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On some variational inequality-constrained control problems

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Abstract

In this paper, by considering some properties associated with scalar functionals of multiple-integral type, we study the well-posedness and generalized well-posedness for a new variational inequality-constrained optimization problems By using the set of approximating solutions, we state some characterization theorems on well-posedness and generalized well-posedness. Also, in order to validate the derived results, some examples are given.

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1 Introduction

To study and solve some optimization problems by using the classical methods, in many situations, can represent a very complicated task and, moreover, such methods may (or may not) ensure exact solutions. In this regard, the well-posedness becomes extremely important for the study of optimization problems. More precisely, it is a useful technique by ensuring the convergence for the sequence of approximating solutions to the exact solution. The notion of well-posedness for unconstrained optimization problems was defined by Tykhonov [34]. Following this concept (see, for instance, [12, 28]), many types of well-posedness for variational problems were introduced, namely: well-posedness of Levitin–Polyak type [11, 17, 19, 20], and extended well-posedness (for instance, [5, 6, 9, 14, 15, 22, 25–27, 38]), α-well-posedness [23, 36], and L-well-posedness [21]. Also, this tool can be useful to investigate the connected problems, namely: fixed-point problems [3], hemivariational inequality [37], variational inequality [2, 7, 18], equilibrium problems [4, 8], Nash equilibrium [24], complementary problems [10], etc.

Jayswal and Shalini [16] studied the well-posedness of some mixed vector variational inequalities. Also, Hu et al. [13] established well-posedness for split variational–hemivariational inequality problems. Bai et al. [1] studied, in a Banach space, generalized mixed elliptic hemivariational–variational inequalities and obtained a well-posedness result for the considered inequality.

In the present paper, well-posedness and generalized well-posedness are studied for new variational control problems defined by functionals of multiple-integral type. For this purpose, we consider the notions of monotonicity, hemicontinuity, pseudomonotonicity, and lower semicontinuity for functionals of multiple-integral type. The approximating solution set is used to formulate and prove some theorems on well-posedness and generalized well-posedness. Next, let us highlight the main merits of this paper. First, most of the former research papers have been investigated in finite-dimensional spaces (Hilbert spaces, Banach spaces, Euclidean spaces). The results derived in this research paper are dynamic generalizations of the static results that exist in the literature. Here, the mathematical context is defined by function spaces of infinite dimension and controlled functionals of multiple-integral type. It represents a completely new element in the field of well-posed control problems. Recently, Treanţă [32] studied some similar optimization problems that imply partial derivatives of second order, but without control functions. Also, the curvilinear case, but for the controlled variational inequality problem, is investigated in Treanţă [33]. In consequence, this paper deals with a special situation in which the optimal control problem has controlled variational inequality as a constraint.

We continue the paper as follows: in Sect. 2, we present the notions of lower semicontinuity, monotonicity, hemicontinuity, and pseudomonotonicity to functionals of multiple-integral type. We establish an auxiliary lemma that is important for the main results formulated in the paper. In Sect. 3, by defining the approximating solution set, we analyze well-posedness and generalized well-posedness. Also, under suitable assumptions, we state that well-posedness is the same as the existence and uniqueness of a solution in the aforesaid problems. Moreover, some sufficient conditions for the generalized well-posedness are provided. Illustrative examples are presented in the paper to validate the theoretical aspects. Finally (see Sect. 4), we state the conclusions of the paper and some immediate research directions.

2 Problem formulation

In this paper, in accordance with Treanţă [29–33], we consider T is a domain (it is supposed to be compact) in $\mathbb{R}^m$, the point $T \ni t = (t^\alpha)$, $\alpha = \overline{1,m}$, means a multivariate evolution parameter, $\boldsymbol{\Theta}$ is the space of piecewise-differentiable *state* functions $\theta : T \to \mathbb{R}^n$, having the norm

$$\|\theta\| = \|\theta\|_\infty + \sum_{\alpha=1}^{m} \|\theta_\alpha\|_\infty, \quad \forall \theta \in \boldsymbol{\Theta},$$

where $\theta_\alpha := \frac{\partial \theta}{\partial t^\alpha}$, $\alpha = \overline{1,m}$. Also, let $\mathcal{P}$ be the space of piecewise-continuous *control* functions $p : T \to \mathbb{R}^k$, together with the norm $\|\cdot\|_\infty$.

In the following, consider $\Theta \times P$ is a nonempty, closed, and convex subset of $\boldsymbol{\Theta} \times \mathcal{P}$, with $(\theta, p)|_{\partial T}$ = given and $\theta_\alpha = X(t, \theta, p)$ = given, and with the inner product

$$\begin{aligned}\langle(\theta,p),(\vartheta,q)\rangle &= \int_T \big[\theta(t)\cdot\vartheta(t) + p(t)\cdot q(t)\big]\,dt \\ &= \int_T \left[\sum_{i=1}^{n}\theta^i(t)\vartheta^i(t) + \sum_{j=1}^{k} p^j(t) q^j(t)\right] dt, \quad \forall (\theta,p),(\vartheta,q) \in \boldsymbol{\Theta}\times\mathcal{P}\end{aligned}$$

and the norm induced by it. Denote by $dt = dt^1 \cdots dt^m$ the element of volume on $\mathbb{R}^m$. Let $J^1(\mathbb{R}^m, \mathbb{R}^n) := \{(t, \theta(t), \theta_\alpha(t)) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{mn}\}$ be the jet bundle associated with $\mathbb{R}^m$

and $\mathbb{R}^n$. By using the real-valued continuously smooth function $g : J^1(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{R}^k \to \mathbb{R}$, define the scalar functional governed by a multiple integral:

$$G : \boldsymbol{\Theta} \times \mathcal{P} \to \mathbb{R}, \quad G(\theta, p) = \int_T g(t, \theta, \theta_\alpha, p)\, dt.$$

Further, we use the notation $(\chi_{\theta,p}(t)) := (t, \theta(t), \theta_\alpha(t), p(t))$.

Definition 2.1 The functional $\int_T g(\chi_{\theta,p}(t))\, dt$ is monotone on $\Theta \times P$ if the following inequality

$$\begin{aligned}\int_T \Big[&\big(\theta(t) - \vartheta(t)\big)\left(\frac{\partial g}{\partial \theta}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big)\right) \\ &+ \big(p(t) - q(t)\big)\left(\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)\right) \\ &+ \mathcal{D}_\alpha\big(\theta(t) - \vartheta(t)\big)\left(\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\right)\Big]\, dt \geq 0\end{aligned}$$

is true, for $\forall(\theta, p), (\vartheta, q) \in \Theta \times P$, where $\mathcal{D}_\alpha := \frac{\partial}{\partial t^\alpha}$ is the operator of the total derivative.

Example 2.1 Consider $n = k = 1$, $m = 2$, and $T = [0, 3]^2$. Let us define

$$g\big(\chi_{\theta,p}(t)\big) = \theta(t) + e^{p(t)} - 1.$$

The functional $\int_T g(\chi_{\theta,p}(t))\, dt$ is monotone on $\Theta \times P = C^1(T, \mathbb{R}) \times C(T, \mathbb{R})$ since

$$\begin{aligned}&\int_T \Big[\big(\theta(t) - \vartheta(t)\big)\left(\frac{\partial g}{\partial \theta}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big)\right) \\ &\quad + \big(p(t) - q(t)\big)\left(\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)\right) \\ &\quad + \mathcal{D}_\alpha\big(\theta(t) - \vartheta(t)\big)\left(\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\right)\Big]\, dt \\ &= \int_T \big(p(t) - q(t)\big)\big(e^{p(t)} - e^{q(t)}\big)\, dt \geq 0, \quad \forall(\theta, p), (\vartheta, q) \in \Theta \times P\end{aligned}$$

is valid.

Definition 2.2 The functional $\int_T g(\chi_{\theta,p}(t))\, dt$ is pseudomonotone on $\Theta \times P$ if the following implication

$$\begin{aligned}&\int_T \Big[\big(\theta(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(p(t) - q(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) \\ &\quad + \mathcal{D}_\alpha\big(\theta(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\Big]\, dt \geq 0 \\ &\Rightarrow \quad \int_T \Big[\big(\theta(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta,p}(t)\big) + \big(p(t) - q(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big) \\ &\qquad + \mathcal{D}_\alpha\big(\theta(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta,p}(t)\big)\Big]\, dt \geq 0\end{aligned}$$

holds, for $\forall(\theta, p), (\vartheta, q) \in \Theta \times P$.

Example 2.2 Consider $n = k = 1$, $m = 2$, and $T = [0,3]^2$. Let us define

$$g(\chi_{\theta,p}(t)) = \sin\theta(t) + p(t)e^{p(t)}.$$

The functional $\int_T g(\chi_{\theta,p}(t))\,dt$ is pseudomonotone on $\Theta \times P = C^1(T,[-1,1]) \times C(T,[-1,1])$ since

$$\begin{aligned}
&\int_T \bigg[\big(\theta(t)-\vartheta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(p(t)-q(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\theta(t)-\vartheta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg]\,dt \\
&\quad = \int_T \big[\big(\theta(t)-\vartheta(t)\big)\cos\vartheta(t) + \big(p(t)-q(t)\big)\big(e^{q(t)} + q(t)e^{q(t)}\big)\big]\,dt \geq 0 \\
&\quad \forall(\theta,p),(\vartheta,q) \in \Theta \times P \\
&\quad \Rightarrow \quad \int_T \bigg[\big(\theta(t)-\vartheta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta,p}(t)\big) + \big(p(t)-q(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big) \\
&\qquad + \mathcal{D}_\alpha\big(\theta(t)-\vartheta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta,p}(t)\big)\bigg]\,dt \\
&\qquad = \int_T \big[\big(\theta(t)-\vartheta(t)\big)\cos\theta(t) + \big(p(t)-q(t)\big)\big(e^{p(t)} + p(t)e^{p(t)}\big)\big]\,dt \geq 0 \\
&\qquad \forall(\theta,p),(\vartheta,q) \in \Theta \times P
\end{aligned}$$

is satisfied. However, it is not monotone on $\Theta \times P$ since

$$\begin{aligned}
&\int_T \bigg[\big(\theta(t)-\vartheta(t)\big)\bigg(\frac{\partial g}{\partial \theta}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big)\bigg) \\
&\quad + \big(p(t)-q(t)\big)\bigg(\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)\bigg) \\
&\quad + \mathcal{D}_\alpha\big(\theta(t)-\vartheta(t)\big)\bigg(\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta,p}(t)\big) - \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg)\bigg]\,dt \\
&\quad = \int_T \big[\big(\theta(t)-\vartheta(t)\big)\big(\cos\theta(t) - \cos\vartheta(t)\big) \\
&\quad + \big(p(t)-q(t)\big)\big(p(t)e^{p(t)} + e^{p(t)} - q(t)e^{q(t)} - e^{q(t)}\big)\big]\,dt \ngeq 0, \\
&\quad \forall(\theta,p),(\vartheta,q) \in \Theta \times P.
\end{aligned}$$

Taking into account Usman and Khan [35], we formulate the next definition.

Definition 2.3 The functional $\int_T g(\chi_{\theta,p}(t))\,dt$ is hemicontinuous on $\Theta \times P$ if

$$\lambda \to \bigg\langle \big(\theta(t),p(t)\big) - \big(\vartheta(t),q(t)\big), \bigg(\frac{\delta G}{\delta \theta_\lambda}(t), \frac{\delta G}{\delta p_\lambda}(t)\bigg)\bigg\rangle, \quad 0 \leq \lambda \leq 1$$

is continuous at 0^+, for $\forall(\theta,p),(\vartheta,q)\in\Theta\times P$, where

$$\begin{aligned}
&\frac{\delta G}{\delta\theta_\lambda}(t):=\frac{\partial g}{\partial\theta}\big(\chi_{\theta_\lambda,p_\lambda}(t)\big)-\mathcal{D}_\alpha\frac{\partial g}{\partial\theta_\alpha}\big(\chi_{\theta_\lambda,p_\lambda}(t)\big)\in\Theta,\\
&\frac{\delta G}{\delta p_\lambda}(t):=\frac{\partial g}{\partial p}\big(\chi_{\theta_\lambda,p_\lambda}(t)\big)\in P,\\
&\theta_\lambda:=\lambda\theta+(1-\lambda)\vartheta,\qquad p_\lambda:=\lambda p+(1-\lambda)q.
\end{aligned}$$

Now, we formulate the following variational inequality-constrained optimization problem (for short, CP):

$$\begin{aligned}
&\text{Minimize }\int_T g(\chi_{\theta,p})\,dt \qquad\qquad \text{(CP)}\\
&\text{subject to }(\theta,p)\in\Lambda,
\end{aligned}$$

where Λ is the set of solutions for the following variational inequality (for short, IP):
Find $(\theta,p)\in\Theta\times P$ *such that*

$$\begin{aligned}
&\int_T\left[\frac{\partial g}{\partial\theta}(\chi_{\theta,p})(\vartheta-\theta)+\frac{\partial g}{\partial\theta_\alpha}(\chi_{\theta,p})\mathcal{D}_\alpha(\vartheta-\theta)\right]dt \qquad\qquad \text{(IP)}\\
&\quad+\int_T\left[\frac{\partial g}{\partial p}(\chi_{\theta,p})(q-p)\right]dt\ge 0,\quad\forall(\vartheta,q)\in\Theta\times P.
\end{aligned}$$

More precisely, the feasible solutions of (IP) are formulated as

$$\begin{aligned}
\Lambda=\bigg\{&(\theta,p)\in\Theta\times P:\int_T\bigg[\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial\theta}\big(\chi_{\theta,p}(t)\big)\\
&+\mathcal{D}_\alpha\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial\theta_\alpha}\big(\chi_{\theta,p}(t)\big)+\big(q(t)-p(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big)\bigg]dt\ge 0,\\
&\forall(\vartheta,q)\in\Theta\times P\bigg\}.
\end{aligned}$$

Lemma 2.1 *Let the functional* $\int_T g(\chi_{\theta,p}(t))\,dt$ *be hemicontinuous and pseudomonotone on the convex and closed set* $\Theta\times P$. *Then,* $(\theta,p)\in\Theta\times P$ *solves* (IP) *if and only if it solves the variational inequality problem*

$$\begin{aligned}
&\int_T\bigg[\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial\theta}\big(\chi_{\vartheta,q}(t)\big)+\big(q(t)-p(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)\\
&\quad+\mathcal{D}_\alpha\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial\theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg]dt\ge 0,\quad\forall(\vartheta,q)\in\Theta\times P.
\end{aligned}$$

Proof First, let us consider that $(\theta,p)\in\Theta\times P$ solves (IP). In consequence, it follows that

$$\begin{aligned}
&\int_T\bigg[\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial\theta}\big(\chi_{\theta,p}(t)\big)+\big(q(t)-p(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big)\\
&\quad+\mathcal{D}_\alpha\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial\theta_\alpha}\big(\chi_{\theta,p}(t)\big)\bigg]dt\ge 0,\quad\forall(\vartheta,q)\in\Theta\times P.
\end{aligned}$$

By considering the property of pseudomonotonicity of the functional $\int_T g(\chi_{\theta,p}(t))\,dt$, the above inequality implies

$$\int_T \Big[\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t)-p(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) + \mathcal{D}_\alpha\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\Big]\,dt \geq 0, \quad \forall(\vartheta,q)\in\Theta\times P.$$

Conversely, let us assume that

$$\int_T \Big[\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t)-p(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) + \mathcal{D}_\alpha\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\Big]\,dt \geq 0, \quad \forall(\vartheta,q)\in\Theta\times P.$$

Further, for $\lambda\in(0,1]$ and $(\vartheta,q)\in\Theta\times P$, we introduce

$$(\vartheta_\lambda,q_\lambda) = \big((1-\lambda)\theta+\lambda\vartheta,(1-\lambda)p+\lambda q\big)\in\Theta\times P.$$

Therefore, the previous inequality can be reformulated as follows

$$\int_T \Big[\big(\vartheta_\lambda(t)-\theta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta_\lambda,q_\lambda}(t)\big) + \big(q_\lambda(t)-p(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta_\lambda,q_\lambda}(t)\big) + \mathcal{D}_\alpha\big(\vartheta_\lambda(t)-\theta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta_\lambda,q_\lambda}(t)\big)\Big]\,dt \geq 0, \quad (\vartheta,q)\in\Theta\times P.$$

Considering $\lambda\to 0$ and using the hemicontinuity property of the functional $\int_T g(\chi_{\theta,p}(t))\,dt$, we obtain

$$\int_T \Big[\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta,p}(t)\big) + \big(q(t)-p(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta,p}(t)\big) + \mathcal{D}_\alpha\big(\vartheta(t)-\theta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta,p}(t)\big)\Big]\,dt \geq 0, \quad \forall(\vartheta,q)\in\Theta\times P,$$

which shows that (θ,p) is a solution of (IP). □

Definition 2.4 The functional $\int_T g(\chi_{\theta,p}(t))\,dt$ is lower semicontinuous at $(\theta_0,p_0)\in\Theta\times P$ if

$$\int_T g\big(\chi_{\theta_0,p_0}(t)\big)\,dt \leq \lim_{(\theta,p)\to(\theta_0,p_0)}\inf \int_T g\big(\chi_{\theta,p}(t)\big)\,dt.$$

3 Well-posedness and generalized well-posedness of (CP)

In this section, taking into account the mathematical tools in Sect. 2, we analyze the well-posedness and generalized well-posedness of (CP).

Let Φ be the solution set of (CP),

$$\Phi = \Big\{(\theta,p)\in\Theta\times P \,\Big|\, \int_T g(\chi_{\theta,p})\,dt \leq \inf_{(\vartheta,q)\in\Lambda}\int_T g(\chi_{\vartheta,q})\,dt \text{ and}$$

$$\int_T \left[(\vartheta - \theta) \frac{\partial g}{\partial \theta}(\chi_{\theta,p}) + (q - p) \frac{\partial g}{\partial p}(\chi_{\theta,p}) \right.$$
$$\left. + \mathcal{D}_\alpha (\vartheta - \theta) \frac{\partial g}{\partial \theta_\alpha}(\chi_{\theta,p}) \right] dt \geq 0, \forall (\vartheta, q) \in \Theta \times P \Big\}.$$

For $\rho, \xi \geq 0$, we define the set of approximating solutions of (CP) as

$$\Phi(\rho, \xi) = \Big\{ (\theta, p) \in \Theta \times P \Big| \int_T g(\chi_{\theta,p})\, dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g(\chi_{\vartheta,q})\, dt + \rho \text{ and}$$
$$\int_T \left[(\vartheta - \theta) \frac{\partial g}{\partial \theta}(\chi_{\theta,p}) + (q - p) \frac{\partial g}{\partial p}(\chi_{\theta,p}) \right.$$
$$\left. + \mathcal{D}_\alpha (\vartheta - \theta) \frac{\partial g}{\partial \theta_\alpha}(\chi_{\theta,p}) \right] dt + \xi \geq 0, \forall (\vartheta, q) \in \Theta \times P \Big\}.$$

Remark 3.1 Clearly, for $(\rho, \xi) = (0,0)$, we have $\Phi = \Phi(\rho, \xi)$ and, for $(\rho, \xi) > (0,0)$, we obtain $\Phi \subseteq \Phi(\rho, \xi)$.

Definition 3.1 A sequence $\{(\theta_n, p_n)\}$ is an approximating sequence of (CP) if there exists a sequence of positive real numbers $\xi_n \to 0$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \sup \int_T g\big(\chi_{\theta_n,p_n}(t)\big)\, dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g\big(\chi_{\vartheta,q}(t)\big)\, dt$$

and

$$\int_T \left[\big(\vartheta(t) - \theta_n(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\theta_n,p_n}(t)\big) + \big(q(t) - p_n(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\theta_n,p_n}(t)\big) \right.$$
$$\left. + \mathcal{D}_\alpha \big(\vartheta(t) - \theta_n(t)\big) \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_n,p_n}(t)\big) \right] dt + \xi_n \geq 0, \quad \forall (\vartheta, q) \in \Theta \times P$$

are satisfied.

Definition 3.2 The variational problem (CP) is well-posed if:

(i) it admits a unique solution (θ_0, p_0);

(ii) an approximating sequence of (CP) converges to (θ_0, p_0).

Definition 3.3 The variational problem (CP) is generalized well-posed if:

(i) $\Phi \neq \phi 0$;

(ii) an approximating sequence of (CP) admits a subsequence that converges to some pair of Φ.

Further, the diameter of a set B is determined as follows

$$\operatorname{diam} B = \sup_{x,y\in B} \|x - y\|.$$

Theorem 3.1 *Let the functional* $\int_T g(\chi_{\theta,p}(t))\, dt$ *be monotone, lower semicontinuous, and hemicontinuous on* $\Theta \times P$. *The variational problem* (CP) *is well-posed if and only if*

$$\Phi(\rho, \xi) \neq \phi, \quad \forall \rho, \xi > 0 \quad \textit{and} \quad \operatorname{diam} \Phi(\rho, \xi) \to 0 \quad \textit{as } (\rho, \xi) \to (0,0).$$

Proof Let us consider that (CP) is well-posed. Thus, it admits a unique solution $(\bar{\theta},\bar{p}) \in \Phi$. Since $\Phi \subseteq \Phi(\rho,\xi), \forall \rho,\xi > 0$, we obtain $\Phi(\rho,\xi) \neq \phi, \forall \rho,\xi > 0$. Suppose that $\operatorname{diam}\Phi(\rho,\xi) \nrightarrow 0$ as $(\rho,\xi) \to (0,0)$. Then, there exist $r > 0$, a natural number m, $\rho_n,\xi_n > 0$ with $\rho_n,\xi_n \to 0$ and $(\theta_n,p_n),(\theta'_n,p'_n) \in \Phi(\rho_n,\xi_n)$ such that

$$\left\|(\theta_n,p_n) - \left(\theta'_n,p'_n\right)\right\| > r, \quad \forall n \geq m. \tag{1}$$

Since $(\theta_n,p_n),(\theta'_n,p'_n) \in \Phi(\rho_n,\xi_n)$, we obtain

$$\begin{aligned}
&\int_T g\big(\chi_{\theta_n,p_n}(t)\big)\,dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g\big(\chi_{\vartheta,q}(t)\big)\,dt + \rho_n, \\
&\int_T \bigg[\big(\vartheta(t) - \theta_n(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta_n,p_n}(t)\big) + \big(q(t) - p_n(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_n,p_n}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_n(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_n,p_n}(t)\big)\bigg]\,dt + \xi_n \geq 0, \quad \forall(\vartheta,q) \in \Theta \times P
\end{aligned}$$

and

$$\begin{aligned}
&\int_T g\big(\chi_{\theta'_n,p'_n}(t)\big)\,dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g\big(\chi_{\vartheta,q}(t)\big)\,dt + \rho_n, \\
&\int_T \bigg[\big(\vartheta(t) - \theta'_n(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta'_n,p'_n}(t)\big) + \big(q(t) - p'_n(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta'_n,p'_n}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta'_n(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta'_n,p'_n}(t)\big)\bigg]\,dt + \xi_n \geq 0, \quad \forall(\vartheta,q) \in \Theta \times P.
\end{aligned}$$

We obtain that $\{(\theta_n,p_n)\}$ and $\{(\theta'_n,p'_n)\}$ are approximating sequences of (CP), converging to $(\bar{\theta},\bar{p})$ (by hypothesis, (CP) is well-posed). By direct computation, we obtain

$$\begin{aligned}
&\left\|(\theta_n,p_n) - \left(\theta'_n,p'_n\right)\right\| \\
&\quad = \left\|(\theta_n,p_n) - (\bar{\theta},\bar{p}) + (\bar{\theta},\bar{p}) - \left(\theta'_n,p'_n\right)\right\| \\
&\quad \leq \left\|(\theta_n,p_n) - (\bar{\theta},\bar{p})\right\| + \left\|(\bar{\theta},\bar{p}) - \left(\theta'_n,p'_n\right)\right\| \leq \xi,
\end{aligned}$$

which contradicts (1), for some $\xi = r$. In consequence, $\operatorname{diam}\Phi(\rho,\xi) \to 0$ as $(\rho,\xi) \to (0,0)$.

Now, conversely, let us consider that $\{(\theta_n,p_n)\}$ is an approximating sequence of (CP). Then, there exists a sequence of positive real numbers $\xi_n \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty}\sup \int_T g\big(\chi_{\theta_n,p_n}(t)\big)\,dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g\big(\chi_{\vartheta,q}(t)\big)\,dt, \tag{2}$$

$$\begin{aligned}
&\int_T \bigg[\big(\vartheta(t) - \theta_n(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta_n,p_n}(t)\big) + \big(q(t) - p_n(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_n,p_n}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_n(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_n,p_n}(t)\big)\bigg]\,dt + \xi_n \geq 0, \quad \forall(\vartheta,q) \in \Theta \times P
\end{aligned} \tag{3}$$

hold, implying $(\theta_n,p_n) \in \Phi(\rho_n,\xi_n)$, for a sequence of positive real numbers $\rho_n \to 0$ as $n \to \infty$. Since we have $\operatorname{diam}\Phi(\rho_n,\xi_n) \to 0$ as $(\rho_n,\xi_n) \to (0,0)$, therefore $\{(\theta_n,p_n)\}$ is a Cauchy sequence converging to some $(\bar{\theta},\bar{p}) \in \Theta \times P$ ($\Theta \times P$ is a closed set).

By hypothesis, $\int_T g(\chi_{\theta,p}(t))\,dt$ is monotone on $\Theta \times P$. Therefore, for $(\bar{\theta},\bar{p}), (\vartheta, q) \in \Theta \times P$, it follows that

$$\begin{aligned}\int_T \Big[& \big(\bar{\theta}(t) - \vartheta(t)\big)\left(\frac{\partial g}{\partial \theta}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) - \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big)\right) \\ & + \big(\bar{p}(t) - q(t)\big)\left(\frac{\partial g}{\partial p}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) - \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big)\right) \\ & + \mathcal{D}_\alpha\big(\bar{\theta}(t) - \vartheta(t)\big)\left(\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) - \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\right)\Big]\,dt \geq 0,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\int_T \Big[& \big(\bar{\theta}(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) + \big(\bar{p}(t) - q(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) \\ & + \mathcal{D}_\alpha\big(\bar{\theta}(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\bar{\theta},\bar{p}}(t)\big)\Big]\,dt \\ \geq \int_T \Big[& \big(\bar{\theta}(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(\bar{p}(t) - q(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) \\ & + \mathcal{D}_\alpha\big(\bar{\theta}(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\Big]\,dt.\end{aligned} \tag{4}$$

Taking the limit in inequality (3), we have

$$\begin{aligned}\int_T \Big[& \big(\bar{\theta}(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) + \big(\bar{p}(t) - q(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) \\ & + \mathcal{D}_\alpha\big(\bar{\theta}(t) - \vartheta(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\bar{\theta},\bar{p}}(t)\big)\Big]\,dt \leq 0.\end{aligned} \tag{5}$$

On combining (4) and (5), we obtain

$$\begin{aligned}\int_T \Big[& \big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - \bar{p}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) \\ & + \mathcal{D}_\alpha\big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\Big]\,dt \geq 0.\end{aligned}$$

Further, by considering Lemma 2.1, it follows that

$$\begin{aligned}\int_T \Big[& \big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) + \big(q(t) - \bar{p}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) \\ & + \mathcal{D}_\alpha\big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\bar{\theta},\bar{p}}(t)\big)\Big]\,dt \geq 0,\end{aligned} \tag{6}$$

which implies that $(\bar{\theta},\bar{p}) \in \Lambda$.

Since the functional $\int_T g(\chi_{\theta,p}(t))\,dt$ is lower semicontinuous, we obtain

$$\int_T g\big(\chi_{\bar{\theta},\bar{p}}(t)\big)\,dt \leq \lim_{n\to\infty}\inf \int_T g\big(\chi_{\theta_n,p_n}(t)\big)\,dt \leq \lim_{n\to\infty}\sup \int_T g\big(\chi_{\theta_n,p_n}(t)\big)\,dt.$$

By using (2), the above inequality reduces to

$$\int_T g\big(\chi_{\bar\theta,\bar p}(t)\big)\,dt \le \inf_{(\vartheta,q)\in\Lambda}\int_T g\big(\chi_{\vartheta,q}(t)\big)\,dt. \tag{7}$$

From (6) and (7), we obtain that $(\bar\theta,\bar p)$ solves (VPC).

Now, let us prove that $(\bar\theta,\bar p)$ is a unique solution of (CP). Suppose that (θ_1,p_1), (θ_2,p_2) are two different solutions of (CP). Then,

$$0 < \big\|(\theta_1,p_1)-(\theta_2,p_2)\big\| \le \operatorname{diam}\Phi(\rho,\xi)\to 0 \quad \text{as } (\rho,\xi)\to(0,0),$$

which is not possible. □

Theorem 3.2 *Let the functional $\int_T g(\chi_{\theta,p}(t))\,dt$ be monotone, lower semicontinuous, and hemicontinuous on $\Theta\times P$. The variational problem* (CP) *is well-posed if and only if* (CP) *admits a unique solution.*

Proof Let us consider that (CP) is well-posed. Therefore, it admits a unique solution (θ_0,p_0). Now, conversely, we consider that (CP) has a sole solution (θ_0,p_0), that is,

$$\begin{aligned}&\int_T g(\chi_{\theta_0,p_0})\,dt \le \inf_{(\vartheta,q)\in\Lambda}\int_T g(\chi_{\vartheta,q})\,dt,\\ &\int_T\bigg[(\vartheta-\theta_0)\frac{\partial g}{\partial\theta}(\chi_{\theta_0,p_0}) + (q-p_0)\frac{\partial g}{\partial p}(\chi_{\theta_0,p_0})\\ &\quad + \mathcal{D}_\alpha(\vartheta-\theta_0)\frac{\partial g}{\partial\theta_\alpha}(\chi_{\theta_0,p_0})\bigg]\,dt\ge 0,\quad \forall(\vartheta,q)\in\Theta\times P,\end{aligned} \tag{8}$$

but it is not well-posed. By Definition 3.2, there exists an approximating sequence $\{(\theta_n,p_n)\}$ of (CP) that does not converge to (θ_0,p_0). On the other hand, there exist a sequence of positive real numbers $\xi_n\to 0$ as $n\to\infty$ such that the following inequalities

$$\limsup_{n\to\infty}\int_T g\big(\chi_{\theta_n,p_n}(t)\big)\,dt \le \inf_{(\vartheta,q)\in\Lambda}\int_T g\big(\chi_{\vartheta,q}(t)\big)\,dt$$

and

$$\begin{aligned}&\int_T\bigg[\big(\vartheta(t)-\theta_n(t)\big)\frac{\partial g}{\partial\theta}\big(\chi_{\theta_n,p_n}(t)\big) + \big(q(t)-p_n(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_n,p_n}(t)\big)\\ &\quad + \mathcal{D}_\alpha\big(\vartheta(t)-\theta_n(t)\big)\frac{\partial g}{\partial\theta_\alpha}\big(\chi_{\theta_n,p_n}(t)\big)\bigg]\,dt+\xi_n\ge 0,\quad\forall(\vartheta,q)\in\Theta\times P\end{aligned} \tag{9}$$

are fulfilled. Further, to establish the boundedness of $\{(\theta_n,p_n)\}$, we proceed by contradiction. Suppose $\{(\theta_n,p_n)\}$ is not bounded, therefore, $\|(\theta_n,p_n)\|\to+\infty$ as $n\to+\infty$. Let us consider $\delta_n=\frac{1}{\|(\theta_n,p_n)-(\theta_0,p_0)\|}$ and $(\underline{\theta}_n,\underline{p}_n)=(\theta_0,p_0)+\delta_n[(\theta_n,p_n)-(\theta_0,p_0)]$. We have that $\{(\underline{\theta}_n,\underline{p}_n)\}$ is bounded in $\Theta\times P$. Hence, if we pass to a subsequence if necessary, we can assume that

$$(\underline{\theta}_n,\underline{p}_n)\to(\underline{\theta},\underline{p})\quad\text{weakly in }\Theta\times P\ne(\theta_0,p_0).$$

It is not difficult to check that $(\underline{\theta},\underline{p}) \neq (\theta_0,p_0)$ as $\|\delta_n[(\theta_n,p_n)-(\theta_0,p_0)\| = 1$ for all $n \in \mathbb{N}$. Since (θ_0,p_0) is a solution of (CP), therefore the inequalities in (8) are satisfied. By Lemma 2.1, we obtain

$$\begin{aligned}
&\int_T g(\chi_{\theta_0,p_0})\,dt \le \inf_{(\vartheta,q)\in\Lambda}\int_T g(\chi_{\vartheta,q})\,dt,\\
&\int_T\biggl[(\vartheta-\theta_0)\frac{\partial g}{\partial\theta}(\chi_{\vartheta,q}) + (q-p_0)\frac{\partial g}{\partial p}(\chi_{\vartheta,q})\\
&\quad + \mathcal{D}_\alpha(\vartheta-\theta_0)\frac{\partial g}{\partial\theta_\alpha}(\chi_{\vartheta,q})\biggr]dt \ge 0, \quad \forall(\vartheta,q)\in\Theta\times P. \qquad (10)
\end{aligned}$$

By hypothesis, the functional $\int_T g(\chi_{\theta,p}(t))\,dt$ is monotone on $\Theta \times P$. Therefore, for $(\theta_n,p_n),(\vartheta,q)\in\Theta\times P$, we have

$$\begin{aligned}
&\int_T\biggl[\bigl(\theta_n(t)-\vartheta(t)\bigr)\biggl(\frac{\partial g}{\partial\theta}\bigl(\chi_{\theta_n,p_n}(t)\bigr)-\frac{\partial g}{\partial\theta}\bigl(\chi_{\vartheta,q}(t)\bigr)\biggr)\\
&\quad + \bigl(p_n(t)-q(t)\bigr)\biggl(\frac{\partial g}{\partial p}\bigl(\chi_{\theta_n,p_n}(t)\bigr)-\frac{\partial g}{\partial\theta}\bigl(\chi_{\vartheta,q}(t)\bigr)\biggr)\\
&\quad + \mathcal{D}_\alpha\bigl(\theta_n(t)-\vartheta(t)\bigr)\biggl(\frac{\partial g}{\partial\theta_\alpha}\bigl(\chi_{\theta_n,p_n}(t)\bigr)-\frac{\partial g}{\partial\theta_\alpha}\bigl(\chi_{\vartheta,q}(t)\bigr)\biggr)\biggr]dt \ge 0,
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\int_T\biggl[\bigl(\vartheta(t)-\theta_n(t)\bigr)\frac{\partial g}{\partial\theta}\bigl(\chi_{\theta_n,p_n}(t)\bigr) + \bigl(q(t)-p_n(t)\bigr)\frac{\partial g}{\partial p}\bigl(\chi_{\theta_n,p_n}(t)\bigr)\\
&\qquad + \mathcal{D}_\alpha\bigl(\vartheta(t)-\theta_n(t)\bigr)\frac{\partial g}{\partial\theta_\alpha}\bigl(\chi_{\theta_n,p_n}(t)\bigr)\biggr]dt\\
&\quad\le \int_T\biggl[\bigl(\vartheta(t)-\theta_n(t)\bigr)\frac{\partial g}{\partial\theta}\bigl(\chi_{\vartheta,q}(t)\bigr) + \bigl(q(t)-p_n(t)\bigr)\frac{\partial g}{\partial p}\bigl(\chi_{\vartheta,q}(t)\bigr)\\
&\qquad + \mathcal{D}_\alpha\bigl(\vartheta(t)-\theta_n(t)\bigr)\frac{\partial g}{\partial\theta_\alpha}\bigl(\chi_{\vartheta,q}(t)\bigr)\biggr]dt. \qquad (11)
\end{aligned}$$

Combining with (9) and (11), we have

$$\begin{aligned}
&\int_T\biggl[\bigl(\vartheta(t)-\theta_n(t)\bigr)\frac{\partial g}{\partial\theta}\bigl(\chi_{\vartheta,q}(t)\bigr) + \bigl(q(t)-p_n(t)\bigr)\frac{\partial g}{\partial p}\bigl(\chi_{\vartheta,q}(t)\bigr)\\
&\quad + \mathcal{D}_\alpha\bigl(\vartheta(t)-\theta_n(t)\bigr)\frac{\partial g}{\partial\theta_\alpha}\bigl(\chi_{\vartheta,q}(t)\bigr)\biggr]dt \ge -\xi_n, \quad \forall(\vartheta,q)\in\Theta\times P.
\end{aligned}$$

Because $\delta_n \to 0$ as $n\to\infty$, we can take $n_0\in\mathbb{N}$ to be large enough so that $\delta_n < 1$, for all $n \ge n_0$. By multiplying the above inequality and (10) by $\delta_n > 0$ and $1-\delta_n > 0$, respectively, we make the sum of the resulting inequalities to obtain

$$\begin{aligned}
&\int_T\biggl[\bigl(\vartheta(t)-\underline{\theta}_n(t)\bigr)\frac{\partial g}{\partial\theta}\bigl(\chi_{\vartheta,q}(t)\bigr) + \bigl(q(t)-\underline{p}_n(t)\bigr)\frac{\partial g}{\partial p}\bigl(\chi_{\vartheta,q}(t)\bigr)\\
&\quad + \mathcal{D}_\alpha\bigl(\vartheta(t)-\underline{\theta}_n(t)\bigr)\frac{\partial g}{\partial\theta_\alpha}\bigl(\chi_{\vartheta,q}(t)\bigr)\biggr]dt \ge -\xi_n, \quad \forall(\vartheta,q)\in\Theta\times P, \forall n\ge n_0.
\end{aligned}$$

Since $(\underline{\theta}_n, \underline{p}_n) \to (\underline{\theta}, \underline{p}) \neq (\theta_0, p_0)$ and $(\underline{\theta}_n, \underline{p}_n) = (\theta_0, p_0) + \delta_n[(\theta_n, p_n) - (\theta_0, p_0)]$, we have

$$\begin{aligned}
&\int_T \Big[\big(\vartheta(t) - \underline{\theta}(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - \underline{p}(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \underline{\theta}(t)\big) \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big) \Big] dt \\
&= \lim_{n\to\infty} \int_T \Big[\big(\vartheta(t) - \underline{\theta}_n(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - \underline{p}_n(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \underline{\theta}_n(t)\big) \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big) \Big] dt \\
&\geq - \lim_{n\to\infty} \xi_n = 0, \quad \forall (\vartheta, q) \in \Theta \times P.
\end{aligned}$$

By using Lemma 2.1 and considering the lower semicontinuity property, we obtain

$$\begin{aligned}
&\int_T g\big(\chi_{\underline{\theta},\underline{p}}(t)\big)\, dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g\big(\chi_{\vartheta,q}(t)\big)\, dt, \\
&\int_T \Big[\big(\vartheta(t) - \underline{\theta}(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\underline{\theta},\underline{p}}(t)\big) + \big(q(t) - \underline{p}(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\underline{\theta},\underline{p}}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \underline{\theta}(t)\big) \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\underline{\theta},\underline{p}}(t)\big) \Big] dt \geq 0, \quad \forall (\vartheta, q) \in \Theta \times P.
\end{aligned} \tag{12}$$

We obtain that $(\underline{\theta}, \underline{p})$ is a solution of (CP), which is a contradiction with the uniqueness of (θ_0, p_0). Therefore, $\{(\theta_n, p_n)\}$ is a bounded sequence having a convergent subsequence $\{(\theta_{n_k}, p_{n_k})\}$ that converges to $(\bar{\theta}, \bar{p}) \in \Theta \times P$ as $k \to \infty$. Again, from monotonicity, for $(\theta_{n_k}, p_{n_k}), (\vartheta, q) \in \Theta \times P$, we have (see (11))

$$\begin{aligned}
&\int_T \Big[\big(\vartheta(t) - \theta_{n_k}(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) + \big(q(t) - p_{n_k}(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big) \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) \Big] dt \\
&\leq \int_T \Big[\big(\vartheta(t) - \theta_{n_k}(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - p_{n_k}(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big) \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big) \Big] dt.
\end{aligned} \tag{13}$$

Also, as a result of (9), we can write

$$\begin{aligned}
&\lim_{k\to\infty} \int_T \Big[\big(\vartheta(t) - \theta_{n_k}(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) + \big(q(t) - p_{n_k}(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) \\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big) \frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) \Big] dt \geq 0.
\end{aligned} \tag{14}$$

Combining (13) and (14), we have

$$\lim_{k\to\infty} \int_T \Big[\big(\vartheta(t) - \theta_{n_k}(t)\big) \frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - p_{n_k}(t)\big) \frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)$$

$$+ \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg] dt \geq 0$$
$$\Rightarrow \quad \int_T \bigg[\big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - \bar{p}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)$$
$$+ \mathcal{D}_\alpha\big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg] dt \geq 0.$$

By using Lemma 2.1 and considering the lower semicontinuity property, we obtain

$$\int_T g\big(\chi_{\bar{\theta},\bar{p}}(t)\big)\, dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g\big(\chi_{\vartheta,q}(t)\big)\, dt,$$
$$\int_T \bigg[\big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\bar{\theta},\bar{p}}(t)\big) + \big(q(t) - \bar{p}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\bar{\theta},\bar{p}}(t)\big)$$
$$+ \mathcal{D}_\alpha\big(\vartheta(t) - \bar{\theta}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\bar{\theta},\bar{p}}(t)\big)\bigg] dt \geq 0,$$

which shows that $(\bar{\theta}, \bar{p})$ is a solution of (CP). Hence, $(\theta_{n_k}, p_{n_k}) \to (\bar{\theta}, \bar{p})$, that is, $(\theta_{n_k}, p_{n_k}) \to (\theta_0, p_0)$, involving $(\theta_n, p_n) \to (\theta_0, p_0)$. □

Theorem 3.3 *Let the functional $\int_T g(\chi_{\theta,p}(t))\, dt$ be hemicontinuous, lower semicontinuous, and monotone on the compact and convex set $\Theta \times P$. The variational control problem* (CP) *is generalized well-posed if and only if Φ is nonempty.*

Proof Let us consider that (CP) is generalized well-posed. Hence, by Definition 3.2, Φ is nonempty. Now, conversely, let $\{(\theta_n, p_n)\}$ be an approximating sequence of (CP). Therefore, there exists a sequence of positive real numbers $\xi_n \to 0$ such that

$$\lim_{n\to\infty} \sup \int_T g\big(\chi_{\theta_n,p_n}(t)\big)\, dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g\big(\chi_{\vartheta,q}(t)\big)\, dt, \tag{15}$$
$$\int_T \bigg[\big(\vartheta(t) - \theta_n(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta_n,p_n}(t)\big) + \big(q(t) - p_n(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_n,p_n}(t)\big)$$
$$+ \mathcal{D}_\alpha\big(\vartheta(t) - \theta_n(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_n,p_n}(t)\big)\bigg] dt + \xi_n \geq 0, \quad \forall(\vartheta, q) \in \Theta \times P \tag{16}$$

are satisfied. Since $\Theta \times P$ is a compact set, $\{(\theta_n, p_n)\}$ has a subsequence $\{(\theta_{n_k}, p_{n_k})\}$, converging to some pair $(\theta_0, p_0) \in \Theta \times P$. Since the functional $\int_T g(\chi_{\theta,p}(t))\, dt$ is monotone on $\Theta \times P$, for

$$(\theta_{n_k}, p_{n_k}), \quad (\vartheta, q) \in \Theta \times P,$$

it follows that

$$\int_T \bigg[\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) + \big(q(t) - p_{n_k}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big)$$
$$+ \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big)\bigg] dt$$
$$\leq \int_T \bigg[\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - p_{n_k}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)$$

$$+ \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg] dt.$$

Taking the limit $k \to \infty$, we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\int_T\bigg[\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) + \big(q(t) - p_{n_k}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big)\\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big)\bigg] dt\\
&\leq \lim_{k\to\infty}\int_T\bigg[\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - p_{n_k}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)\\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg] dt. \qquad (17)
\end{aligned}$$

Since $\{(\theta_{n_k}, p_{n_k})\}$ is an approximating subsequence in $\Theta \times P$, by (16), we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\int_T\bigg[\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big) + \big(q(t) - p_{n_k}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big)\\
&+ \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_{n_k},p_{n_k}}(t)\big)\bigg] dt \geq 0, \quad \forall(\vartheta, q) \in \Theta \times P. \qquad (18)
\end{aligned}$$

Combining (17) and (18), we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\int_T\bigg[\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - p_{n_k}(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)\\
&+ \mathcal{D}_\alpha\big(\vartheta(t) - \theta_{n_k}(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg] dt \geq 0, \quad \forall\big(\vartheta(t), q(t)\big) \in \Theta \times P\\
&\Rightarrow \quad \int_T\bigg[\big(\vartheta(t) - \theta_0(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\vartheta,q}(t)\big) + \big(q(t) - p_0(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\vartheta,q}(t)\big)\\
&\qquad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_0(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\vartheta,q}(t)\big)\bigg] dt \geq 0, \quad \forall(\vartheta, q) \in \Theta \times P.
\end{aligned}$$

By using Lemma 2.1 and considering the lower semicontinuity property, we obtain

$$\begin{aligned}
&\int_T g\big(\chi_{\theta_0,p_0}(t)\big)\, dt \leq \inf_{(\vartheta,q)\in\Lambda}\int_T g\big(\chi_{\vartheta,q}(t)\big)\, dt,\\
&\int_T\bigg[\big(\vartheta(t) - \theta_0(t)\big)\frac{\partial g}{\partial \theta}\big(\chi_{\theta_0,p_0}(t)\big) + \big(q(t) - p_0(t)\big)\frac{\partial g}{\partial p}\big(\chi_{\theta_0,p_0}(t)\big)\\
&\quad + \mathcal{D}_\alpha\big(\vartheta(t) - \theta_0(t)\big)\frac{\partial g}{\partial \theta_\alpha}\big(\chi_{\theta_0,p_0}(t)\big)\bigg] dt \geq 0, \quad \forall(\vartheta, q) \in \Theta \times P,
\end{aligned}$$

which shows that $(\theta_0, p_0) \in \Phi$. □

Theorem 3.4 *Let the functional $\int_T g(\chi_{\theta,p}(t))\, dt$ be lower semicontinuous, hemicontinuous, and monotone on the compact and convex set $\Theta \times P$. The variational control problem* (CP) *is generalized well-posed if there exists $\xi > 0$ so that $\Phi(\xi, \xi)$ is* (*nonempty*) *bounded.*

Proof Let $\xi > 0$ be such that $\Phi(\xi,\xi)$ is bounded (nonempty). Let us consider that $\{(\theta_n, p_n)\}$ is an approximating sequence of (CP). Hence, there exists a sequence of positive real numbers $\xi_n \to 0$ such that

$$\lim_{n\to\infty} \sup \int_T g(\chi_{\theta_n,p_n}(t))\,dt \leq \inf_{(\vartheta,q)\in\Lambda} \int_T g(\chi_{\vartheta,q}(t))\,dt,$$

$$\int_T \Big[(\vartheta(t) - \theta_n(t)) \frac{\partial g}{\partial \theta}(\chi_{\theta_n,p_n}(t)) + (q(t) - p_n(t)) \frac{\partial g}{\partial p}(\chi_{\theta_n,p_n}(t)) + \mathcal{D}_\alpha(\vartheta(t) - \theta_n(t)) \frac{\partial g}{\partial \theta_\alpha}(\chi_{\theta_n,p_n}(t)) \Big]\,dt + \xi_n \geq 0, \quad \forall (\vartheta,q) \in \Theta \times P$$

are satisfied, involving that $(\theta_n, p_n) \in \Phi(\xi,\xi)$, $\forall n > m$ (m is a natural number). Therefore, we obtain that $\{(\theta_n, p_n)\}$ is a bounded sequence having a convergent subsequence $\{(\theta_{n_k}, p_{n_k})\}$, weakly converging to (θ_0, p_0) as $k \to \infty$. Proceeding in a similar way as in Theorem 3.3, we obtain $(\theta_0, p_0) \in \Phi$. □

Next, we provide a concrete application that can be studied only with the mathematical tools and results developed in the current paper.

Illustrative Application. Minimize the mass of the flat plate $[0,3]^2 = [0,3] \times [0,3]$, having a controlled density given by $p^4(t) + e^{\theta(t)} - \theta(t)$, that depends on the current point, such that the following controlled dynamical system $\theta_\alpha(t) = p(t)$, $\forall t \in [0,3]^2$, together with the boundary conditions $(\theta,p)|_{\partial T} = 0$, and the positivity property

$$\int_{[0,3]^2} \big[4(q(t) - p(t))p^3(t) + (\vartheta(t) - \theta(t))(e^{\theta(t)} - 1)\big]\,dt \geq 0, \quad \forall (\vartheta,q) \in C^1(T,[-5,5]) \times C(T,[-5,5])$$

are satisfied.

In order to solve the above concrete mechanical-physics problem, we take $m = 2$, $n = k = 1$, $T = [0,3]^2$ (see Sect. 2), and consider

$$g(\chi_{\theta,p}(t)) = p^4(t) + e^{\theta(t)} - \theta(t)$$

and the variational inequality-constrained control problem:

$$\text{Minimize} \int_T g(\chi_{\theta,p}(t))\,dt \tag{CP-1}$$

subject to

$$\int_T \big[4(q(t) - p(t))p^3(t) + (\vartheta(t) - \theta(t))(e^{\theta(t)} - 1)\big]\,dt \geq 0,$$

$$(\theta,p)|_{\partial T} = 0, \qquad \theta_\alpha = p, \quad \forall (\vartheta,q) \in \Theta \times P = C^1(T,[-5,5]) \times C(T,[-5,5]).$$

We have $\Phi = \{(0,0)\}$ and, also, it can be easily seen that $\int_T g(\chi_{\theta,p}(t))\,dt$ is monotone, lower semicontinuous, and hemicontinuous on $\Theta \times P$. Since Theorem 3.2 is fulfilled, we conclude that the variational problem (CP-1) is well-posed. Moreover, we have $\Phi(\rho,\xi) = \{(0,0)\}$ and, consequently, $\Phi(\rho,\xi) \neq \phi$ and diam $\Phi(\rho,\xi) \to 0$ as $(\rho,\xi) \to (0,0)$. Taking into account Theorem 3.1, the variational problem (CP-1) is well-posed.

4 Conclusions

In this paper, we have studied the well-posedness and generalized well-posedness for new variational control problems. Namely, by using the concepts of lower semicontinuity, pseudomonotonicity, monotonicity, and hemicontinuity associated with functionals of multiple-integral type, under suitable assumptions, we have established that the well-posedness is characterized in terms of the existence and uniqueness of their solutions. Moreover, sufficient conditions were provided for the generalized well-posedness by assuming the nonemptiness and boundedness of the approximating solution set. A concrete application, which can be studied only with the mathematical tools and results developed in the current paper, was presented.

As immediate further developments of this paper, we mention the following two: (a) reformulating the main results derived in this paper by using the variational/functional derivative of integral functionals; (b) the study of the saddle-point optimality criteria associated with this type of constrained optimization problems.

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