1+\varpi} - 3^{1+\varpi}}{6^{1+\varpi}} + \frac{-3^{1+\varpi} + 4^{1+\varpi}}{6^{1+\varpi}}\right], \\
\Theta_3 &= \int_{\frac{2}{3}}^{1}\left|\left(\kappa^{\varpi} - \frac{7}{8}\right)\right| d\kappa \\
&= \frac{7}{96} + \frac{1}{\varpi+1}\left[1 - \left(\frac{8}{7}\right)^{-1-\varpi} - \left(\frac{7}{8}\right)^{1+\varpi} + \left(\frac{3}{2}\right)^{-1-\varpi}\right].
\end{aligned}
$$

We will use integration by parts to solve the following identities that will be utilized for our new results and enhance the beauty of the paper:

$$
\begin{aligned}
\Upsilon_1 &= \int_0^{\frac{1}{3}}\kappa\left|\left(\kappa^{\varpi} - \frac{1}{8}\right)\right| d\kappa \\
&= -\frac{23}{4608} - \frac{1}{(2+\varpi)}\left[8^{-\varpi-2} - (3^{-\varpi-2} - 8^{-\varpi-2}\right], \\
\Upsilon_2 &= \int_0^{\frac{1}{3}}(1-\kappa)\left|\left(\kappa^{\varpi} - \frac{1}{8}\right)\right| d\kappa \\
&= -\frac{25}{4608} - \frac{1}{(2+\varpi)}\left[3^{-\varpi-2} - 8^{-\varpi-2}\right] \\
&\quad + \frac{1}{(1+\varpi)}\left[3^{-\varpi-1} - 8^{-\varpi-1}\right] - \frac{2^{-3(2+\varpi)}(15+7\varpi)}{2+3\varpi+\varpi^2}, \\
\Upsilon_3 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\kappa\left|\left(\kappa^{\varpi} - \frac{1}{2}\right)\right| d\kappa \\
&= -\frac{1}{72} + \frac{1}{\varpi+2}\left[\frac{2^{2+\varpi} - 3^{2+\varpi}}{6^{2+\varpi}} + \frac{-3^{2+\varpi} + 4^{2+\varpi}}{6^{2+\varpi}}\right],
\end{aligned}
$$

$$\begin{aligned}
\Upsilon_4 &= \int_{\frac{1}{3}}^{\frac{2}{3}} (1-\kappa)\left|\left(\kappa^{\varpi} - \frac{1}{2}\right)\right| d\kappa \\
&= -\frac{1}{72} + \frac{1}{\varpi+2}\left[-\frac{2^{2+\varpi} - 3^{2+\varpi}}{6^{2+\varpi}} + \frac{-3^{2+\varpi} + 4^{2+\varpi}}{6^{2+\varpi}}\right] \\
&\quad + \frac{1}{\varpi+1}\left[-\frac{2^{1+\varpi} - 3^{1+\varpi}}{6^{1+\varpi}} + \frac{-3^{1+\varpi} + 4^{1+\varpi}}{6^{1+\varpi}}\right], \\
\Upsilon_5 &= \int_{\frac{2}{3}}^{1} \kappa\left|\left(\kappa^{\varpi} - \frac{7}{8}\right)\right| d\kappa \\
&= \frac{175}{4608} + \frac{1}{\varpi+2}\left[1 - \left(\frac{8}{7}\right)^{-2-\varpi} - \left(\frac{7}{8}\right)^{2+\varpi} + \left(\frac{3}{2}\right)^{-2-\varpi}\right], \\
\Upsilon_6 &= \int_{\frac{2}{3}}^{1} (1-\kappa)\left|\left(\kappa^{\varpi} - \frac{7}{8}\right)\right| d\kappa \\
&= \frac{161}{4608} + \frac{1}{\varpi+1}\left[1 - \left(\frac{8}{7}\right)^{-1-\varpi} - \left(\frac{7}{8}\right)^{1+\varpi} + \left(\frac{3}{2}\right)^{-1-\varpi}\right] \\
&\quad - \frac{1}{\varpi+2}\left[1 - \left(\frac{8}{7}\right)^{-2-\varpi} - \left(\frac{7}{8}\right)^{2+\varpi} + \left(\frac{3}{2}\right)^{-2-\varpi}\right].
\end{aligned}$$

By using computations Θ_1–Θ_3 and Υ_1–Υ_6 in equation (25), we obtain the required result. □

Remark 3 If we take $\lambda = 2$ in Theorem 3.1 then we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a}) + 3\psi\left(\Omega_1+\Omega_2 - \frac{2\mathfrak{a}+\mathfrak{b}}{3}\right) + 3\psi\left(\Omega_1+\Omega_2 - \frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)\right.\right. \\
&\quad \left.\left. + \psi(\Omega_1+\Omega_2-\mathfrak{b})\right] - \frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^{\varpi}} J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\right| \\
&\le (\mathfrak{b}-\mathfrak{a})\big[\big(\Theta_1|\psi'(\Omega_1)| + \Theta_1|\psi'(\Omega_2)| - \big(\Upsilon_1|\psi'(\mathfrak{a})| + \Upsilon_2|\psi'(\mathfrak{b})|\big)\big) \\
&\quad + \big(\Theta_2|\psi'(\Omega_1)| + \Theta_2|\psi'(\Omega_2)| - \big(\Upsilon_3|\psi'(\mathfrak{a})| + \Upsilon_4|\psi'(\mathfrak{b}_\Theta)|\big)\big) \\
&\quad + \big(\Theta_3|\psi'(\Omega_1)| + \Theta_3|\psi'(\Omega_2)| - \big(\Upsilon_5|\psi'(\mathfrak{a})| + \Upsilon_6|\psi'(\mathfrak{b})|\big)\big)\big],
\end{aligned} \tag{26}$$

which is a new inequality in the literature.

- If we take $\varpi = 1$ in equation (26) then we have

$$\begin{aligned}
&\left|\frac{1}{8}\left\{\psi(\Omega_1+\Omega_2-\mathfrak{a}) + 3\psi\left(\Omega_1+\Omega_2 - \left(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)\right)\right.\right. \\
&\quad \left. + 3\psi\left(\Omega_1+\Omega_2 - \left(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)\right) + \psi(\Omega_1+\Omega_2-\mathfrak{b})\right\} \\
&\quad \left. - \frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{(\Omega_1+\Omega_2-\mathfrak{b})}^{(\Omega_1+\Omega_2-\mathfrak{a})} \psi(\varkappa)\,d\varkappa\right| \\
&\le (\mathfrak{b}-\mathfrak{a})\left[\left(\frac{17}{576}|\psi'(\Omega_1)| + \frac{17}{576}|\psi'(\Omega_2)|\right.\right. \\
&\quad \left.\left. - \left(\frac{251}{41{,}472}|\psi'(\mathfrak{a})| + \frac{973}{41{,}472}|\psi'(\mathfrak{b})|\right)\right)\right.
\end{aligned}$$

$$+\left(\frac{1}{36}\left|\psi'(\Omega_1)\right|+\frac{1}{36}\left|\psi'(\Omega_2)\right|-\left(\frac{1}{72}\left|\psi'(\mathfrak{a})\right|+\frac{1}{72}\left|\psi'(\mathfrak{b})\right|\right)\right)$$
$$+\left(\frac{17}{576}\left|\psi'(\Omega_1)\right|+\frac{17}{576}\left|\psi'(\Omega_2)\right|\right.$$
$$\left.\left.-\left(\frac{973}{41{,}472}\left|\psi'(\mathfrak{a})\right|+\frac{251}{41{,}472}\left|\psi'(\mathfrak{b})\right|\right)\right)\right], \tag{27}$$

which is a new inequality in the literature.

- If we take $\Omega_1=\mathfrak{a}$, $\Omega_2=\mathfrak{b}$ and $\varpi=1$ in equation (26), then we have

$$\left|\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi(\varkappa)\,d\varkappa-\left(\frac{3}{8}\right)\psi\left(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)\right.$$
$$\left.-\left(\frac{3}{8}\right)\psi\left(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)-\frac{\psi(\mathfrak{a})+\psi(\mathfrak{b})}{8}\right|$$
$$\le(\mathfrak{b}-\mathfrak{a})\left(\frac{25}{576}\right)\left[\left|\psi'(\mathfrak{b})\right|+\left|\psi'(\mathfrak{a})\right|\right],$$

which appeared in [9].

Now, we give results for Simpson-type inequalities by employing Lemma 2.2.

Theorem 3.2 *Under the assumptions of Lemma 2.2, if the mapping $|\psi'|$ is continuous convex on I, then we have the following inequality*:

$$\left|\frac{1}{6}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]\right.$$
$$\left.-\frac{\Gamma(1+\varpi)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)^\varpi}J^{(\varpi)}_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|$$
$$\le\left(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\right)\left[G_1\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|-Q_1\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|-Q_2\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|\right]$$
$$+\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\left[G_2\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|-Q_3\sum_{\upsilon=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|-Q_4\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|\right],$$

where G_1, G_2, and Q_1–Q_4 are the computations explained below.

Proof By taking the modulus on both sides in Lemma 2.2 and using the Niezgoda–Jensen–Mercer inequality, we have

$$\left|\frac{1}{6}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]\right.$$
$$\left.-\frac{\Gamma(1+\varpi)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)^\varpi}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|$$
$$\le\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_0^1\left|p^\varpi(\kappa)\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta)\right)\right|d\kappa$$

$$
\begin{aligned}
&= \sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta) \int_0^{\frac{1}{2}} \left| \left(\kappa^{\varpi} - \frac{1}{6} \right) \right| \\
&\quad \times \left[\sum_{\Theta=1}^{\lambda} \left| \psi'(\Omega_\Theta) \right| - \left(\kappa \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{a}_\Theta) \right| + (1-\kappa) \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{b}_\Theta) \right| \right) \right] d\kappa \\
&\quad + \sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta - \mathfrak{a}_\Theta) \int_{\frac{1}{2}}^{1} \left| \left(\kappa^{\varpi} - \frac{5}{6} \right) \right| \\
&\quad \times \left[\sum_{\Theta=1}^{\lambda} \left| \psi'(\Omega_\Theta) \right| - \left(\kappa \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{a}_\Theta) \right| + (1-\kappa) \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{b}_\Theta) \right| \right) \right] d\kappa. \qquad (28)
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
G_1 &= \int_0^{\frac{1}{2}} \left| \left(\kappa^{\varpi} - \frac{1}{6} \right) \right| d\kappa \\
&= -\frac{1}{36} - \frac{2}{\varpi+1} \left(\frac{1}{6} \right)^{\varpi+1} + \frac{1}{\varpi+1} \left(\frac{1}{2} \right)^{\varpi+1}, \\
G_2 &= \int_{\frac{1}{2}}^{1} \left| \left(\kappa^{\varpi} - \frac{5}{6} \right) \right| d\kappa \\
&= \frac{5}{36} + \frac{1}{\varpi+1} \left[-2 \left(\frac{5}{6} \right)^{\varpi+1} + \left(\frac{1}{2} \right)^{\varpi+1} + 1 \right].
\end{aligned}
$$

Now, by using integration by parts we have

$$
\begin{aligned}
Q_1 &= \int_0^{\frac{1}{2}} \kappa \left| \left(\kappa^{\varpi} - \frac{1}{6} \right) \right| d\kappa \\
&= -\frac{7}{432} - \frac{2}{\varpi+2} \left(\frac{1}{6} \right)^{\varpi+2} + \frac{1}{\varpi+2} \left(\frac{1}{2} \right)^{\varpi+2}, \\
Q_2 &= \int_0^{\frac{1}{2}} (1-\kappa) \left| \left(\kappa^{\varpi} - \frac{1}{6} \right) \right| d\kappa \\
&= -\frac{5}{432} - \frac{1}{\varpi+1} \left[2 \left(\frac{1}{6} \right)^{\varpi+1} - \left(\frac{1}{2} \right)^{\varpi+1} \right] + \frac{1}{\varpi+2} \left[2 \left(\frac{1}{6} \right)^{\varpi+2} - \left(\frac{1}{2} \right)^{\varpi+2} \right], \\
Q_3 &= \int_{\frac{1}{2}}^{1} \kappa \left| \left(\kappa^{\varpi} - \frac{5}{6} \right) \right| d\kappa \\
&= \frac{25}{432} + \frac{1}{\varpi+2} \left[-2 \left(\frac{5}{6} \right)^{\varpi+2} + \left(\frac{1}{2} \right)^{\varpi+2} + 1 \right], \\
Q_4 &= \int_{\frac{1}{2}}^{1} (1-\kappa) \left| \left(\kappa^{\varpi} - \frac{5}{6} \right) \right| d\kappa \\
&= \frac{35}{432} + \frac{1}{\varpi+1} \left[-2 \left(\frac{5}{6} \right)^{\varpi+1} + \left(\frac{1}{2} \right)^{\varpi+1} + 1 \right] \\
&\quad - \frac{1}{\varpi+2} \left[-2 \left(\frac{5}{6} \right)^{\varpi+2} + \left(\frac{1}{2} \right)^{\varpi+2} + 1 \right].
\end{aligned}
$$

Using computations G_1, G_2, and Q_1–Q_4 in (28), we obtain the required result. □

Remark 4 If we take $\lambda = 2$ in Theorem 3.2, we have

$$\begin{aligned} &\left|\frac{1}{6}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]\right. \\ &\quad\left.-(\mathfrak{b}-\mathfrak{a})^{\varpi}J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\right| \\ &\leq(\mathfrak{b}-\mathfrak{a})\left[G_1\left|\psi'(\Omega_1)\right|+G_1\left|\psi'(\Omega_2)\right|-Q_1\left|\psi'(\mathfrak{a})\right|-Q_2\left|\psi'(\mathfrak{b})\right|\right] \\ &\quad+(\mathfrak{b}-\mathfrak{a})\left[G_2\left|\psi'(\Omega_1)\right|+G_2\left|\psi'(\Omega_2)\right|\right]-Q_3\left|\psi'(\mathfrak{a})\right|-Q_4\left|\psi'(\mathfrak{b})\right|\right], \end{aligned} \tag{29}$$

which is a new inequality in the literature.

- If we take $\varpi = 1$ in (29), we have

$$\begin{aligned} &\left|\frac{1}{6}\left\{\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right\}\right. \\ &\quad\left.-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\Omega_1+\Omega_2-\mathfrak{b}}^{\Omega_1+\Omega_2-\mathfrak{a}}\psi(\varkappa)d(\varkappa)\right| \\ &\leq(\mathfrak{b}-\mathfrak{a})\left\{\left(\frac{5}{72}\right)\left|\psi'(\Omega_1)\right|+\left(\frac{5}{72}\right)\left|\psi'(\Omega_2)\right|-\left(\frac{61}{1296}\left|\psi'(\mathfrak{a})\right|+\frac{29}{1296}\left|\psi'(\mathfrak{b})\right|\right)\right. \\ &\quad\left.+\left(\frac{5}{72}\right)\left|\psi'(\Omega_1)\right|+\left(\frac{5}{72}\right)\left|\psi'(\Omega_2)\right|-\left(\frac{29}{1296}\left|\psi'(\mathfrak{a})\right|+\frac{61}{1296}\left|\psi'(\mathfrak{b})\right|\right)\right\}, \end{aligned} \tag{30}$$

  which is a new inequality in the literature.

- If we take $\Omega_1 = \mathfrak{a}$, $\Omega_2 = \mathfrak{b}$ and $\varpi = 1$ in (29), we have

$$\begin{aligned} &\left|\frac{1}{6}\left\{\psi(\mathfrak{a})+4\psi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\mathfrak{b})\right\}-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi(\varkappa)\,d\varkappa\right| \\ &\leq\left(\frac{5}{72}\right)(\mathfrak{b}-\mathfrak{a})\left[\left|\psi'(\mathfrak{a})\right|+\left|\psi'(\mathfrak{b})\right|\right], \end{aligned} \tag{31}$$

  which was given by Alomari in [10].

Theorem 3.3 *Under the assumptions of Lemma 2.1, if the mapping $|\psi'|^q$ with $q > 1$ is continuous convex on the interval I, then we have the following inequality*:

$$\begin{aligned} &\left|\frac{1}{8}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\right.\right. \\ &\quad\left.+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right] \\ &\quad\left.-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi}}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right| \\ &\leq\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\left[(\Theta_1)^{1-\frac{1}{q}}\left(\Theta_1\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q\right.\right. \\ &\quad\left.\left.-\left(\Upsilon_1\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+\Upsilon_2\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)^{\frac{1}{q}}\right. \end{aligned}$$

$$+ (\Theta_2)^{1-\frac{1}{q}} \left(\Theta_2 \sum_{\Theta=1}^{\lambda} \left| \psi'(\Omega_\Theta) \right|^q - \left(\Upsilon_3 \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{a}_\Theta) \right|^q + \Upsilon_4 \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{b}_\Theta) \right|^q \right) \right)^{\frac{1}{q}}$$

$$+ (\Theta_3)^{1-\frac{1}{q}} \left(\Theta_3 \sum_{\Theta=1}^{\lambda} \left| \psi'(\Omega_\Theta) \right|^q - \left(\Upsilon_5 \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{a}_\Theta) \right|^q + \Upsilon_6 \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{b}_\Theta) \right|^q \right) \right)^{\frac{1}{q}} \Bigg],$$

where notions Θ_1–Θ_3 *and* Υ_1–Υ_6 *are defined in Theorem* 3.1.

Proof By taking the modulus on both sides in Lemma 2.1, we have

$$\Bigg| \frac{1}{8} \Bigg[\psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{a}_\Theta \right) + 3\psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \frac{2\mathfrak{a}_\Theta + \mathfrak{b}_\Theta}{3} \right)$$

$$+ 3\psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \frac{\mathfrak{a}_\Theta + 2\mathfrak{b}_\Theta}{3} \right) + \psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{b}_\Theta \right) \Bigg]$$

$$- \frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1} (\mathfrak{b}_\Theta - \mathfrak{a}_\Theta))^\varpi} J_{(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{a}_\Theta)^-} \psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{b}_\Theta \right) \Bigg|$$

$$\leq \sum_{\Theta=1}^{\lambda-1} (\mathfrak{b}_\Theta - \mathfrak{a}_\Theta) \int_0^{\frac{1}{3}} \left| \kappa^\varpi - \frac{1}{8} \right| \left| \psi' \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} (\kappa \mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta) \right) \right| d\kappa$$

$$+ \sum_{\Theta=1}^{\lambda-1} (\mathfrak{b}_\Theta - \mathfrak{a}_\Theta) \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \kappa^\varpi - \frac{1}{2} \right| \left| \psi' \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} (\kappa \mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta) \right) \right| d\kappa$$

$$+ \sum_{\Theta=1}^{\lambda-1} (\mathfrak{b}_\Theta - \mathfrak{a}_\Theta) \int_{\frac{2}{3}}^{1} \left| \kappa^\varpi - \frac{7}{8} \right| \left| \psi' \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} (\kappa \mathfrak{a}_\Theta + (1-\kappa)\mathfrak{b}_\Theta) \right) \right| d\kappa$$

by using the power-mean inequality and Niezgoda–Jensen–Mercer inequality, we have

$$\Bigg| \frac{1}{8} \Bigg[\psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{a}_\Theta \right) + 3\psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \frac{2\mathfrak{a}_\Theta + \mathfrak{b}_\Theta}{3} \right)$$

$$+ 3\psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \frac{\mathfrak{a}_\Theta + 2\mathfrak{b}_\Theta}{3} \right) + \psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{b}_\Theta \right) \Bigg]$$

$$- \frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1} (\mathfrak{b}_\Theta - \mathfrak{a}_\Theta))^\varpi} J_{(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{a}_\Theta)^-} \psi \left(\sum_{\Theta=1}^{\lambda} \Omega_\Theta - \sum_{\Theta=1}^{\lambda-1} \mathfrak{b}_\Theta \right) \Bigg|$$

$$\leq \sum_{\Theta=1}^{\lambda-1} (\mathfrak{b}_\Theta - \mathfrak{a}_\Theta) \Bigg[\left(\int_0^{\frac{1}{3}} \left| \kappa^\varpi - \frac{1}{8} \right| \right)^{1-\frac{1}{q}} \Bigg(\int_0^{\frac{1}{3}} \left| \kappa^\varpi - \frac{1}{8} \right| \Bigg(\sum_{\Theta=1}^{\lambda} \left| \psi'(\Omega_\Theta) \right|^q$$

$$- \left(\kappa \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{a}_\Theta) \right|^q + (1-\kappa) \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{b}_\Theta) \right|^q \right) \Bigg) d\kappa \Bigg)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \kappa^\varpi - \frac{1}{2} \right| d\kappa \right)^{1-\frac{1}{q}} \Bigg(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \kappa^\varpi - \frac{1}{2} \right| \Bigg(\sum_{\Theta=1}^{\lambda} \left| \psi'(\Omega_\Theta) \right|^q$$

$$- \left(\kappa \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{a}_\Theta) \right|^q + (1-\kappa) \sum_{\Theta=1}^{\lambda-1} \left| \psi'(\mathfrak{b}_\Theta) \right|^q \right) \Bigg) d\kappa \Bigg)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left|\kappa^{\varpi}-\frac{7}{8}\right|d\kappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|\kappa^{\varpi}-\frac{7}{8}\right|\left(\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_{\Theta})\right|^{q}\right.\right.$$
$$\left.\left.-\left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_{\Theta})\right|^{q}+(1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_{\Theta})\right|^{q}\right)\right)d\kappa\right)^{\frac{1}{q}}\Bigg]. \quad (32)$$

By using computations Θ_1–Θ_3 and Υ_1–Υ_6 given in Theorem 3.1, the proof is completed. □

Remark 5 If we take $\lambda = 2$ in Theorem 3.3, then we have

$$\left|\frac{1}{8}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+3\psi\left(\Omega_1+\Omega_2-\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)+3\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)\right.\right.$$
$$\left.\left.+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]-\frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^{\varpi}}J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\right|$$
$$\leq(\mathfrak{b}-\mathfrak{a})\big[(\Theta_1)^{1-\frac{1}{q}}\big(\Theta_1\left|\psi'(\Omega_1)\right|^q+\Theta_1\left|\psi'(\Omega_2)\right|^q-\big(\Upsilon_1\left|\psi'(\mathfrak{a})\right|^q+\Upsilon_2\left|\psi'(\mathfrak{b})\right|^q\big)\big)^{\frac{1}{q}}$$
$$+(\Theta_2)^{1-\frac{1}{q}}\big(\Theta_2\left|\psi'(\Omega_1)\right|^q+\Theta_2\left|\psi'(\Omega_2)\right|^q-\big(\Upsilon_3\left|\psi'(\mathfrak{a})\right|^q+\Upsilon_4\left|\psi'(\mathfrak{b})\right|^q\big)\big)^{\frac{1}{q}}$$
$$+(\Theta_3)^{1-\frac{1}{q}}\big(\Theta_3\left|\psi'(\Omega_1)\right|^q+\Theta_3\left|\psi'\Omega_2\right|^q-\big(\Upsilon_5\left|\psi'(\mathfrak{a})\right|^q+\Upsilon_6\left|\psi'(\mathfrak{b})\right|^q\big)\big)^{\frac{1}{q}}\big], \quad (33)$$

which is a new inequality in the literature.

- If we take $\varpi = 1$ in (33), then we have

$$\left|\frac{1}{8}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+3\psi\left(\Omega_1+\Omega_2-\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)+3\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)\right.\right.$$
$$\left.\left.+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{(\Omega_1+\Omega_2-\mathfrak{b})}^{(\Omega_1+\Omega_2-\mathfrak{a})}\psi(\varkappa)\,d\varkappa\right|$$
$$\leq(\mathfrak{b}-\mathfrak{a})\left[\left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left[\left(\frac{17}{576}\right)\left|\psi'(\Omega_1)\right|^q+\left(\frac{17}{576}\right)\left|\psi'(\Omega_2)\right|^q\right.\right.$$
$$\left.-\left(\frac{251}{41{,}472}\left|\psi'(\mathfrak{a})\right|^q+\frac{973}{41{,}472}\left|\psi'(\mathfrak{b})\right|^q\right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{1}{36}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{36}\right)\left|\psi'(\Omega_1)\right|^q+\left(\frac{1}{36}\right)\left|\psi'(\Omega_2)\right|^q\right.$$
$$\left.-\left(\frac{1}{72}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{72}\left|\psi'(\mathfrak{b})\right|^q\right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left[\left(\frac{17}{576}\right)\left|\psi'(\Omega_1)\right|^q+\left(\frac{17}{576}\right)\left|\psi'(\Omega_2)\right|^q\right.$$
$$\left.\left.-\left(\frac{973}{41{,}472}\left|\psi'(\mathfrak{a})\right|^q+\frac{251}{41{,}472}\left|\psi'(\mathfrak{b})\right|^q\right)\right]^{\frac{1}{q}}\right],$$

which is a new inequality in the literature.

- If we take $\Omega_1 = \mathfrak{a}$, $\Omega_2 = \mathfrak{b}$ and $\varpi = 1$ in (33), then we have

$$\begin{aligned}
&\left|\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi(\varkappa)\,d\varkappa - \left(\frac{3}{8}\right)\psi\left(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)\right.\\
&\qquad\left. - \left(\frac{3}{8}\right)\psi\left(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right) - \frac{\psi(\mathfrak{a})+\psi(\mathfrak{b})}{8}\right|\\
&\quad\leq \frac{(\mathfrak{b}-\mathfrak{a})}{36}\left[\left(\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left[\frac{251}{1152}\left|\psi'(\mathfrak{b})\right|^q + \frac{973}{1152}\left|\psi'(\mathfrak{a})\right|^q\right]^{\frac{1}{q}}\right)\right.\\
&\qquad + \left[\frac{1}{2}\left|\psi'(\mathfrak{b})\right|^q + \frac{1}{2}\left|\psi'(\mathfrak{a})\right|^q\right]^{\frac{1}{q}}\\
&\qquad\left. + \left(\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left[\frac{973}{1152}\left|\psi'(\mathfrak{b})\right|^q + \frac{251}{1152}\left|\psi'(\mathfrak{a})\right|^q\right]^{\frac{1}{q}}\right)\right],
\end{aligned}$$

which appeared in [9].

Theorem 3.4 *Under the assumptions of Lemma 2.2, if the mapping $|\psi'|^q$ is continuous convex with $q > 1$ on the interval I, then we have the following inequality*:

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right) + 4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right) + \psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]\right.\\
&\qquad\left. - \frac{\Gamma(1+\varpi)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)^\varpi} J^{(\varpi)}_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta - \sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|\\
&\quad\leq \sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)(G_1)^{1-\frac{1}{q}}\left[G_1\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q - Q_1\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q - Q_2\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right]^{\frac{1}{q}}\\
&\qquad + \sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)(G_2)^{1-\frac{1}{q}}\\
&\qquad \times\left[G_2\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q - Q_3\sum_{\upsilon=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q - Q_4\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right]^{\frac{1}{q}},
\end{aligned}$$

where G_1, G_2, and Q_1–Q_4 are the computations explained in Theorem 3.2.

Proof The proof of the theorem is similar to the proof of Theorem 3.3. □

Remark 6 If we take $\lambda = 2$ in Theorem 3.4, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a}) + 4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) + \psi(\Omega_1+\Omega_2-\mathfrak{b})\right]\right.\\
&\qquad\left. - \frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^\varpi} J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\right|\\
&\quad\leq (\mathfrak{b}-\mathfrak{a})(G_1)^{1-\frac{1}{q}}\left[G_1\left|\psi'(\Omega_1)\right|^q + G_1\left|\psi'(\Omega_2)\right|^q - Q_1\left|\psi'(\mathfrak{a})\right|^q - Q_2\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}\\
&\qquad + (\mathfrak{b}-\mathfrak{a})(G_2)^{1-\frac{1}{q}}\left[G_2\left|\psi'(\Omega_1)\right|^q + G_2\left|\psi'(\Omega_2)\right|^q - Q_3\left|\psi'(\mathfrak{a})\right|^q - Q_4\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}. \qquad (34)
\end{aligned}$$

- If we take $\varpi = 1$ in (34), we have

$$\begin{aligned}&\left|\frac{1}{6}\left\{\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right\}\right.\\&\quad\left.-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\Omega_1+\Omega_2-\mathfrak{b}}^{\Omega_1+\Omega_2-\mathfrak{a}}\psi(\varkappa)\,d\varkappa\right|\\&\le(\mathfrak{b}-\mathfrak{a})\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left[\frac{5}{72}\left|\psi'(\Omega_1)\right|^q+\frac{5}{72}\left|\psi'(\Omega_2)\right|^q\right.\right.\\&\quad\left.-\left(\frac{29}{1296}\left|\psi'(\mathfrak{a})\right|^q+\frac{61}{1296}\left|\psi'(\mathfrak{b})\right|^q\right)\right]^{\frac{1}{q}}\\&\quad\left.+\left[\frac{5}{72}\left|\psi'(\Omega_1)\right|^q+\frac{5}{72}\left|\psi'(\Omega_2)\right|^q-\left(\frac{61}{1296}\left|\psi'(\mathfrak{a})\right|^q+\frac{29}{1296}\left|\psi'(\mathfrak{b})\right|^q\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

  which is a new inequality in the literature.
- If we take $\Omega_1 = \mathfrak{a}$, $\Omega_2 = \mathfrak{b}$ $\varpi = 1$ in (34), we have

$$\begin{aligned}&\left|\frac{1}{6}\left\{\psi(\mathfrak{a})+4\psi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\mathfrak{b})\right\}-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi(\varkappa)\,d\varkappa\right|\\&\le(\mathfrak{b}-\mathfrak{a})\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left[\frac{29}{1296}\left|\psi'(\mathfrak{a})\right|^q+\frac{61}{1296}\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}\right.\\&\quad\left.+\left[\frac{61}{1296}\left|\psi'(\mathfrak{a})\right|^q+\frac{29}{1296}\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}\right\},\end{aligned}$$

  which was proved by Alomari in [10].

Theorem 3.5 *Under the assumptions of Lemma 2.1, if the mapping $|\psi'|^q$ is continuous convex on the interval I, then*:

$$\begin{aligned}&\left|\frac{1}{8}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\right.\right.\\&\quad\left.+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]\\&\quad\left.-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi}}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|\\&\le\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\left[\left(\frac{6-8^{\frac{1}{\varpi}}}{3.8^{\frac{1+\varpi p}{\varpi}}}+\frac{1}{(\varpi p+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1}-2.\left(\frac{1}{8^{\frac{1}{\varpi}}}\right)^{\varpi p+1}\right]\right)^{\frac{1}{p}}\right.\\&\quad\times\left(\frac{1}{3}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\left(\frac{1}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+\frac{5}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)^{\frac{1}{q}}\\&\quad+\left(\frac{6-2^{\frac{1}{\varpi}}}{3.2^{\frac{1+\varpi p}{\varpi}}}+\frac{1}{(p\varpi+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1}-2.\left(\frac{1}{2^{\frac{1}{\varpi}}}\right)^{\varpi p+1}+\left(\frac{2}{3}\right)^{\varpi p+1}\right]\right)^{\frac{1}{p}}\end{aligned}$$

$$\times\left(\frac{1}{3}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\left(\frac{1}{6}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+\frac{1}{6}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\left(\left(2.\left(\frac{7}{8}\right)^{\frac{1}{\varpi}}-1\right)\left(\frac{7}{8}\right)^p+\frac{1}{(p\varpi+1)}\left[-2.\left(\frac{7}{8}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{2}{3}\right)^{\varpi p+1}+1\right]\right)^{\frac{1}{p}}$$

$$\left.\times\left(\frac{1}{3}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\left(\frac{5}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+\frac{1}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)^{\frac{1}{q}}\right]$$

holds, where p and q are conjugate exponents with $p,q>1$.

Proof Utilizing Lemma 2.1 along with the modulus property yields,

$$\left|\frac{1}{8}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\right.\right.$$
$$\left.+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]$$
$$\left.-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^\varpi}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|$$
$$\leq\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_0^{\frac{1}{3}}\left|\kappa^\varpi-\frac{1}{8}\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)\right|d\kappa$$
$$+\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\kappa^\varpi-\frac{1}{2}\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)\right|d\kappa$$
$$+\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_{\frac{2}{3}}^{1}\left|\kappa^\varpi-\frac{7}{8}\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\bigl(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta\bigr)\right)\right|d\kappa.$$

By using Hölder's inequality and the Niezgoda–Jensen–Mercer inequality,

$$\left|\frac{1}{8}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\right.\right.$$
$$\left.+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]$$
$$\left.-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^\varpi}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|$$
$$\leq\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\left[\left(\int_0^{\frac{1}{3}}\left|\kappa^\varpi-\frac{1}{8}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q\right.\right.\right.$$
$$\left.\left.-\left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+(1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)d\kappa\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\kappa^\varpi-\frac{1}{2}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q\right.\right.$$

$$
\begin{aligned}
&- \left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q + (1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\Bigg) d\kappa\Bigg)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{2}{3}}^{1}\left|\kappa^{\varpi}-\frac{7}{8}\right|^p d\kappa\right)^{\frac{1}{p}}\Bigg(\int_{\frac{2}{3}}^{1}\Bigg(\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q \\
&- \left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q + (1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\Bigg) d\kappa\Bigg)^{\frac{1}{q}}\Bigg]. \qquad (35)
\end{aligned}
$$

Consider

$$
\begin{aligned}
&\int_0^{\frac{1}{3}}\left|\kappa^{\varpi}-\frac{1}{8}\right|^p d\kappa \\
&= \int_0^{(\frac{1}{8})^{\frac{1}{\varpi}}}\left(\frac{1}{8}-\kappa^{\varpi}\right)^p d\kappa + \int_{(\frac{1}{8})^{\varpi}}^{\frac{1}{3}}\left(\kappa^{\varpi}-\frac{1}{8}\right)^p d\kappa \\
&\leq \int_0^{(\frac{1}{8})^{\frac{1}{\varpi}}}\left(\frac{1}{8^p}-\kappa^{\varpi p}\right) d\kappa + \int_{(\frac{1}{8})^{\varpi}}^{\frac{1}{3}}\left(\kappa^{\varpi p}-\frac{1}{8^p}\right) d\kappa \\
&= \frac{6-8^{\frac{1}{\varpi}}}{3.8^{\frac{1+\varpi p}{\varpi}}} + \frac{1}{(\varpi p+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1} - 2.\left(\frac{1}{8^{\frac{1}{\varpi}}}\right)^{\varpi p+1}\right],
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\kappa^{\varpi}-\frac{1}{2}\right|^p d\kappa \\
&\leq \frac{6-2^{\frac{1}{\varpi}}}{3.2^{\frac{1+\varpi p}{\varpi}}} + \frac{1}{(p\varpi+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1} - 2.\left(\frac{1}{2^{\frac{1}{\varpi}}}\right)^{\varpi p+1} + \left(\frac{2}{3}\right)^{\varpi p+1}\right], \\
&\int_{\frac{2}{3}}^{1}\left|\kappa^{\varpi}-\frac{7}{8}\right|^p d\kappa \\
&\leq \left(2.\left(\frac{7}{8}\right)^{\frac{1}{\varpi}}-1\right)\left(\frac{7}{8}\right)^p + \frac{1}{(p\varpi+1)}\left[-2.\left(\frac{7}{8}\right)^{\frac{\varpi p+1}{\varpi}} + \left(\frac{2}{3}\right)^{\varpi p+1}+1\right].
\end{aligned}
$$

It follows from $(\Lambda_1-\Lambda_2)^p \leq (\Lambda_1^p-\Lambda_2^p)$ for any $\Lambda_1 \geq \Lambda_2 \geq 0$ and $p > 1$. □

Remark 7 If we take $\lambda = 2$ in Theorem 3.5, then we have

$$
\begin{aligned}
&\Bigg|\frac{1}{8}\Bigg[\psi(\Omega_1+\Omega_2-\mathfrak{a}) + 3\psi\left(\Omega_1+\Omega_2-\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right) + 3\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right) \\
&\quad + \psi(\Omega_1+\Omega_2-\mathfrak{b})\Bigg] - \frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^{\varpi}} J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\Bigg| \\
&\leq (\mathfrak{b}-\mathfrak{a})\Bigg[\left(\frac{6-8^{\frac{1}{\varpi}}}{3.8^{\frac{1+\varpi p}{\varpi}}} + \frac{1}{(\varpi p+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1} - 2.\left(\frac{1}{8^{\frac{1}{\varpi}}}\right)^{\varpi p+1}\right]\right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{3}\left[\left|\psi'(\Omega_1)\right|^q + \left|\psi'(\Omega_2)\right|^q\right] - \left(\frac{1}{18}\left|\psi'(\mathfrak{a})\right|^q + \frac{5}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\frac{6-2^{\frac{1}{\varpi}}}{3.2^{\frac{1+\varpi p}{\varpi}}}+\frac{1}{(p\varpi+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1}-2.\left(\frac{1}{2^{\frac{1}{\varpi}}}\right)^{\varpi p+1}+\left(\frac{2}{3}\right)^{\varpi p+1}\right]\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{3}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\left(\frac{1}{6}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{6}\left|\psi'(\mathfrak{b})\right|^q\right)\right)^{\frac{1}{q}}$$
$$+\left(\left(2.\left(\frac{7}{8}\right)^{\frac{1}{\varpi}}-1\right)\left(\frac{7}{8}\right)^p+\frac{1}{(p\varpi+1)}\left[-2.\left(\frac{7}{8}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{2}{3}\right)^{\varpi p+1}+1\right]\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{3}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\left(\frac{5}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right)^{\frac{1}{q}}\Bigg], \tag{36}$$

which is a new inequality in the literature.

- If we take $\varpi = 1$ in (36), then we have

$$\left|\frac{1}{8}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+3\psi\left(\Omega_1+\Omega_2-\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)+3\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)\right.\right.$$
$$\left.\left.+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{(\Omega_1+\Omega_2-\mathfrak{b})}^{(\Omega_1+\Omega_2-\mathfrak{a})}\psi(\varkappa)\,d\varkappa\right|$$
$$\leq(\mathfrak{b}-\mathfrak{a})\Bigg[\left(\frac{-2}{3.8^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{8}\right)^{p+1}\right]\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{3}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\left(\frac{1}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{5}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{4}{3.2^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{2}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}\right]\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{3}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\left(\frac{1}{6}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{6}\left|\psi'(\mathfrak{b})\right|^q\right)\right)^{\frac{1}{q}}$$
$$+\left(\left(\frac{3}{4}\right)\left(\frac{7}{8}\right)^p+\frac{1}{(p+1)}\left[-2.\left(\frac{7}{8}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}+1\right]\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{3}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\left(\frac{5}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right)^{\frac{1}{q}}\Bigg],$$

which is a new inequality in the literature.

- If we take $\Omega_1 = \mathfrak{a}$, $\Omega_2 = \mathfrak{b}$ and $\varpi = 1$ in (36), then we have

$$\left|\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi(\varkappa)\,d\varkappa-\left(\frac{3}{8}\right)\psi\left(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)\right.$$
$$\left.-\left(\frac{3}{8}\right)\psi\left(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)-\frac{\psi(\mathfrak{a})+\psi(\mathfrak{b})}{8}\right|$$
$$\leq(\mathfrak{b}-\mathfrak{a})\Bigg\{\left(\frac{-2}{3.8^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{8}\right)^{p+1}\right]\right)^{\frac{1}{p}}$$
$$\times\left(\frac{5}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{18}\left|\psi'(\mathfrak{b})\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{4}{3.2^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{2}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}\right]\right)^{\frac{1}{p}}$$

$$\times\left(\frac{1}{6}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{6}\left|\psi'(\mathfrak{b})\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\left(\frac{3}{4}\right)\left(\frac{7}{8}\right)^p+\frac{1}{(p+1)}\left[-2.\left(\frac{7}{8}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}+1\right]\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{5}{18}\left|\psi'(\mathfrak{b})\right|^q\right)^{\frac{1}{q}}\Bigg\}.$$

Theorem 3.6 *Under the assumptions of Lemma 2.2, if the mapping $|\psi'|^q$ is continuous convex on the interval I, then we have the following inequality*:

$$\left|\frac{1}{6}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]\right.$$
$$\left.-\frac{\Gamma(1+\varpi)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)^\varpi}J^{(\varpi)}_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|$$
$$\leq\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\left[\left[\left(\frac{2}{6^{\frac{1}{\varpi}}}-\frac{1}{2}\right).\frac{1}{6^p}+\frac{1}{\varpi p+1}\left(-2\left(\frac{1}{6}\right)^{\frac{\varpi p+1}{\varpi}}+1\right)\right]^{\frac{1}{p}}\right.$$
$$\times\left[\frac{1}{2}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\frac{1}{8}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q-\frac{3}{8}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right]^{\frac{1}{q}}$$
$$+\left[\left(2.\left(\frac{5}{6}\right)^{\frac{1}{\varpi}}-\frac{3}{2}\right).\left(\frac{5}{6}\right)^p+\frac{1}{\varpi p+1}\left(-2\left(\frac{5}{6}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{1}{2}\right)^{\varpi p+1}+1\right)\right]^{\frac{1}{p}}$$
$$\left.\times\left[\frac{1}{2}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\frac{3}{8}\sum_{\upsilon=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q-\frac{1}{8}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right]^{\frac{1}{q}}\right],$$

where p and q are conjugate exponents with $p,q>1$.

Proof The proof of the theorem is similar to the proof of Theorem 3.5. □

Remark 8 If we take $\lambda=2$ in Theorem 3.6, we have

$$\left|\frac{1}{6}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]\right.$$
$$\left.-\frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^\varpi}J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\right|$$
$$\leq(\mathfrak{b}-\mathfrak{a})\left[\left[\left(\frac{2}{6^{\frac{1}{\varpi}}}-\frac{1}{2}\right).\frac{1}{6^p}+\frac{1}{\varpi p+1}\left(-2\left(\frac{1}{6}\right)^{\frac{\varpi p+1}{\varpi}}+1\right)\right]^{\frac{1}{p}}\right.$$
$$\times\left[\frac{1}{2}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\frac{1}{8}\left|\psi'(\mathfrak{a})\right|^q-\frac{3}{8}\|\psi'(\mathfrak{b})|^q\right]^{\frac{1}{q}}$$
$$+\left[\left(2.\left(\frac{5}{6}\right)^{\frac{1}{\varpi}}-\frac{3}{2}\right).\left(\frac{5}{6}\right)^p+\frac{1}{\varpi p+1}\left(-2\left(\frac{5}{6}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{1}{2}\right)^{\varpi p+1}+1\right)\right]^{\frac{1}{p}}$$
$$\left.\times\left[\frac{1}{2}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\frac{3}{8}\left|\psi'(\mathfrak{a})\right|^q-\frac{1}{8}\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}\right],\tag{37}$$

which is a new inequality in the literature.

- If we take $\varpi = 1$ in equation (37), we have

$$\begin{aligned}&\bigg|\frac{1}{6}\left\{\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right\}\\&\quad-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\Omega_1+\Omega_2-\mathfrak{b}}^{\Omega_1+\Omega_2-\mathfrak{a}}\psi(\varkappa)\,d\varkappa\bigg|\\&\quad\le(\mathfrak{b}-\mathfrak{a})\left\{\left[-\frac{1}{6^{p+1}}+\frac{1}{p+1}\left(-2\left(\frac{1}{6}\right)^{p+1}+1\right)\right]^{\frac{1}{p}}\right.\\&\qquad\times\left[\frac{1}{2}\left|\psi'(\Omega_1)\right|^q+\frac{1}{2}\left|\psi'(\Omega_2)\right|^q-\left(\frac{1}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{3}{8}\left|\psi'(\mathfrak{b})\right|^q\right)\right]^{\frac{1}{q}}\\&\qquad+\left[\left(\frac{1}{6}\right).\left(\frac{5}{6}\right)^p+\frac{1}{p+1}\left(-2\left(\frac{5}{6}\right)^{p+1}+\left(\frac{1}{2}\right)^{p+1}+1\right)\right]^{\frac{1}{p}}\\&\qquad\left.\times\left[\frac{1}{2}\left|\psi'(\Omega_1)\right|^q+\frac{1}{2}\left|\psi'(\Omega_2)\right|^q-\left(\frac{3}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{8}\left|\psi'(\mathfrak{b})\right|^q\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

  which is a new inequality in the literature.
- If we take $\Omega_1=\mathfrak{a}$, $\Omega_2=\mathfrak{b}$ $\varpi = 1$ in (37), we have

$$\begin{aligned}&\left|\frac{1}{6}\left\{\psi(\mathfrak{a})+4\psi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\mathfrak{b})\right\}-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi(\varkappa)\,d\varkappa\right|\\&\quad\le(\mathfrak{b}-\mathfrak{a})\left\{\left[-\frac{1}{6^{p+1}}+\frac{1}{p+1}\left(-2\left(\frac{1}{6}\right)^{p+1}+1\right)\right]^{\frac{1}{p}}\left[\frac{3}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{8}\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}\right.\\&\qquad+\left[\left(\frac{1}{6}\right).\left(\frac{5}{6}\right)^p+\frac{1}{p+1}\left(-2\left(\frac{5}{6}\right)^{p+1}+\left(\frac{1}{2}\right)^{p+1}+1\right)\right]^{\frac{1}{p}}\\&\qquad\left.\times\left[\frac{1}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{3}{8}\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}\right\},\end{aligned}$$

  which is a new inequality in the literature.

Theorem 3.7 *Under the assumptions of Lemma 2.1 and if the mapping $|\psi'|^q$ is continuous convex on the interval I, then we have the following inequality*:

$$\begin{aligned}&\Bigg|\frac{1}{8}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\right.\\&\quad\left.+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]\\&\quad-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi}}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg|\\&\quad\le\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\left[\frac{1}{p}\left(\frac{6-8^{\frac{1}{\varpi}}}{3.8^{\frac{1+\varpi p}{\varpi}}}+\frac{1}{(\varpi p+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1}-2.\left(\frac{1}{8^{\frac{1}{\varpi}}}\right)^{\varpi p+1}\right]\right)\right.\\&\quad+\frac{1}{q}\left(\frac{1}{3}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\left(\frac{1}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+\frac{5}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)\end{aligned}$$

$$+\frac{1}{p}\left(\frac{6-2^{\frac{1}{\varpi}}}{3.2^{\frac{1+\varpi p}{\varpi}}}+\frac{1}{(p\varpi+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1}-2.\left(\frac{1}{2^{\frac{1}{\varpi}}}\right)^{\varpi p+1}+\left(\frac{2}{3}\right)^{\varpi p+1}\right]\right)$$
$$+\frac{1}{q}\left(\frac{1}{3}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\left(\frac{1}{6}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+\frac{1}{6}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)$$
$$+\frac{1}{p}\left(\left(2.\left(\frac{7}{8}\right)^{\frac{1}{\varpi}}-1\right)\left(\frac{7}{8}\right)^p+\frac{1}{(p\varpi+1)}\left[-2.\left(\frac{7}{8}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{2}{3}\right)^{\varpi p+1}+1\right]\right)$$
$$+\frac{1}{q}\left(\frac{1}{3}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\left(\frac{5}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+\frac{1}{18}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)\Bigg]$$

holds, where p and q are conjugate exponents with $p,q>1$.

Proof Utilizing Lemma 2.1 along with the modulus property yields,

$$\begin{aligned}
&\Bigg|\frac{1}{8}\Bigg[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\\
&\quad+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg]\\
&\quad-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi}}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg|\\
&\le\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_0^{\frac{1}{3}}\left|\kappa^{\varpi}-\frac{1}{8}\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta)\right)\right|d\kappa\\
&\quad+\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\kappa^{\varpi}-\frac{1}{2}\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta)\right)\right|d\kappa\\
&\quad+\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\int_{\frac{2}{3}}^{1}\left|\kappa^{\varpi}-\frac{7}{8}\right|\left|\psi'\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}(\kappa\mathfrak{a}_\Theta+(1-\kappa)\mathfrak{b}_\Theta)\right)\right|d\kappa
\end{aligned}$$

and by using Young's inequality and the Niezgoda–Jensen–Mercer inequality we have

$$\begin{aligned}
&\Bigg|\frac{1}{8}\Bigg[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{2\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{3}\right)\\
&\quad+3\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+2\mathfrak{b}_\Theta}{3}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg]\\
&\quad-\frac{\Gamma(1+\varpi)}{(\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta))^{\varpi}}J_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\Bigg|\\
&\le\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\Bigg[\frac{1}{p}\left(\int_0^{\frac{1}{3}}\left|\kappa^{\varpi}-\frac{1}{8}\right|^p d\kappa\right) \qquad (38)\\
&\quad+\frac{1}{q}\left(\int_0^{\frac{1}{3}}\left(\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q+(1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right)\right)d\kappa\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\kappa^{\varpi}-\frac{1}{2}\right|^{p}d\kappa\right)\\
&+\frac{1}{q}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_{\Theta})\right|^{q}-\left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_{\Theta})\right|^{q}+(1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_{\Theta})\right|^{q}\right)\right)d\kappa\right) \quad (39)\\
&+\frac{1}{p}\left(\int_{\frac{2}{3}}^{1}\left|\kappa^{\varpi}-\frac{7}{8}\right|^{p}d\kappa\right)\\
&+\frac{1}{q}\left(\int_{\frac{2}{3}}^{1}\left(\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_{\Theta})\right|^{q}-\left(\kappa\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_{\Theta})\right|^{q}+(1-\kappa)\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_{\Theta})\right|^{q}\right)\right)d\kappa\right)\Bigg].
\end{aligned}
\tag{40}
$$

It follows from $(\Lambda_1-\Lambda_2)^p \le (\Lambda_1^p-\Lambda_2^p)$ for any $\Lambda_1 \ge \Lambda_2 \ge 0$ and $p>1$. Hence, simple calculations complete the proof. □

Remark 9 If we take $\lambda = 2$ in Theorem 3.7, then we have

$$
\begin{aligned}
&\Bigg|\frac{1}{8}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+3\psi\left(\Omega_1+\Omega_2-\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)+3\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)\right.\\
&\quad\left.+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]-\frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^{\varpi}}J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\Bigg|\\
&\le(\mathfrak{b}-\mathfrak{a})\left\{\frac{1}{p}\left(\frac{6-8^{\frac{1}{\varpi}}}{3.8^{\frac{1+\varpi p}{\varpi}}}+\frac{1}{(\varpi p+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1}-2.\left(\frac{1}{8^{\frac{1}{\varpi}}}\right)^{\varpi p+1}\right]\right)\right.\\
&\quad+\frac{1}{q}\left[\frac{1}{3}\left(\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right)-\left(\frac{1}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{5}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right]\\
&\quad+\frac{1}{p}\left(\frac{6-2^{\frac{1}{\varpi}}}{3.2^{\frac{1+\varpi p}{\varpi}}}+\frac{1}{(p\varpi+1)}\left[\left(\frac{1}{3}\right)^{\varpi p+1}-2.\left(\frac{1}{2^{\frac{1}{\varpi}}}\right)^{\varpi p+1}+\left(\frac{2}{3}\right)^{\varpi p+1}\right]\right)\\
&\quad+\frac{1}{q}\left[\frac{1}{3}\left(\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right)-\left(\frac{1}{6}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{6}\left|\psi'(\mathfrak{b})\right|^q\right)\right]\\
&\quad+\frac{1}{p}\left(\left(2.\left(\frac{7}{8}\right)^{\frac{1}{\varpi}}-1\right)\left(\frac{7}{8}\right)^{p}+\frac{1}{(p\varpi+1)}\left[-2.\left(\frac{7}{8}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{2}{3}\right)^{\varpi p+1}+1\right]\right)\\
&\quad\left.+\frac{1}{q}\left[\frac{1}{3}\left(\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right)-\left(\frac{5}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right]\right\},
\end{aligned}
\tag{41}
$$

which is a new inequality in the literature.

- If we take $\varpi = 1$ in (41), then we have

$$
\begin{aligned}
&\Bigg|\frac{1}{8}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+3\psi\left(\Omega_1+\Omega_2-\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)+3\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)\right.\\
&\quad\left.+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{(\Omega_1+\Omega_2-\mathfrak{b})}^{(\Omega_1+\Omega_2-\mathfrak{a})}\psi(\varkappa)\,d\varkappa\Bigg|\\
&\le(\mathfrak{b}-\mathfrak{a})\left\{\frac{1}{p}\left(\frac{-2}{3.8^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{8}\right)^{p+1}\right]\right)\right.\\
&\quad+\frac{1}{q}\left[\frac{1}{3}\left(\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right)-\left(\frac{1}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{5}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{p}\left(\frac{4}{3.2^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{2}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}\right]\right)\\
&+\frac{1}{q}\left[\frac{1}{3}\left(\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right)-\left(\frac{1}{6}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{6}\left|\psi'(\mathfrak{b})\right|^q\right)\right]\\
&+\frac{1}{p}\left(\left(\frac{3}{4}\right)\left(\frac{7}{8}\right)^p+\frac{1}{(p+1)}\left[-2.\left(\frac{7}{8}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}+1\right]\right)\\
&+\frac{1}{q}\left[\frac{1}{3}\left(\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right)-\left(\frac{5}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right]\Bigg\},
\end{aligned}
$$

which is a new inequality in the literature.

- If we take $\Omega_1=\mathfrak{a}$, $\Omega_2=\mathfrak{b}$ and $\varpi=1$ in (41), then we have

$$
\begin{aligned}
&\left|\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi(\varkappa)\,d\varkappa-\left(\frac{3}{8}\right)\psi\left(\frac{2\mathfrak{a}+\mathfrak{b}}{3}\right)\right.\\
&\qquad\left.-\left(\frac{3}{8}\right)\psi\left(\frac{\mathfrak{a}+2\mathfrak{b}}{3}\right)-\frac{\psi(\mathfrak{a})+\psi(\mathfrak{b})}{8}\right|\\
&\le(\mathfrak{b}-\mathfrak{a})\left[\frac{1}{p}\left(\frac{-2}{3.8^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{8}\right)^{p+1}\right]\right)\right.\\
&\qquad+\frac{1}{q}\left(\frac{5}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\\
&\qquad+\frac{1}{p}\left(\frac{4}{3.2^{1+p}}+\frac{1}{(p+1)}\left[\left(\frac{1}{3}\right)^{p+1}-2.\left(\frac{1}{2}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}\right]\right)\\
&\qquad+\frac{1}{q}\left(\frac{1}{6}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{6}\left|\psi'(\mathfrak{b})\right|^q\right)\\
&\qquad+\frac{1}{p}\left(\left(\frac{3}{4}\right)\left(\frac{7}{8}\right)^p+\frac{1}{(p+1)}\left[-2.\left(\frac{7}{8}\right)^{p+1}+\left(\frac{2}{3}\right)^{p+1}+1\right]\right)\\
&\qquad\left.+\frac{1}{q}\left(\frac{1}{18}\left|\psi'(\mathfrak{a})\right|^q+\frac{5}{18}\left|\psi'(\mathfrak{b})\right|^q\right)\right],
\end{aligned}
$$

Theorem 3.8 *Under the assumptions of Lemma 2.2 and if the mapping $|\psi'|^q$ is continuous convex on the interval I, then we have the following inequality*:

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta\right)+4\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\frac{\mathfrak{a}_\Theta+\mathfrak{b}_\Theta}{2}\right)+\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right]\right.\\
&\qquad\left.-\frac{\Gamma(1+\varpi)}{\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)^\varpi}J^{(\varpi)}_{(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{a}_\Theta)^-}\psi\left(\sum_{\Theta=1}^{\lambda}\Omega_\Theta-\sum_{\Theta=1}^{\lambda-1}\mathfrak{b}_\Theta\right)\right|\\
&\le\sum_{\Theta=1}^{\lambda-1}(\mathfrak{b}_\Theta-\mathfrak{a}_\Theta)\left[\frac{1}{p}\left[\left(\frac{2}{6^{\frac{1}{\varpi}}}-\frac{1}{2}\right).\frac{1}{6^p}+\frac{1}{\varpi p+1}\left(-2\left(\frac{1}{6}\right)^{\frac{\varpi p+1}{\varpi}}+1\right)\right]\right.\\
&\qquad+\frac{1}{q}\left[\frac{1}{2}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\frac{1}{8}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q-\frac{3}{8}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right]
\end{aligned}
$$

$$+\frac{1}{p}\left[\left(2.\left(\frac{5}{6}\right)^{\frac{1}{\varpi}}-\frac{3}{2}\right).\left(\frac{5}{6}\right)^{p}+\frac{1}{\varpi p+1}\left(-2\left(\frac{5}{6}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{1}{2}\right)^{\varpi p+1}+1\right)\right]$$
$$+\frac{1}{q}\left[\frac{1}{2}\sum_{\Theta=1}^{\lambda}\left|\psi'(\Omega_\Theta)\right|^q-\frac{3}{8}\sum_{\upsilon=1}^{\lambda-1}\left|\psi'(\mathfrak{a}_\Theta)\right|^q-\frac{1}{8}\sum_{\Theta=1}^{\lambda-1}\left|\psi'(\mathfrak{b}_\Theta)\right|^q\right]\Bigg],$$

where p and q are conjugate exponents with $p,q>1$.

Proof The proof of the theorem is similar to the proof of Theorem 3.7. □

Remark 10 If we take $\lambda = 2$ in Theorem 3.8, we have

$$\begin{aligned}&\Bigg|\frac{1}{6}\left[\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right]\\
&\quad-\frac{\Gamma(1+\varpi)}{(\mathfrak{b}-\mathfrak{a})^{\varpi}}J^{(\varpi)}_{(\Omega_1+\Omega_2-\mathfrak{a})^-}\psi(\Omega_1+\Omega_2-\mathfrak{b})\Bigg|\\
&\le(\mathfrak{b}-\mathfrak{a})\Bigg[\frac{1}{p}\left[\left(\frac{2}{6^{\frac{1}{\varpi}}}-\frac{1}{2}\right).\frac{1}{6^p}+\frac{1}{\varpi p+1}\left(-2\left(\frac{1}{6}\right)^{\frac{\varpi p+1}{\varpi}}+1\right)\right]\\
&\quad+\frac{1}{q}\left[\frac{1}{2}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\frac{1}{8}\left|\psi'(\mathfrak{a})\right|^q-\frac{3}{8}\|\psi'(\mathfrak{b})|^q\right]\\
&\quad+\frac{1}{p}\left[\left(2.\left(\frac{5}{6}\right)^{\frac{1}{\varpi}}-\frac{3}{2}\right).\left(\frac{5}{6}\right)^{p}+\frac{1}{\varpi p+1}\left(-2\left(\frac{5}{6}\right)^{\frac{\varpi p+1}{\varpi}}+\left(\frac{1}{2}\right)^{\varpi p+1}+1\right)\right]\\
&\quad+\frac{1}{q}\left[\frac{1}{2}\left[\left|\psi'(\Omega_1)\right|^q+\left|\psi'(\Omega_2)\right|^q\right]-\frac{3}{8}\left|\psi'(\mathfrak{a})\right|^q-\frac{1}{8}\left|\psi'(\mathfrak{b})\right|^q\right]^{\frac{1}{q}}\Bigg],\end{aligned}\tag{42}$$

which is a new inequality in the literature.

- If we take $\varpi = 1$ in equation (42), we have

$$\begin{aligned}&\Bigg|\frac{1}{6}\left\{\psi(\Omega_1+\Omega_2-\mathfrak{a})+4\psi\left(\Omega_1+\Omega_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\Omega_1+\Omega_2-\mathfrak{b})\right\}\\
&\quad-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\Omega_1+\Omega_2-\mathfrak{b}}^{\Omega_1+\Omega_2-\mathfrak{a}}\psi(\varkappa)\,d\varkappa\Bigg|\\
&\le(\mathfrak{b}-\mathfrak{a})\Bigg\{\frac{1}{p}\left[-\frac{1}{6^{p+1}}+\frac{1}{p+1}\left(-2\left(\frac{1}{6}\right)^{p+1}+1\right)\right]\\
&\quad+\frac{1}{q}\left[\frac{1}{2}\left|\psi'(\Omega_1)\right|^q+\frac{1}{2}\left|\psi'(\Omega_2)\right|^q-\left(\frac{1}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{3}{8}\left|\psi'(\mathfrak{b})\right|^q\right)\right]\\
&\quad+\frac{1}{p}\left[\left(\frac{1}{6}\right).\left(\frac{5}{6}\right)^{p}+\frac{1}{p+1}\left(-2\left(\frac{5}{6}\right)^{p+1}+\left(\frac{1}{2}\right)^{p+1}+1\right)\right]\\
&\quad+\frac{1}{q}\left[\frac{1}{2}\left|\psi'(\Omega_1)\right|^q+\frac{1}{2}\left|\psi'(\Omega_2)\right|^q-\left(\frac{3}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{8}\left|\psi'(\mathfrak{b})\right|^q\right)\right]\Bigg\},\end{aligned}$$

which is a new inequality in the literature.

- If we take $\Omega_1 = \mathfrak{a}$, $\Omega_2 = \mathfrak{b}$ and $\varpi = 1$ in (42), we have

$$\begin{aligned}&\left|\frac{1}{6}\left\{\psi(\mathfrak{a})+4\psi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\psi(\mathfrak{b})\right\}-\frac{1}{(\mathfrak{b}-\mathfrak{a})}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi(\varkappa)\,d\varkappa\right|\\&\leq(\mathfrak{b}-\mathfrak{a})\left\{\frac{1}{p}\left[-\frac{1}{6^{p+1}}+\frac{1}{p+1}\left(-2\left(\frac{1}{6}\right)^{p+1}+1\right)\right]+\frac{1}{q}\left[\frac{3}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{1}{8}\left|\psi'(\mathfrak{b})\right|^q\right]\right.\\&\quad+\left[\frac{1}{p}\left(\frac{1}{6}\right).\left(\frac{5}{6}\right)^p+\frac{1}{p+1}\left(-2\left(\frac{5}{6}\right)^{p+1}+\left(\frac{1}{2}\right)^{p+1}+1\right)\right]\\&\quad\left.+\frac{1}{q}\left[\frac{1}{8}\left|\psi'(\mathfrak{a})\right|^q+\frac{3}{8}\left|\psi'(\mathfrak{b})\right|^q\right]\right\},\end{aligned}$$

which is a new inequality in the literature.

4 Applications

4.1 Applications to numerical quadrature rule

We now look at how the integral inequalities created in the previous section can be utilized to approximate composite quadrature rules in which it turns out to have a significantly lower error than what can be achieved using older techniques [28, 29].

Theorem 4.1 *Under the assumptions of Theorem 3.1 for $\varpi = 1$, if $I_\epsilon : \Omega_1 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{\epsilon-1} < \xi_\epsilon = \Omega_2$ is a partition of $[\Omega_1, \Omega_2]$ and $\xi_{\gamma,1}, \xi_{\gamma,2} \in [\xi_\gamma, \xi_{\gamma+1}]$, with $\mathfrak{h}_\gamma = \xi_{\gamma+1} - \xi_\gamma$ for all $\gamma = 0, 1, \ldots, \epsilon - 1$, then we have*:

$$\int_{(\xi_0+\xi_\epsilon-\xi_2)}^{(\xi_0+\xi_\epsilon-\xi_1)}\psi(\xi)\,d\xi = B(I_\epsilon,\psi)+R(I_\epsilon,\psi),$$

where

$$\begin{aligned}B(I_\epsilon,\psi)&=\left(\frac{3}{8}\right)\sum_{\gamma=0}^{\epsilon-1}\psi\left(\xi_\gamma+\xi_{\gamma+1}-\left(\frac{2\xi_{\gamma,1}+\xi_{\gamma,2}}{3}\right)\right)\mathfrak{h}_\gamma\\&\quad+\left(\frac{3}{8}\right)\sum_{\gamma=0}^{\epsilon-1}\psi\left(\xi_\gamma+\xi_{\gamma+1}-\left(\frac{\xi_{\gamma,1}+2\xi_{\gamma,2}}{3}\right)\right)\mathfrak{h}_\gamma\\&\quad+\sum_{\gamma=0}^{\epsilon-1}\frac{\psi(\xi_\gamma+\xi_{\gamma+1}-\xi_{\gamma,1})+\psi(\xi_\gamma+\xi_{\gamma+1}-\xi_{\gamma,2})}{8}\mathfrak{h}_\gamma\end{aligned}$$

and the remainder term satisfies the estimation:

$$\begin{aligned}\left|R(I_\epsilon,\psi)\right|&\leq\left[\left\{\frac{17}{576}\sum_{\gamma=0}^{\epsilon-1}\left[\left|\psi'(\xi_\gamma)\right|+\left|\psi'(\xi_{\gamma+1})\right|\right]\mathfrak{h}_\gamma^2\right.\right.\\&\quad\left.-\left(\frac{251}{41{,}472}\sum_{\gamma=0}^{\epsilon-1}\left|\psi'(\xi_{\gamma,1})\right|\mathfrak{h}_\gamma^2+\frac{973}{41{,}472}\sum_{\gamma=0}^{\epsilon-1}\left|\psi'(\xi_{\gamma,2})\right|\mathfrak{h}_\gamma^2\right)\right\}\\&\quad+\left\{\frac{1}{36}\sum_{\gamma=0}^{\epsilon-1}\left[\left|\psi'(\xi_\gamma)\right|+\left|\psi'(\xi_{\gamma+1})\right|\right]\mathfrak{h}_\gamma^2\right.\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{1}{72} \sum_{\gamma=0}^{\epsilon-1} \left| \psi'(\xi_{\gamma,1}) \right| \mathfrak{h}_\gamma^2 + \frac{1}{72} \sum_{\gamma=0}^{\epsilon-1} \left| \psi'(\xi_{\gamma,2}) \right| \mathfrak{h}_\gamma^2 \right) \Bigg\} \\
&+ \Bigg\{ \frac{17}{576} \sum_{\gamma=0}^{\epsilon-1} \left[\left| \psi'(\xi_\gamma) \right| + \left| \psi'(\xi_{\gamma+1}) \right| \right] \mathfrak{h}_\gamma^2 \\
&- \left(\frac{973}{41{,}472} \sum_{\gamma=0}^{\epsilon-1} \left| \psi'(\xi_{\gamma,1}) \right| \mathfrak{h}_\gamma^2 + \frac{251}{41{,}472} \sum_{\gamma=0}^{\epsilon-1} \left| \psi'(\xi_{\gamma,2}) \right| \mathfrak{h}_\gamma^2 \right) \Bigg\} \Bigg].
\end{aligned}
$$

Proof Applying Theorem 3.1 with $\lambda = 2$ and $\varpi = 1$ on interval $[\xi_\gamma, \xi_{\gamma+1}]$, $\gamma = 0, 1, \dots, \epsilon - 1$, we obtain

$$
\begin{aligned}
&\Bigg| \left(\frac{3}{8} \right) \psi \left(\xi_\gamma + \xi_{\gamma+1} - \left(\frac{2\xi_{\gamma,1} + \xi_{\gamma,2}}{3} \right) \right) \mathfrak{h}_\gamma + \left(\frac{3}{8} \right) \psi \left(\xi_\gamma + \xi_{\gamma+1} + \left(\frac{\xi_{\gamma,1} + 2\xi_{\gamma,2}}{3} \right) \right) \mathfrak{h}_\gamma \\
&\quad + \frac{\psi(\xi_\gamma + \xi_{\gamma+1} - \xi_{\gamma,1}) + \psi(\xi_\gamma + \xi_{\gamma+1} - \xi_{\gamma,2})}{8} \mathfrak{h}_\gamma - \int_{(\xi_\gamma + \xi_{\gamma+1} - \xi_{\gamma,2})}^{(\xi_\gamma + \xi_{\gamma+1} - \xi_{\gamma,1})} \psi(\xi)\, d\xi \Bigg| \\
&\leq \Bigg[\Bigg\{ \frac{17}{576} \left[\left| \psi'(\xi_\gamma) \right| + \left| \psi'(\xi_{\gamma+1}) \right| \right] \mathfrak{h}_\gamma^2 \\
&\quad - \left(\frac{251}{41{,}472} \left| \psi'(\xi_{\gamma,1}) \right| \mathfrak{h}_\gamma^2 + \frac{973}{41{,}472} \left| \psi'(\xi_{\gamma,2}) \right| \mathfrak{h}_\gamma^2 \right) \Bigg\} \\
&\quad + \Bigg\{ \frac{1}{36} \left[\left| \psi'(\xi_\gamma) \right| + \left| \psi'(\xi_{\gamma+1}) \right| \right] \mathfrak{h}_\gamma^2 \\
&\quad - \left(\frac{1}{72} \left| \psi'(\xi_{\gamma,1}) \right| \mathfrak{h}_\gamma^2 + \frac{1}{72} \left| \psi'(\xi_{\gamma,2}) \right| \mathfrak{h}_\gamma^2 \right) \Bigg\} \\
&\quad + \Bigg\{ \frac{17}{576} \left[\left| \psi'(\xi_\gamma) \right| + \left| \psi'(\xi_{\gamma+1}) \right| \right] \mathfrak{h}_\gamma^2 \\
&\quad - \left(\frac{973}{41{,}472} \left| \psi'(\xi_{\gamma,1}) \right| \mathfrak{h}_\gamma^2 + \frac{251}{41{,}472} \left| \psi'(\xi_{\gamma,2}) \right| \mathfrak{h}_\gamma^2 \right) \Bigg\} \Bigg]. \qquad (43)
\end{aligned}
$$

Summing (43) over 0 to $\epsilon - 1$ and using the triangular inequality we obtain the above estimation. □

Theorem 4.2 *Under the assumptions of Theorem 3.2 for $\varpi = 1$, if $I_\epsilon : \Omega_1 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{\epsilon-1} < \xi_\epsilon = \Omega_2$ is a partition of $[\Omega_1, \Omega_2]$ and $\xi_{\gamma,1}, \xi_{\gamma,2} \in [\xi_\gamma, \xi_{\gamma+1}]$, with $\mathfrak{h}_\gamma = \xi_{\gamma+1} - \xi_\gamma$ for all $\gamma = 0, 1, \dots, \epsilon - 1$, then we have:*

$$
\int_{(\xi_0 + \xi_\epsilon - \xi_2)}^{(\xi_0 + \xi_\epsilon - \xi_1)} \psi(\xi)\, d\xi = B(I_\epsilon, \psi) + R(I_\epsilon, \psi),
$$

where

$$
\begin{aligned}
B(I_\epsilon, \psi) = \frac{1}{6} \Bigg[&\sum_{\gamma=0}^{\epsilon-1} \psi(\xi_\gamma + \xi_{\gamma+1} - \xi_{\gamma,1}) \mathfrak{h}_\gamma + 4 \sum_{\gamma=0}^{\epsilon-1} \psi \left(\xi_\gamma + \xi_{\gamma+1} - \frac{\xi_{\gamma,1} + x_{i,2}}{2} \right) \mathfrak{h}_\gamma \\
&+ \sum_{\gamma=0}^{\epsilon-1} \psi(\xi_\gamma + \xi_{\gamma+1} - \xi_{\gamma,2}) \mathfrak{h}_\gamma \Bigg]
\end{aligned}
$$

and the remainder term satisfies the estimation:

$$\begin{aligned}|R_{\varpi}(I_\epsilon,\psi)| \le{}& \left[\frac{5}{72}\sum_{\gamma=0}^{\epsilon-1}\left[|\psi'(\xi_\gamma)|+|\psi'(\xi_{\gamma+1})|\right]\mathfrak{h}_\gamma^2\right.\\&\left.-\frac{61}{1296}\sum_{\gamma=0}^{\epsilon-1}|\psi'(\xi_{\gamma,1})|\mathfrak{h}_\gamma^2-\frac{29}{1296}\sum_{\gamma=0}^{\epsilon-1}|\psi'(\xi_{\gamma,2})|\mathfrak{h}_\gamma^2\right]\\&+\left[\frac{5}{72}\sum_{\gamma=0}^{\epsilon-1}\left[|\psi'(\xi_\gamma)|+|\psi'(\xi_{\gamma+1})|\right]\mathfrak{h}_\gamma^2\right.\\&\left.-\frac{29}{1296}\sum_{\gamma=0}^{\epsilon-1}|\psi'(\xi_{\gamma,1})|\mathfrak{h}_\gamma^2-\frac{61}{1296}\sum_{\gamma=0}^{\epsilon-1}|\psi'(\xi_{\gamma,2})|\mathfrak{h}_\gamma^2\right].\end{aligned}$$

Proof Applying Theorem 3.2 with $\lambda = 2$ and $\varpi = 1$ on interval $[\xi_\gamma,\xi_{\gamma+1}]$, $\gamma = 0,1,\dots,\epsilon-1$, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\xi_\gamma+\xi_{\gamma+1}-\xi_{\gamma,1})\mathfrak{h}_\gamma+4\psi\left(\xi_\gamma+\xi_{\gamma+1}-\frac{\xi_{\gamma,1}+\xi_{\gamma,2}}{2}\right)\mathfrak{h}_\gamma\right.\right.\\&\quad\left.\left.+\psi(\xi_\gamma+\xi_{\gamma+1}-\xi_{\gamma,2})\mathfrak{h}_\gamma\right]-\int_{(\xi_\gamma+\xi_{\gamma+1}-\xi_{\gamma,2})}^{(\xi_\gamma+\xi_{\gamma+1}-\xi_{\gamma,1})}\psi(\xi)\,d\xi\right|\\&\le\left[\frac{5}{72}\left[|\psi'(\xi_\gamma)|+|\psi'(\xi_{\gamma+1})|\right]\mathfrak{h}_\gamma^2-\frac{61}{1296}|\psi'(\xi_{\gamma,1})|\mathfrak{h}_\gamma^2-\frac{29}{1296}|\psi'(\xi_{\gamma,2})|\mathfrak{h}_\gamma^2\right]\\&\quad+\left[\frac{5}{72}\left[|\psi'(\xi_\gamma)|+|\psi'(\xi_{\gamma+1})|\right]\mathfrak{h}_\gamma^2-\frac{29}{1296}|\psi'(\xi_{\gamma,1})|\mathfrak{h}_\gamma^2-\frac{61}{1296}|\psi'(\xi_{\gamma,2})|\mathfrak{h}_\gamma^2\right].\end{aligned}\tag{44}$$

Summing (44) over 0 to $\epsilon - 1$ and utilizing the triangular inequality we obtain the above result. □

5 Conclusion

In this study, the main finding is novel Newton–Simpson-type inequalities involving a fractional integral operator via majorization. This novel framework is the convolution of the majorization concept and estimation of definite integrals. Adopting the novel approach, we extended the study of Newton–Simpson-type integral inequalities using power-mean, Young's, and Hölder's integral inequalities. Finally, some applications to the quadrature rule are presented. Our approach may have further implementations in the theory of majorization. It is interesting to extend such findings for other convexities. We presume that our newly announced concept will be the focus of much research in this fascinating field of inequalities and analysis. The incredible methods and marvellous concepts in this article can be expanded and enlarged to coordinates and fractional integrals. Our long-term plan is to continue in this way with our research.

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