Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Normalized generalized Bessel function and its geometric properties

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Abstract

The normalization of the generalized Bessel functions $\mathrm{U}_{\sigma,r}$ ($\sigma, r \in \mathbb{C}$) defined by

$$\mathrm{U}_{\sigma,r}(z) = z + \sum_{j=1}^{\infty} \frac{(-r)^j}{4^j (1)_j (\sigma)_j} z^{j+1}$$

was introduced, and some of its geometric properties have been presented previously. The main purpose of the present paper is to complete the results given in the literature by employing a new procedure. We first used an identity for the logarithmic of the gamma function as well as an inequality for the digamma function to establish sufficient conditions on the parameters so that $\mathrm{U}_{\sigma,r}$ is starlike or convex of order α ($0 \le \alpha \le 1$) in the open unit disk. Moreover, the starlikeness and convexity of $\mathrm{U}_{\sigma,r}$ have been considered where the leading concept of the proofs comes from the starlikeness of the power series $f(z) = \sum_{j=1}^{\infty} A_j z^j$ and the classical Alexander theorem between the classes of starlike and convex functions. We gave a simple proof to show that our conditions are not contradictory. Ultimately, the close-to-convexity of $(z \cos \sqrt{z}) * \mathrm{U}_{\sigma,r}$ and $(\sin z) * \frac{\mathrm{U}_{\sigma,r}(z^2)}{z}$ have been determined, where "$*$" stands for the convolution between the power series.

MSC: 30C45; 33C10; 30C50

Keywords: Univalent; Starlike; Convex and close-to convex functions; Convolution (Hadamard product); Modified Bessel function; Digamma function; Generalized Struve and Lommel–Wright function; Error function

1 Introduction and preliminaries

It is well-known that the special functions, and, in particular, the generalized Bessel function, play a crucial role in different fields of mathematical physics and engineering. These functions received particular attention for providing solutions of the differential equations and systems used as mathematical models, as well as numerous classes of transcendental functions, as special ones, which appear in many branches, including the Geometric Function Theory (GFT).

Geometric Function Theory concerned with the interplay between the geometric properties of the image domain and the analytic properties of the mapping function. The origin of the GFT was founded at the turn of the 20*th* century by the famous mathematician Riemann in his doctoral thesis. The Riemann mapping theorem is known as one of the most fundamental contributions of Complex Analysis, and it allows the mathematicians to solve problems for the simply connected domain in the particular case of the open unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ without loss of the generality. The theorem states that every simply connected domain D of the complex plane that is a proper subset of the complex plane $\mathbb{C}$ can be mapped conformally onto $\mathbb{U}$. Furthermore, there is unique conformal mapping $f : D \to \mathbb{U}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

The cornerstone of GFT is the theory of univalent functions initiated by Koebe [10]. Before the Bieberbach conjecture [4] on the coefficients of a power series expansion of normalized univalent functions was proved, many papers dealing with the relevance between the theory of univalent functions and these special functions were published in the literature. In a series of studies, Kreyszig and Todd [11–13] investigated the univalence of the error function $\mathrm{Erf}(z)$, the function $\exp(z^2) \cdot \mathrm{Erf}(z)$, as well as the Bessel function $z^{1-\nu} \cdot \mathrm{J}_\nu(z)$. Merkes and Scott [18] investigated the starlikeness of Gaussian hypergeometric functions using the continued fraction of Gauss. In [3], Carlson and Shaffer defined an operator involving an incomplete beta function and obtained interesting results for starlike and prestarlike functions. In addition, the order of starlikeness of hypergeometric functions was investigated by Ruscheweyh and Singh [28] using a refined version of continued fractions like those used by Merkes and Scott.

More recently, there has been an extensive bibliography on the geometric properties of some normalized special functions, like the univalence, starlikeness, convexity, and close-to-convexity in the open unit disk. Regarding treatises on this investigation, we refer, for example, for the hypergeometric function to [19, 25–27], for the Bessel function to [1, 2], for the generalized Struve function to [21, 22, 31, 33], for the Lommel function to [30], for the generalized Lommel–Wright function to [32], for the Fox–Wright function to [16], and to [17] for the Le Roy-type Mittag-Leffler function. These results would enrich the understanding of the geometrical properties of such functions as tools in such applications of GFT.

The content of the paper is summarized in the following way. First of all, we outline several well-known mathematical facts to be used in the sequel. Further, we complete the results given in [1, 2, 20] by applying a new procedure first using an identity for the logarithm of the gamma function, as well as an inequality for the digamma function proved by [8], to establish sufficient conditions on the parameters such that $\mathrm{U}_{\sigma,r}$ is starlike or convex of order α $(0 \leq \alpha \leq 1)$. Moreover, the starlikeness and convexity of $\mathrm{U}_{\sigma,r}$ have been considered where the leading concept of the proofs comes from the starlikeness of the power series $f(z) = \sum_{j=1}^{\infty} A_j z^j$ and the classical Alexander theorem between the classes of starlike and convex functions followed by a simple proof showing that our conditions are not contradictory. Ultimately, the close-to-convexity of $(z\cos\sqrt{z}) * \mathrm{U}_{\sigma,r}$ and $(\sin z) * \frac{\mathrm{U}_{\sigma,r}(z^2)}{z}$ have been determined, where "$*$" represents the convolution between the power series.

Throughout this paper, let $\mathcal{H}$ stand for the class of all functions that are analytic in $\mathbb{U}$, while $\mathcal{A}$ denote the subfamily of $\mathcal{H}$ consisting of functions that have the form $f(z) = z + \sum_{j=2}^{\infty} A_j z^j$, $z \in \mathbb{U}$, and by $\mathcal{S}$ the subfamily of $\mathcal{A}$, which are univalent in $\mathbb{U}$.

If $g \in \mathcal{A}$ has the form $g(z) = z + \sum_{j=2}^{\infty} B_j z^j$, $z \in \mathbb{U}$, then the *Hadamard product (or convolution)* of f and g, denoted by $f * g$, is given by

$$(f * g)(z) := z + \sum_{j=2}^{\infty} A_j B_j z^j, \quad z \in \mathbb{U},$$

and the above definition of the *Hadamard product* originates from (see [5])

$$(f * g)\left(r^2 e^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} f\left(re^{i(\theta-t)}\right) g\left(re^{it}\right) \mathrm{d}t, \quad r < 1.$$

One of the most important concepts of the univalent function theory are the families of starlike and convex functions, which are subfamilies of $\mathcal{H}$. More importantly, these classes admit geometrical and analytical characteristics that do not pass in the case of those functions that are used in the mathematical analysis. We refer the interested readers to [5, 7, 9, 24] for further information. Naturally, a domain $D \subset \mathbb{C}$ is called a *starlike with respect to an interior point* z_0 if every line segment joining z_0 to any other point in D lies completely in D. In particular, if $z_0 = 0$, then D is called a *starlike domain*. A function $f \in \mathcal{A}$ is called a *starlike with respect to the origin (or briefly starlike)*, denoted by $\mathcal{S}^*$, if $f(\mathbb{U})$ is a starlike domain, that is,

$$\mathcal{S}^* := \{f \in \mathcal{A} : f \text{ is a starlike function}\}.$$

The following theorem gives an analytic description of the starlike functions:

Theorem A *If $f \in \mathcal{A}$, then f is a starlike function if and only if*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{U}.$$

Further, if every line segment joining any two points of $D \subset \mathbb{C}$ lies completely in D, then D is called a *convex domain*. A function $f \in \mathcal{A}$ *is called a convex function* if $f(\mathbb{U})$ is a convex domain, that is,

$$\mathcal{K} := \{f \in \mathcal{A} : f \text{ is a convex function}\}.$$

The well-known analytical characterization of convexity is given by the following theorem:

Theorem B *If $f \in \mathcal{A}$, then f is a convex function if and only if*

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0, \quad z \in \mathbb{U}.$$

It is well-known that $\mathcal{S}^*$ and $\mathcal{K}$ have particular interest if more restrictions are enjoined, and it gives us several types of subclasses of univalent functions. Moreover, the positivity of $\operatorname{Re} \frac{zf'(z)}{f(z)}$ and $1 + \operatorname{Re} \frac{zf''(z)}{f'(z)}$ for $\mathcal{S}^*$ and $\mathcal{K}$, respectively, helps us to study different families of conformal transformation with other motivating geometric properties.

On the other hand, $f \in \mathcal{A}$ is a *starlike functions of order* α, denoted by $\mathcal{S}^*(\alpha)$, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{U},$$

where $0 \le \alpha \le 1$, and is in the class of *convex functions of order* α, denoted by $\mathcal{K}(\alpha)$, if and only if

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \alpha, \quad z \in \mathbb{U}.$$

It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K}(\alpha) \subset \mathcal{K} := \mathcal{K}(0)$. Further, $f \in \mathcal{H}$ is *close-to-convex in* $\mathbb{U}$ if it is univalent, and the range $f(\mathbb{U})$ is a *close-to-convex domain*, that is the complement of $f(\mathbb{U})$ can be expressed as the union of non-interesting half-lines. In addition, a normalized $f \in \mathcal{H}$ is *close-to-convex* with respect to a fixed starlike function $g \in \mathcal{S}^*$, not necessarily normalized, denoted by $\mathcal{C}_g$, if and only if

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in \mathbb{U}.$$

The well-known observation that all the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, and $\mathcal{C}_g$ are subsets of $\mathcal{S}$ can be easily verified.

A widely investigated homogeneous second-order differential equation is given by (see, for details, [1])

$$z^2\omega''(z) + qz\omega'(z) + \left[rz^2 - p^2 + (1-q)p\right]\omega(z) = 0, \tag{1.1}$$

whose solutions are extensions of the generalized Bessel function, $p, q \in \mathbb{R}$ and $r \in \mathbb{C}$. The generalized Bessel function of order p is the particular solution of (1.1), which has the power series expansion

$$\omega_{p,q,r}(z) = \sum_{j=0}^{\infty} \frac{(-r)^j}{\Gamma(j+1)\Gamma(p+j+\frac{q+1}{2})} \left(\frac{z}{2}\right)^{2j+p}. \tag{1.2}$$

It is worth mentioning that the above differential equation (1.1) has a particular interest. It allows us to know more information regarding the Bessel, modified Bessel, and spherical Bessel functions. In addition, the series (1.2) is convergent everywhere while it is not univalent in $\mathbb{U}$. Considering also that special values of the parameters p, q, and r will give us the well-known Bessel, modified Bessel, and spherical Bessel functions. For instance, putting $q = c = 1$, the Bessel function will follow, which defined as

$$J_p(z) := \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+p+1)} \left(\frac{z}{2}\right)^{p+2j}, \quad z \in \mathbb{C}.$$

For $q = 1$ and $c = -1$, we get the modified Bessel function defined by

$$I_p(z) := \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(j+p+1)} \left(\frac{z}{2}\right)^{p+2j}, \quad z \in \mathbb{C},$$

while for $q = 2$ and $c = -1$, we get the spherical Bessel function defined by

$$S_p(z) := \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(j+p+\frac{3}{2})}\left(\frac{z}{2}\right)^{p+2j}, \quad z \in \mathbb{C}.$$

One can observe that $\omega_{p,q,r} \notin \mathcal{A}$; therefore, we consider the following transformation (see [22])

$$\mathrm{u}_{p,q,r}(z) := 2^p \Gamma\left(p + \frac{q+2}{2}\right) z^{-\frac{p}{2}} \omega_{p,q,r}(\sqrt{z}). \tag{1.3}$$

From (1.3), the series expansion of $\mathrm{u}_{p,q,r}$ has the form

$$\mathrm{u}_{p,q,r}(z) = \sum_{j=0}^{\infty} \frac{(-r)^j}{4^j (1)_j (p + \frac{q+2}{2})_j} z^j,$$

where $p + (q+2)/2 \notin \{0, -1, -2, \dots\}$, and $(\rho)_n$ represents the *Pochhammer symbol* defined by

$$(\rho)_n := \begin{cases} 1 & \text{if } n = 0, \\ \rho(\rho+1)(\rho+2)\dots(\rho+n-1) & \text{if } n \in \mathbb{N} := \{1, 2, \dots\}. \end{cases}$$

Based on the previous representations, we formulate the following definition:

Definition 1.1 For $p, q, r \in \mathbb{C}$, the normalization of the of generalized Bessel functions $\mathrm{U}_{\sigma,r}$ is defined by

$$\mathrm{U}_{\sigma,r}(z) := z \cdot \mathrm{u}_{p,q,r}(z) = z + \sum_{j=1}^{\infty} \frac{(-r)^j}{4^j (1)_j (\sigma)_j} z^{j+1}, \quad z \in \mathbb{U}, \tag{1.4}$$

where $\sigma := p + (q+2)/2 \notin \{0, -1, -2, \dots\}$.

The following lemmas will be beneficial to get the main results:

Lemma 1.1 ([8]) *The following inequality holds for $t \in (0, \infty)$:*

$$\ln t - \frac{1}{t} < \psi(t) < \ln t - \frac{1}{2t}, \tag{1.5}$$

where ψ represents the digamma function, *that is the derivative of the logarithm of Γ function.*

Lemma 1.2 ([6, Satz IX]) *If $\{A_j\}_{n\in\mathbb{N}}$ is a nonnegative real sequence with $A_1 = 1$, such that $\{jA_j\}_{j\in\mathbb{N}}$ and $\{jA_j - (j+1)A_{j+1}\}_{j\in\mathbb{N}}$ are nonincreasing sequences, then $f(z) = \sum_{j=1}^{\infty} A_j z^j$ is starlike in $\mathbb{U}$.*

Lemma 1.3 ([23, Corollary 7 and Theorem 8′]) *Assume that $0 \le jA_j \le \dots \le 2A_2 \le 1$, or $2 \ge jA_j \ge \dots \ge 2A_2 \ge 1$, where f can be expressed by $f(z) = z + \sum_{j=2}^{\infty} A_j z^j$, $z \in \mathbb{U}$, then the function f is close-to-convex with respect to $-\log(1-z)$.*

The following lemma is a special case of [23, Corollary 9] for the odd functions of the form $f(z) = z + \sum_{j=1}^{\infty} A_{2j+1} z^{2j+1}$, $z \in \mathbb{U}$, (see also [23, Theorem 10]), and it deals with f to be close-to-convex with respect to

$$g_*(z) := \frac{1}{2} \log\left(\frac{1+z}{1-z}\right).$$

Lemma 1.4 *Assume that* $f(z) = z + \sum_{j=1}^{\infty} A_{2j+1} z^{2j+1}$, $z \in \mathbb{U}$, *is an odd function such that* $0 \le (1+2j)A_{2j+1} \le \cdots \le 3A_3 \le 1$, *or* $2 \ge (1+2j)A_{2j+1} \ge \cdots \ge 3A_3 \ge 1$, *for all* $n \in \mathbb{N}$. *Then,* $f \in \mathcal{C}_{g_*} \subset \mathcal{S}$.

2 Main results

The first two theorems of this section contain some interesting and useful results involving the order of starlikeness and the order of convexity of $\mathrm{U}_{\sigma,r}$. The proofs use the inequalities for the digamma function and its derivative that have been proved in [8].

Theorem 2.1 *Let* $\sigma \in (-1,0) \cup (0,+\infty)$ *and* $r \in \mathbb{C}$, *such that*

$$\ln(1+\sigma) + \ln 2 - \frac{1}{1+\sigma} - \frac{3}{2} - \ln|r| \ge 0, \tag{2.1}$$

and

$$0 \le \alpha \le \frac{3|\sigma| - 2|r|}{3|\sigma| - |r|}. \tag{2.2}$$

Then, $\mathrm{U}_{\sigma,r} \in \mathcal{S}^*(\alpha)$.

Proof It is obvious that the inequality

$$\left| \frac{z\mathrm{U}'_{\sigma,r}(z)}{\mathrm{U}_{\sigma,r}(z)} - 1 \right| < 1 - \alpha, \quad z \in \mathbb{U}, \tag{2.3}$$

implies that $\mathrm{U}_{\sigma,r} \in \mathcal{S}^*(\alpha)$, where $\alpha < 1$.

From the well-known triangle inequality and the theorem of the maximum of the modulus for an analytic function, we get

$$\begin{aligned} \left| \mathrm{U}'_{\sigma,r}(z) - \frac{\mathrm{U}_{\sigma,r}(z)}{z} \right| &= \left| \sum_{j=1}^{\infty} \frac{j(-r)^j}{4^j j!(\sigma)_j} z^j \right| < \sup_{\theta \in [0,2\pi]} \left| \sum_{j=1}^{\infty} \frac{j(-r)^j}{4^j j!(\sigma)_j} e^{ij\theta} \right| \\ &\le \frac{\Gamma(\sigma+1)}{|\sigma|} \sum_{j=1}^{\infty} \frac{j|r|^j}{4^j \Gamma(\sigma+j)\Gamma(j+1)}, \quad z \in \mathbb{U}. \end{aligned}$$

Let the function $X_{\sigma,r} : [1,+\infty) \to (0,+\infty)$ defined by

$$X_{\sigma,r}(t) = \frac{t|r|^t}{\Gamma(\sigma+t)\Gamma(t+1)}, \quad t \ge 1. \tag{2.4}$$

It is well-known that

$$\ln\Gamma(z) = -\gamma z - \ln z + \sum_{m=1}^{\infty} \left[\frac{z}{m} - \ln\left(1 + \frac{z}{m}\right) \right], \tag{2.5}$$

where γ is the *Euler–Mascheroni constant* given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log n\right) = 0.5772156649\ldots.$$

Taking the natural logarithm on both sides of (2.4) and using (2.5), we get

$$\begin{aligned}\ln X_{\sigma,r}(t) = {} & \ln t + t\ln|r| + \gamma(\sigma+t) + \ln(\sigma+t) - \sum_{m=1}^{\infty}\left[\frac{\sigma+t}{m} - \ln\left(1+\frac{\sigma+t}{m}\right)\right] \\ & + \gamma(t+1) + \ln(t+1) - \sum_{m=1}^{\infty}\left[\frac{t+1}{m} - \ln\left(1+\frac{t+1}{m}\right)\right],\end{aligned} \tag{2.6}$$

and differentiating the both sides of (2.6), it follows that

$$\begin{aligned}\frac{X'_{\sigma,r}(t)}{X_{\sigma,r}(t)} = {} & \frac{1}{t} + \ln|r| + 2\gamma + \frac{1}{\sigma+t} + \frac{1}{t+1} \\ & - \sum_{m=1}^{\infty}\left[\frac{1}{m} - \frac{1}{\sigma+t+m}\right] - \sum_{m=1}^{\infty}\left[\frac{1}{m} - \frac{1}{t+m+1}\right] =: \widetilde{X}_{\sigma,r}(t).\end{aligned}$$

Differentiating the function $\widetilde{X}_{\sigma,r}$ we obtain

$$\begin{aligned}\widetilde{X}'_{\sigma,r}(t) = {} & -\frac{1}{t^2} - \frac{1}{(\sigma+t)^2} - \frac{1}{(t+1)^2} \\ & - \sum_{m=1}^{\infty}\frac{1}{(\sigma+t+m)^2} - \sum_{m=1}^{\infty}\frac{1}{(t+m+1)^2} < 0, \quad t\in[1,+\infty),\end{aligned}$$

for each $\sigma\in(-1,0)\cup(0,+\infty)$, which implies that $\widetilde{X}_{\sigma,r}$ is strictly decreasing on $[1,+\infty)$.

Since $\widetilde{X}_{\sigma,r}$ is strictly decreasing, if we show that $\widetilde{X}_{\sigma,r}(1) < 0$, this implies that $\widetilde{X}_{\sigma,r}(t) < 0$ for each $t \geq 1$ and $\sigma\in(-1,0)\cup(0,+\infty)$ so that $X'_{\sigma,r}(t) = X_{\sigma,r}(t)\widetilde{X}_{\sigma,r}(t) < 0$, that is $X_{\sigma,r}$ is a strictly decreasing function on $[1,+\infty)$. Thus, we shall establish conditions on σ and r such that $\widetilde{X}_{\sigma,r}(1)$ is non-positive.

Keeping in mind that ψ represents the well-known digamma function defined by

$$\psi(z) = \frac{\partial}{\partial z}\left[\log\Gamma(x)\right] = \frac{\Gamma'(z)}{\Gamma(z)},$$

and using the fact that

$$\psi(z+1) = -\gamma + \sum_{s=1}^{\infty}\left(\frac{1}{s} - \frac{1}{s+z}\right), \quad z\in\mathbb{C}\setminus\{-1,-2,\ldots\},$$

the function $\widetilde{X}_{\sigma,r}$ can be expressed as

$$\widetilde{X}_{\sigma,r}(t) = \frac{1}{t} + \frac{1}{t+\sigma} + \frac{1}{t+1} - \psi(\sigma+1+t) - \psi(t+2) + \ln|r|. \tag{2.7}$$

Since

$$\psi(z+1)=\frac{1}{z}+\psi(z),$$

relation (2.7) becomes

$$\widetilde{X}_{\sigma,r}(t)=\frac{1}{t}-\psi(t+\sigma)-\psi(t+1)+\ln|r|.$$

Now, the fact that $\widetilde{X}_{\sigma,r}$ is a strictly decreasing function on $[1,+\infty)$ can be used to get

$$M\bigl(|r|,\sigma\bigr):=\sup\bigl\{\widetilde{X}_{\sigma,r}(t):t\geq 1\bigr\}=\widetilde{X}_{\sigma,r}(1)=1+\ln|r|-\psi(1+\sigma)-\psi(2),$$

such that $M(|r|,\sigma)\leq 0$. From (1.5), we have

$$M\bigl(|r|,\sigma\bigr)=1+\ln|r|-\psi(\sigma+1)-\psi(2)<\frac{3}{2}+\ln|r|-\ln(1+\sigma)-\ln 2+\frac{1}{1+\sigma}\leq 0,$$

and the last inequality represents assumption (2.1) of the theorem. Therefore,

$$\left|\mathrm{U}'_{\sigma,r}(z)-\frac{\mathrm{U}_{\sigma,r}(z)}{z}\right|\leq\frac{\Gamma(\sigma+1)}{|\sigma|}\sum_{j=1}^{\infty}\frac{1}{4^j}\cdot\frac{|r|}{\Gamma(\sigma+1)\Gamma(2)}=\frac{|r|}{3|\sigma|},\quad z\in\mathbb{U}. \tag{2.8}$$

Moreover, from the theorem of the maximum of the modulus for an analytic function, we get

$$\begin{aligned}\left|\frac{\mathrm{U}_{\sigma,r}(z)}{z}\right|&=\left|1+\sum_{j=1}^{\infty}\frac{(-r)^j}{4^j j!(\sigma)_j}z^j\right|>1-\left|\sum_{j=1}^{\infty}\frac{(-r)^j}{4^j j!(\sigma)_j}e^{ij\theta}\right|\\&\geq 1-\frac{\Gamma(\sigma+1)}{|\sigma|}\sum_{j=1}^{\infty}\frac{|r|^j}{4^j\Gamma(j+1)\Gamma(j+\sigma)},\quad z\in\mathbb{U},\end{aligned}$$

where $\theta\in\mathbb{R}$.

The function $X_{\sigma,r}$ is strictly decreasing on $[1,+\infty)$, hence the function $\frac{|r|^j}{\Gamma(j+1)\Gamma(j+\sigma)}$ is strictly decreasing for $j\geq 1$, that leads to

$$\left|\frac{\mathrm{U}_{\sigma,r}(z)}{z}\right|>1-\frac{\Gamma(\sigma+1)}{|\sigma|}\sum_{j=1}^{\infty}\frac{1}{4^j}\cdot\frac{|r|}{\Gamma(\sigma+1)\Gamma(2)}=\frac{3|\sigma|-|r|}{3|\sigma|},\quad z\in\mathbb{U}. \tag{2.9}$$

Since

$$\left|\frac{z\mathrm{U}'_{\sigma,r}(z)}{\mathrm{U}_{\sigma,r}(z)}-1\right|=\left|\mathrm{U}'_{\sigma,r}(z)-\frac{\mathrm{U}_{\sigma,r}(z)}{z}\right|\cdot\left|\frac{z}{\mathrm{U}_{\sigma,r}(z)}\right|,\quad z\in\mathbb{U},$$

from (2.8) and (2.9), according to assumption (2.2), we deduce that

$$\left|\frac{z\mathrm{U}'_{\sigma,r}(z)}{\mathrm{U}_{\sigma,r}(z)}-1\right|<\frac{|r|}{3|\sigma|-|r|}\leq 1-\alpha,\quad z\in\mathbb{U}.$$

Finally, from (2.3), it follows that $\mathrm{U}_{\sigma,r}\in\mathcal{S}^*(\alpha)$. □

Analogously, we will prove in the following result that deals with sufficient conditions on the parameters σ and r such that $\mathrm{U}_{\sigma,r} \in \mathcal{K}(\alpha)$.

Theorem 2.2 *Let* $\sigma \in (-1,0) \cup (0,+\infty)$ *and* $r \in \mathbb{C}$, *such that*

$$\ln(\sigma+1) + \ln 2 - \frac{1}{\sigma+1} - 2 - \ln|r| \geq 0, \tag{2.10}$$

and

$$0 \leq \alpha \leq \frac{3|\sigma| - 4|r|}{3|\sigma| - 2|r|}. \tag{2.11}$$

Then, $\mathrm{U}_{\sigma,r} \in \mathcal{K}(\alpha)$.

Proof We could check immediately that

$$\left|\frac{z\mathrm{U}''_{\sigma,r}(z)}{\mathrm{U}'_{\sigma,r}(z)}\right| < 1-\alpha, \quad z \in \mathbb{U}, \tag{2.12}$$

implies $\mathrm{U}_{\sigma,r} \in \mathcal{K}(\alpha)$, where $\alpha < 1$.

Using the triangle's inequality and the maximum modulus theorem of an analytic function, we get

$$\begin{aligned} \left|z\mathrm{U}''_{\sigma,r}(z)\right| &= \left|\sum_{j=1}^{\infty} \frac{j(j+1)(-r)^j}{4^j j!(\sigma)_j} z^j\right| < \left|\sum_{j=1}^{\infty} \frac{j(j+1)(-r)^j}{4^j j!(\sigma)_j} e^{ij\theta}\right| \\ &\leq \frac{\Gamma(\sigma+1)}{|\sigma|} \sum_{j=1}^{\infty} \frac{j(j+1)|r|^j}{4^j \Gamma(j+1)\Gamma(j+\sigma)}, \quad z \in \mathbb{U}, \end{aligned}$$

where $\theta \in \mathbb{R}$.

Letting the function $Y_{\sigma,r} : [1,+\infty) \to (0,+\infty)$ defined by

$$Y_{\sigma,r}(t) = \frac{t(t+1)|r|^t}{\Gamma(t+1)\Gamma(t+\sigma)}, \quad t \geq 1, \tag{2.13}$$

and using relation (2.5), from (2.13), we get

$$\begin{aligned} \ln Y_{\sigma,r}(t) = {} & \ln t + \ln(t+1) + t\ln|r| + \gamma(\sigma+t) + \ln(\sigma+t) + \gamma(t+1) + \ln(t+1) \\ & - \sum_{m=1}^{\infty}\left[\frac{\sigma+t}{m} - \ln\left(1+\frac{\sigma+t}{m}\right)\right] - \sum_{m=1}^{\infty}\left[\frac{t+1}{m} - \ln\left(1+\frac{t+1}{m}\right)\right]. \end{aligned} \tag{2.14}$$

Differentiating (2.14), we have

$$\begin{aligned} \frac{Y'_{\sigma,r}(t)}{Y_{\sigma,r}(t)} = {} & \frac{1}{t} + \frac{1}{t+1} + \ln|r| + 2\gamma + \frac{1}{\sigma+t} + \frac{1}{t+1} \\ & - \sum_{m=1}^{\infty}\left[\frac{1}{m} - \frac{1}{\sigma+t+m}\right] - \sum_{m=1}^{\infty}\left[\frac{1}{m} - \frac{1}{t+m+1}\right] =: \widetilde{Y}_{\sigma,r}(t), \end{aligned}$$

as well as the function $\widetilde{Y}_{\sigma,r}$, we obtain

$$\widetilde{Y}'_{\sigma,r}(t) = -\frac{1}{t^2} - \frac{1}{(t+1)^2} - \frac{1}{(\sigma+t)^2} - \frac{1}{(t+1)^2} \\ -\sum_{m=1}^{\infty}\frac{1}{(\sigma+t+m)^2} - \sum_{m=1}^{\infty}\frac{1}{(t+m+1)^2} < 0, \quad t \in [1,+\infty),$$

for all $\sigma \in (-1,0)\cup(0,+\infty)$ so that $\widetilde{Y}_{\sigma,r}$ is a strictly decreasing function on $[1,+\infty)$. Since $\widetilde{Y}_{\sigma,r}$ can be expressed as

$$\widetilde{Y}_{\sigma,r}(t) = \frac{1}{t} + \frac{1}{t+1} - \psi(t+\sigma) - \psi(t+1) + \ln|r|,$$

and $\widetilde{Y}_{\sigma,r}$ is a strictly decreasing function on $[1,+\infty)$, it follows

$$N\bigl(|r|,\sigma\bigr) := \sup\bigl\{\widetilde{Y}_{\sigma,r}(t) : t \geq 1\bigr\} = \widetilde{Y}_{\sigma,r}(1) = \frac{3}{2} + \ln|r| - \psi(\sigma+1) - \psi(2).$$

Using the inequality (1.5), we obtain

$$N\bigl(|r|,\sigma\bigr) = \frac{3}{2} + \ln|r| - \psi(\sigma+1) - \psi(2) < 2 + \ln|r| - \ln(1+\sigma) - \ln 2 + \frac{1}{\sigma+1} \leq 0,$$

and the last inequality is in fact assumption (2.10). Hence,

$$\bigl|z\mathrm{U}''_{\sigma,r}(z)\bigr| \leq \frac{\Gamma(\sigma+1)}{|\sigma|}\sum_{j=1}^{\infty}\frac{1}{4^j}\cdot\frac{2|r|}{\Gamma(\sigma+1)\Gamma(2)} = \frac{2|r|}{3|\sigma|}, \quad z \in \mathbb{U}. \tag{2.15}$$

Furthermore, from the theorem of the maximum of the modulus for an analytic function, we have

$$\bigl|\mathrm{U}'_{\sigma,r}(z)\bigr| = \left|1 + \sum_{j=1}^{\infty}\frac{(j+1)(-r)^j}{4^j(1)_j(\sigma)_j}z^j\right| > 1 - \left|\sum_{j=1}^{\infty}\frac{(j+1)(-r)^j}{4^j(1)_j(\sigma)_j}e^{ij\theta}\right| \\ \geq 1 - \frac{\Gamma(\sigma+1)}{|\sigma|}\sum_{j=1}^{\infty}\frac{(j+1)|r|^j}{4^j\Gamma(j+1)\Gamma(j+\sigma)}, \quad z \in \mathbb{U},$$

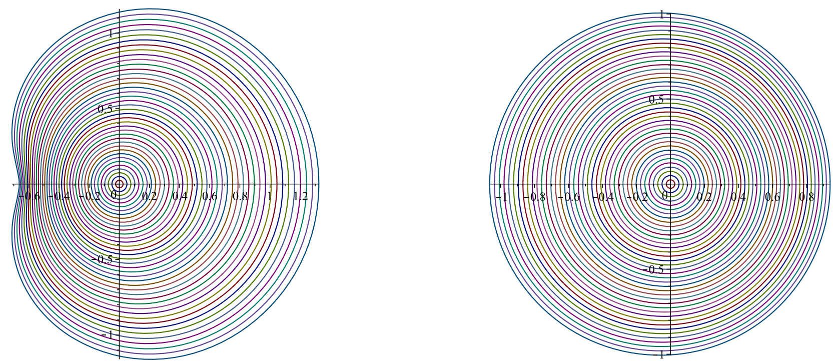
where $\theta \in \mathbb{R}$.

The function $Y_{\sigma,r}$ is strictly decreasing on $[1,+\infty)$, hence the function $\frac{(j+1)|r|^j}{\Gamma(j+1)\Gamma(j+\sigma)}$ is strictly decreasing for $n \geq 1$, thus

$$\bigl|\mathrm{U}'_{\sigma,r}(z)\bigr| > 1 - \frac{\Gamma(\sigma+1)}{|\sigma|}\sum_{j=1}^{\infty}\frac{1}{4^j}\cdot\frac{2|r|}{\Gamma(\sigma+1)\Gamma(2)} = \frac{3|\sigma|-2|r|}{3|\sigma|}, \quad z \in \mathbb{U}. \tag{2.16}$$

Since

$$\left|\frac{z\mathrm{U}''_{\sigma,r}(z)}{\mathrm{U}'_{\sigma,r}(z)}\right| = \bigl|z\mathrm{U}''_{\sigma,r}(z)\bigr|\cdot\left|\frac{1}{\mathrm{U}'_{\sigma,r}(z)}\right|, \quad z \in \mathbb{U},$$



(A) The image of $\mathbb{U}$ by $\mathrm{U}_{\sigma,r}$ for $\sigma = -0.1$ and $r = 0.132$

(B) The image of $\mathbb{U}$ by $\mathrm{U}_{\sigma,r}$ for $\sigma = 8$ and $r = 2$

Figure 1 Figures for Remark 2.1

from (2.15) and (2.16), and using the assumption (2.11), we deduce that

$$\left|\frac{z\mathrm{U}''_{\sigma,r}(z)}{\mathrm{U}'_{\sigma,r}(z)}\right| < \frac{2|r|}{3|\sigma| - 2|r|} \le 1 - \alpha, \quad z \in \mathbb{U},$$

and according to (2.12), it follows that $\mathrm{U}_{\sigma,r} \in \mathcal{K}(\alpha)$. □

Remark 2.1 1. Taking the values $r = 0.132$, $\sigma = -0.1$ and $\alpha = 0.2$, it is easy to check that assumptions (2.1) and (2.2) are satisfied. Then, according to Theorem 2.1, we get $\mathrm{U}_{\sigma,r} \in \mathcal{S}^*(\alpha)$, and from Fig. 1(A), we can see that $\mathrm{U}_{\sigma,r} \notin \mathcal{K}(0)$, hence it is not a convex function.

2. For the values $r = 2$, $\sigma = 8$ and $\alpha = 0.7$, we could easily see that assumptions (2.10) and (2.11) are satisfied. According to Theorem 2.2, we get $\mathrm{U}_{\sigma,r} \in \mathcal{K}(\alpha)$, and from Fig. 1(B), we can see the image of the unit disc by this function.

Theorem 2.3 *Let $\sigma \ge r$ with $r \in (0, +\infty)$. Then, the function $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ is starlike in $\mathbb{U}$.*

Proof From (1.4) and the power series expansion

$$\frac{z}{1+z} = z + \sum_{j=1}^{\infty}(-1)^j z^{j+1}, \quad z \in \mathbb{U},$$

we have

$$\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z) = \sum_{j=1}^{\infty} A_j z^j, \quad z \in \mathbb{U}, \tag{2.17}$$

where

$$A_j = \frac{r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}}, \quad j \in \mathbb{N}. \tag{2.18}$$

To prove our result, according to Lemma 1.2, it is enough to show that the inequalities $jA_j \ge (j+1)A_{j+1}$ and $jA_j + (j+2)A_{j+2} \ge 2(j+1)A_{j+1}$ hold for all $j \in \mathbb{N}$.

Since

$$jA_j-(j+1)A_{j+1}=\Gamma(\sigma)\left[\frac{jr^{j-1}}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)}-\frac{(j+1)r^j}{4^j\Gamma(j+1)\Gamma(\sigma+j)}\right],$$

and $j>(j+1)/4$, $j\in\mathbb{N}$, it follows

$$\begin{aligned}jA_j-(j+1)A_{j+1}&=\Gamma(\sigma)\left[\frac{jr^{j-1}}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)}-\frac{(j+1)r^j}{4^j j\Gamma(j)(\sigma+j-1)\Gamma(\sigma+j-1)}\right]\\&=\frac{\Gamma(\sigma)r^{j-1}}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)}\left[j-\frac{(j+1)r}{4j(\sigma+j-1)}\right]\\&>\frac{(j+1)\Gamma(\sigma)r^{j-1}}{4^j\Gamma(j)\Gamma(\sigma+j-1)}\left[1-\frac{r}{j(\sigma+j-1)}\right]\\&=\frac{(j+1)\Gamma(\sigma)r^{j-1}}{4^j\Gamma(j)\Gamma(\sigma+j-1)}\varphi(j),\end{aligned}$$

where

$$\varphi(j):=1-\frac{r}{j(\sigma+j-1)},\quad j\in\mathbb{N}. \tag{2.19}$$

Using the fact that φ is an increasing function on $\mathbb{N}$, it follows that

$$\min\left\{\varphi(j):j\in\mathbb{N}\right\}=\varphi(1)=\frac{\sigma-r}{\sigma}\geq 0,$$

under the assumptions $\sigma\geq r>0$. Using the inequality

$$\frac{(j+1)\Gamma(\sigma)r^{j-1}}{4^j\Gamma(j)\Gamma(\sigma+j-1)}>0,$$

we deduce that $jA_j-(j+1)A_{j+1}\geq 0$ for all $j\in\mathbb{N}$, that is the sequence $\{jA_j\}_{j\in\mathbb{N}}$ is decreasing.

Since $A_{j+2}>0$ for all $j\in\mathbb{N}$, we get

$$jA_j-2(j+1)A_{j+1}+(j+2)A_{j+2}>jA_j-2(j+1)A_{j+1},\quad j\in\mathbb{N},$$

and because $j\geq(j+1)/2$ for each $j\in\mathbb{N}$, we deduce that

$$\begin{aligned}jA_j-2(j+1)A_{j+1}&=\frac{\Gamma(\sigma)r^{j-1}}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)}\left[j-\frac{2(j+1)r}{4j(\sigma+j-1)}\right]\\&\geq\frac{(j+1)\Gamma(\sigma)r^{j-1}}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)}\frac{1}{2}\left[1-\frac{r}{j(\sigma+j-1)}\right]\\&=\frac{1}{2}\frac{(j+1)\Gamma(\sigma)r^{j-1}}{4^j\Gamma(j)\Gamma(\sigma+j-1)}\varphi(j),\end{aligned}$$

where φ is defined by (2.19). We have already proved that under our assumption $\varphi(j)\geq 0$, $j\in\mathbb{N}$, and using the above-mentioned reasons, it follows that $jA_j-2(j+1)A_{j+1}+(j+2)A_{j+2}>0$ for all $j\in\mathbb{N}$, hence the proof of the theorem is complete. □

Theorem 2.4 *Let $\sigma \geq \frac{r}{2}$ with $r \in (0, +\infty)$ and*

$$32\sigma^2 + 32(1-r)\sigma - 32r + 3r^2 \geq 0, \tag{2.20}$$

Then, $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ is starlike in $\mathbb{U}$.

Proof Using power series expansion (2.17), relation (2.18), and Lemma 1.2, it is enough to prove that $jA_j \geq (j+1)A_{j+1}$ and $jA_j + (j+2)A_{j+2} \geq 2(j+1)A_{j+1}$, for all $j \in \mathbb{N}$.

To show that the inequality $jA_j \geq (j+1)A_{j+1}$ holds for all $j \in \mathbb{N}$, it is easy to observe that

$$\begin{aligned} jA_j - (j+1)A_{j+1} &= \frac{jr^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}} - \frac{(j+1)r^j}{4^j(1)_j(\sigma)_j} \\ &= \frac{r^{j-1}\Gamma(\sigma)}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)} \cdot \frac{4j^2(\sigma+j-1)-(j+1)r}{4j(\sigma+j-1)}. \end{aligned}$$

Let the function $\Phi : [1, +\infty) \to \mathbb{R}$ defined by

$$\Phi(j) := 4j^2(\sigma+j-1) - (j+1)r,$$

and we shall proceed to show that $\Phi(j) \geq 0$ for all $j \in [1, +\infty)$ using the mathematical induction.

First, for $j = 1$, we have that $\Phi(1) = 4\sigma - 2r \geq 0$ if and only if $\sigma \geq \frac{r}{2}$, as we assumed in the statement of the theorem.

Second, let us assume that $\Phi(m) \geq 0$ for a fixed $m \in [1, +\infty)$. Since

$$\Phi(m+1) = \Phi(m) + \varphi(m), \quad \text{where } \varphi(m) := 4m^2 + 4(2m+1)(\sigma+m) - r,$$

using the fact that the function φ is increasing on $\mathbb{N}$, we have $\varphi(m) \geq \varphi(1) = 4(3\sigma+4) - r \geq 0$ if and only if

$$\sigma \geq \frac{r-16}{12}. \tag{2.21}$$

It follows that under assumption (2.21), we have $\Phi(m+1) \geq 0$; therefore, from the mathematical induction, it follows that $\Phi(j) \geq 0$ for all $j \in \mathbb{N}$.

Concluding, if $\sigma \geq \max\{\frac{r}{2}; \frac{r-16}{12}\} = \frac{r}{2}$ whenever $r \in (0, +\infty)$, then we have $jA_j \geq (j+1)A_{j+1}$ for all $j \in \mathbb{N}$.

On the other hand,

$$\begin{aligned} & jA_j - 2(j+1)A_{j+1} + (j+2)A_{j+2} \\ &\quad = \frac{r^{j-1}\Gamma(\sigma)}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)} \\ &\quad\quad \times \frac{16j^2(j+1)(\sigma+j)(\sigma+j-1) - 8(j+1)^2(\sigma+j)r + (j+2)r^2}{16j(j+1)(\sigma+j)(\sigma+j-1)}, \end{aligned}$$

that is

$$jA_j - 2(j+1)A_{j+1} + (j+2)A_{j+2} = \frac{r^{j-1}\Gamma(\sigma)}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)}$$

$$\times \frac{\widetilde{\Phi}(j)}{16j(j+1)(\sigma+j)(\sigma+j-1)}, \tag{2.22}$$

and we will use the mathematical induction again to prove the nonnegativity of the function $\widetilde{\Phi}:[1,+\infty)\to\mathbb{R}$ defined by

$$\widetilde{\Phi}(j) := 16j^2(j+1)(\sigma+j)(\sigma+j-1) - 8(j+1)^2(\sigma+j)r + (j+2)r^2.$$

First, for $j=1$, we have

$$\widetilde{\Phi}(1) = 32(\sigma+1)(\sigma-r) + 3r^2 = 32\sigma^2 + 32(1-r)\sigma - 32r + 3r^2 \geq 0,$$

according to assumption (2.20).

Now, suppose that $\widetilde{\Phi}(m) \geq 0$ for a fixed $m\in\mathbb{N}$. It is easy to check that

$$\widetilde{\Phi}(m+1) = \widetilde{\Phi}(m) + \widetilde{\varphi}(m), \tag{2.23}$$

where

$$\begin{aligned}\widetilde{\varphi}(m) := {}& 80m^4 + (128\sigma+160)m^3 + \big(48\sigma^2 - 24r + 240\sigma + 112\big)m^2\\ &+ \big(80\sigma^2 + (144-16r)\sigma - 56r + 32\big)m\\ &+ 32\sigma^2 + (32-24r)\sigma + r^2 - 32r, \quad m\in\mathbb{N}.\end{aligned}$$

If we define the function $G:[1,+\infty)\to\mathbb{R}$ by

$$\begin{aligned}G(x) := {}& 80x^4 + (128\sigma+160)x^3 + \big(48\sigma^2 - 24r + 240\sigma + 112\big)x^2\\ &+ \big(80\sigma^2 + (144-16r)\sigma - 56r + 32\big)x\\ &+ 32\sigma^2 + (32-24r)\sigma + r^2 - 32r, \quad x\in[1,+\infty),\end{aligned}$$

then $\widetilde{\varphi} = G|_{\mathbb{N}}$. Since $\sigma \geq \frac{r}{2} > 0$, we have

$$\begin{aligned}&48\sigma^2 - 24r + 240\sigma + 112 \geq 12r^2 + 96r + 112 > 0,\\ &80\sigma^2 + (144-16r)\sigma - 56r + 32 = 16\sigma(5\sigma+9) - 8r(2\sigma+7) + 3r^2\\ &\quad > 16\sigma(2\sigma+7) - 8r(2\sigma+7) + 3r^2 = 8(2\sigma+7)(2\sigma-r) + 3r^2 > 0,\end{aligned}$$

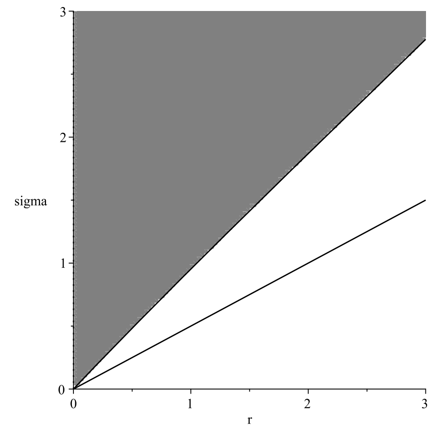
and we see that the coefficients of

$$\begin{aligned}G'(x) = {}& 320x^3 + 3(128\sigma+160)x^2 + 2\big(48\sigma^2 - 24r + 240\sigma + 112\big)x\\ &+ \big(80\sigma^2 + (144-16r)\sigma - 56r + 32\big)\end{aligned}$$

are positive numbers, hence $G'(x) > 0$ for all $x\in[1,+\infty)$, and consequently G is a strictly increasing function on $[1,+\infty)$. From here, using again the assumptions $\sigma \geq \frac{r}{2}$, $\sigma > 0$, it follows that

$$G(x) \geq G(1) = r^2 - (40\sigma+112)r + 160\sigma^2 + 544\sigma + 384$$

Figure 2 The assumptions for Theorem 2.4



$$\geq r^2 + 80\sigma^2 + 320\sigma + 384 > 0,$$

that is $G(x) > 0$, $x \in [1, +\infty)$. Therefore, we get $G(x) > 0$ for all $x \in [1, +\infty)$, and thus $\varphi(m) > 0$, $n \in \mathbb{N}$. From here, relation (2.23) implies $\widetilde{\Phi}(m + 1) > 0$, and using the mathematical induction, we conclude that $\widetilde{\Phi}(j) > 0$ for all $j \in \mathbb{N}$.

Finally, the above last result and relation (2.22) lead us to $jA_j + (j + 2)A_{j+2} \geq 2(j + 1)A_{j+1}$, for all $j \in \mathbb{N}$, and the proof is complete. □

Remark 2.2 1. As we can see in Fig. 2, the assumptions $\sigma \geq \frac{r}{2}$, with $r \in (0, +\infty)$, and (2.20) are not contradictory: the points of the region colored with "grey" colour satisfy all these conditions, or, for example, $r = 1$ and $\sigma = 100$ satisfies both of these assumptions.

2. As it is shown in the above figure, we presume that for $r, \sigma > 0$, we have

$$\begin{aligned}&\left\{(r,\sigma) \in \mathbb{R}^2 : 32\sigma^2 + 32(1 - r)\sigma - 32r + 3r^2 \geq 0, r, \sigma > 0\right\}\\ &\quad\subset \left\{(r,\sigma) \in \mathbb{R}^2 : \sigma \geq \frac{r}{2}, r, \sigma > 0\right\},\end{aligned}$$

hence, we shall try to prove the following implication:

If $r, \sigma > 0$, then

$$32\sigma^2 + 32(1 - r)\sigma - 32r + 3r^2 \geq 0 \quad \Rightarrow \quad \sigma \geq \frac{r}{2},$$

or equivalently

$$\sigma \in \left(0, \frac{r}{2}\right), \quad r > 0 \quad \Rightarrow \quad U(\sigma) := 32\sigma^2 + 32(1 - r)\sigma - 32r + 3r^2 < 0. \tag{2.24}$$

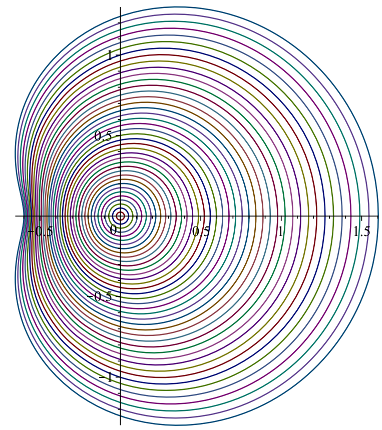
Since

$$\lim_{\sigma \to 0} U(\sigma) = r(3r - 32) =: V(r),$$

and

$$V(r) \not< 0, \quad \text{for all } r \in (0, +\infty),$$

Figure 3 The image of $\mathbb{U}$ by $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ for $\sigma = 5.1$, $r = 10$



implication (2.24) is not true, hence

$$\left\{(r,\sigma) \in \mathbb{R}^2 : 32\sigma^2 + 32(1-r)\sigma - 32r + 3r^2 \geq 0, r, \sigma > 0\right\} \not\subset \left\{(r,\sigma) \in \mathbb{R}^2 : \sigma \geq \frac{r}{2}, r, \sigma > 0\right\}.$$

3. The values $r = 10$, $\sigma = 5.1$ and $\alpha = 0.2$ satisfy the assumptions $\sigma \geq \frac{r}{2} > 0$ and (2.20). Then, according to Theorem 2.4, we get that $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ is starlike in $\mathbb{U}$. From Fig. 3, we can see that $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z) \notin \mathcal{K}(0)$, hence it is not a convex function.

Theorem 2.5 *Let $\sigma \geq 2r$ with $r \in (0, +\infty)$. Then, the function $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ is convex in $\mathbb{U}$.*

Proof To prove this result, we shall use the classical Alexander theorem between the classes of starlike and convex functions, which asserts that $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. Thus, it is sufficient to prove that the function $\frac{z}{1+z} * (z\mathrm{U}'_{\sigma,r}(z))$ is starlike in $\mathbb{U}$.

Assuming that $\mathrm{U}_{\sigma,r}$ has the form (1.4), a simple computation shows that

$$\frac{z}{1+z} * \left(z\mathrm{U}'_{\sigma,r}(z)\right) = \sum_{j=1}^{\infty} B_j z^j, \quad z \in \mathbb{U}, \tag{2.25}$$

where

$$B_j = \frac{jr^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}}, \quad j \in \mathbb{N}. \tag{2.26}$$

According to Lemma 1.3, it is sufficient to prove that $jB_j \geq (j+1)B_{j+1}$ and $jB_j + (j+2)B_{j+2} \geq 2(j+1)B_{j+1}$ for all $j \in \mathbb{N}$.

A simple computation shows that

$$jB_j - (j+1)B_{j+1} = \frac{r^{j-1}}{4^{j-1}(j-1)!(\sigma)_{j-1}} \cdot \frac{4j^3 + (4\sigma - r - 4)j^2 - 2rj - r}{4j(\sigma + j - 1)}, \tag{2.27}$$

and to show that $jB_j - (j+1)B_{j+1} \geq 0$, $j \in \mathbb{N}$, it is sufficient to prove that the function $\phi : [1,+\infty) \to \mathbb{R}$ defined by

$$\phi(x) := 4x^3 + (4\sigma - r - 4)x^2 - 2rx - r,$$

is nonnegative on $[1,+\infty)$. Since

$$\phi'(x) = 2\big[6x^2 + (4\sigma - r - 4)x - r\big],$$

the function ϕ' attained its minimum at the point

$$x_m = \frac{r + 4 - 4\sigma}{12} < 1, \quad \text{whenever } \sigma \geq 2r > 0,$$

hence it is a strictly increasing function on $[1,+\infty)$. Therefore, since $\sigma \geq 2r > 0$, we have

$$\phi'(x) \geq \phi'(1) = 4(2\sigma - r + 1) \geq 4(3r + 1) > 0, \quad x \geq 1,$$

and consequently, the function ϕ is strictly increasing on $[1,+\infty)$. Concluding, for $\sigma \geq 2r > 0$, we obtain that

$$\phi(x) \geq \phi(1) = 4(\sigma - r) \geq 2r > 0, \quad x \geq 1,$$

hence (2.27) leads to

$$jB_j - (j+1)B_{j+1} = \frac{r^{j-1}}{4^{j-1}(j-1)!(\sigma)_{j-1}} \cdot \frac{\phi(j)}{4j(\sigma + j - 1)} \geq 0, \quad j \in \mathbb{N},$$

that is $jB_j \geq (j+1)B_{j+1}$ for all $j \in \mathbb{N}$.

As $B_{j+2} > 0$, $j \in \mathbb{N}$, we have

$$jB_j - 2(j+1)B_{j+1} + (j+2)B_{j+2} > jB_j - 2(j+1)B_{j+1}, \tag{2.28}$$

and to show that $jB_j - 2(j+1)B_{j+1} + (j+2)B_{j+2} \geq 0$, $j \in \mathbb{N}$, it is sufficient to prove that

$$\frac{jB_j}{2(j+1)B_{j+1}} = \frac{2j^3(\sigma + j - 1)}{(j+1)^2 r} =: \psi(j) \geq 1, \quad j \in \mathbb{N}. \tag{2.29}$$

Since

$$\psi(1) = \frac{\sigma}{2r} \geq 1 \quad \text{and} \quad \psi(2) = \frac{16(\sigma + 1)}{9r} > 1, \quad \text{whenever } \sigma \geq 2r > 0,$$

as well as using the inequality

$$\frac{j^2}{(j+1)^2} > \frac{1}{2}, \quad j \geq 3,$$

and $\sigma \geq 2r > 0$, it follows that

$$\psi(j) > \frac{j(\sigma + j - 1)}{r} =: F(j), \quad j \geq 3.$$

However, since $\sigma, r > 0$, the function $F(j)$ is a strictly increasing function as a product of two strictly increasing and positive functions

$$G_1(j) := j \quad \text{and} \quad G_2(j) := \sigma + j - 1, \quad j \geq 3.$$

Hence, using again the assumption $\sigma \geq 2r > 0$, we deduce

$$F(j) = \frac{j(\sigma + j - 1)}{r} \geq F(3) = \frac{3(\sigma + 2)}{r} \geq \frac{3(2r + 2)}{c} = \frac{6(r + 1)}{r} > 1.$$

Therefore, according to (2.29), we conclude that $jB_j - 2(j + 1)B_{j+1} \geq 0$, $j \in \mathbb{N}$, and from (2.28), it follows that $jB_j - 2(j + 1)B_{j+1} + (j + 2)B_{j+2} > 0$ for all $j \in \mathbb{N}$. □

Theorem 2.6 *Let $\sigma \geq r$ with $r \in (0, +\infty)$, and suppose that*

$$32\sigma^2 + 32(1 - 2r)\sigma + 9r^2 - 64r \geq 0, \tag{2.30}$$

Then, $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ is convex function in $\mathbb{U}$.

Proof Using the power series expansion (2.25) where the coefficients are given by (2.26), according to Lemma 1.2, it is enough to prove that $jB_j \geq (j + 1)B_{j+1}$ and $jB_j - 2(j + 1)B_{j+1} + (j + 2)B_{j+2} \geq 0$ for all $j \in \mathbb{N}$.

To show that the inequality $jB_j \geq (j + 1)B_{j+1}$ holds for all $j \in \mathbb{N}$, it is easy to observe that

$$\begin{aligned} jB_j - (j + 1)B_{j+1} &= \frac{j^2 r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}} - \frac{(j + 1)^2 r^j}{4^j(1)_j(\sigma)_j} \\ &= \frac{r^{j-1}\Gamma(\sigma)}{4^{j-1}\Gamma(j)\Gamma(\sigma + j - 1)} \cdot \frac{4j^3(\sigma + j - 1) - (j + 1)^2 r}{4j(\sigma + j - 1)}. \end{aligned}$$

If we define the function $\Psi : [1, +\infty) \to \mathbb{R}$ by

$$\Psi(j) := 4j^3(\sigma + j - 1) - (j + 1)^2 r,$$

we shall proceed to show that $\Psi(j) \geq 0$ for all $j \in \mathbb{N}$ using the mathematical induction.

First, for $j = 1$, we have that $\Psi(1) = 4(\sigma - r) \geq 0$ if and only if $\sigma \geq r$, as we assumed in the statement of the theorem.

Second, let us assume that $\Psi(m) \geq 0$ for a fixed $m \in \mathbb{N}$. A simple computation shows that

$$\Psi(m + 1) = \Psi(m) + \phi(m),$$

where $\phi(m) := 16m^3 + 12(\sigma + 1)m^2 + 3(6\sigma - r + 2)m + 4\sigma - 3r$. For $\sigma \geq r > 0$, we have

$$\begin{aligned} \phi(m) &= 16m^3 + 12(\sigma + 1)m^2 + 3(6\sigma - r + 2)m + 4\sigma - 3r \\ &\geq 16m^3 + 12(r + 1)m^2 + 3(5r + 2)m + r > 0, \quad m \in \mathbb{N}, \end{aligned}$$

therefore $\Psi(m + 1) > \Psi(m)$, and using the mathematical induction, it follows that $jB_j \geq (j + 1)B_{j+1}$ for all $n \in \mathbb{N}$.

On the other hand,

$$\begin{aligned} & jB_j - 2(j+1)B_{j+1} + (j+2)B_{j+2} \\ & \quad = \frac{r^{j-1}\Gamma(\sigma)}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)} \\ & \qquad \times \frac{16j^3(j+1)(\sigma+j)(\sigma+j-1) - 8(j+1)^3(\sigma+j)r + (j+2)r^2}{16j(j+1)(\sigma+j)(\sigma+j-1)}, \end{aligned}$$

that is

$$\begin{aligned} jB_j - 2(j+1)B_{j+1} + (j+2)B_{j+2} = & \frac{r^{j-1}\Gamma(\sigma)}{4^{j-1}\Gamma(j)\Gamma(\sigma+j-1)} \\ & \times \frac{\widetilde{\Psi}(j)}{16j(j+1)(\sigma+j)(\sigma+j-1)}, \end{aligned} \tag{2.31}$$

and we will use the mathematical induction again to prove the nonnegativity of the function $\widetilde{\Psi} : [1, +\infty) \to \mathbb{R}$ defined by

$$\widetilde{\Psi}(j) := 16n^3(j+1)(\sigma+j)(\sigma+j-1) - 8(j+1)^3(\sigma+j)r + (j+2)^2r^2.$$

First, for $j = 1$, according to assumption (2.30), we have

$$\widetilde{\Psi}(1) = 32\sigma^2 + 32(1-2r)\sigma + 9r^2 - 64r \geq 0.$$

Now, suppose that $\widetilde{\Psi}(m) \geq 0$ for a fixed $m \in \mathbb{N}$. It is easy to check that

$$\widetilde{\Psi}(m+1) = \widetilde{\Psi}(m) + \widetilde{\phi}(m), \tag{2.32}$$

where

$$\begin{aligned} \widetilde{\phi}(m) :=& 96m^5 + (160\sigma + 240)m^4 + \left(64\sigma^2 - 32r + 384\sigma + 256\right)m^3 \\ & + \left(144\sigma^2 + (368 - 24r)\sigma - 120r + 144\right)m^2 \\ & + \left(112\sigma^2 + (176 - 72r)\sigma + 2r^2 - 152r + 32\right)m \\ & + 32\sigma^2 + (32 - 56r)\sigma + 5r^2 - 64r, \quad m \in \mathbb{N}. \end{aligned}$$

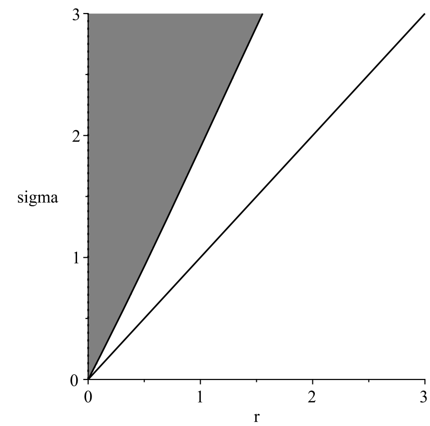
If we define the function $H : [1, +\infty) \to \mathbb{R}$ by

$$\begin{aligned} H(x) :=& 96x^5 + (160\sigma + 240)x^4 + \left(64\sigma^2 - 32r + 384\sigma + 256\right)x^3 \\ & + \left(144\sigma^2 + (368 - 24r)\sigma - 120r + 144\right)x^2 \\ & + \left(112\sigma^2 + (176 - 72r)\sigma + 2c^2 - 152r + 32\right)x \\ & + 32\sigma^2 + (32 - 56r)\sigma + 5r^2 - 64r, \quad x \in [1, +\infty), \end{aligned}$$

then $\widetilde{\phi} = H|_{\mathbb{N}}$. Since $\sigma \geq r > 0$, we have

$$64\sigma^2 - 32r + 384\sigma + 256 \geq 64r^2 + 252r + 256 > 0,$$

Figure 4 The assumptions for Theorem 2.6



$$\begin{aligned}
&144\sigma^2 + (368 - 24r)\sigma - 120r + 144 \geq 144\sigma r + (368 - 24r)\sigma - 120r + 144 \\
&\quad = 120\sigma r - 120r + 368\sigma + 144 \geq 120\sigma r + 248\sigma + 144 > 0, \\
&112\sigma^2 + (176 - 72r)\sigma + 2r^2 - 152r + 32 \geq 112\sigma r + (176 - 72r)\sigma + 2r^2 - 152r + 32 \\
&\quad = 40\sigma r + 176\sigma + 2r^2 - 152r + 32 \geq 42r^2 + 24r + 32 > 0,
\end{aligned}$$

and we see that the coefficients of

$$\begin{aligned}
H'(x) := 480x^4 &+ 4(160\sigma + 240)x^3 + 3\bigl(64\sigma^2 - 32r + 384\sigma + 256\bigr)x^2 \\
&+ 2\bigl(144\sigma^2 + (368 - 24r)\sigma - 120r + 144\bigr)x \\
&+ 112\sigma^2 + (176 - 72r)\sigma + 2r^2 - 152r + 32
\end{aligned}$$

are positive numbers, hence $H'(x) > 0$ for all $x \in [1, +\infty)$, and therefore H is a strictly increasing function on $[1, +\infty)$. From here, using again the assumptions $\sigma \geq r$, $\sigma > 0$, it follows that

$$\begin{aligned}
H(x) \geq H(1) &= 352\sigma^2 + (1120 - 152r)\sigma + 7r^2 - 368r + 768 \\
&\geq 200\sigma r + 1120\sigma + 7r^2 - 368r + 768 \geq 200\sigma r + 7r^2 + 752r + 768 > 0,
\end{aligned}$$

that is $H(x) > 0$, $x \in [1, +\infty)$. Therefore, we get $H(x) > 0$ for all $x \in [1, +\infty)$, and thus $\widetilde{\phi}(m) > 0$, $m \in \mathbb{N}$. Consequently, relation (2.32) implies $\widetilde{\Psi}(m + 1) > 0$, and using the mathematical induction, we conclude that $\widetilde{\Psi}(j) > 0$ for all $j \in \mathbb{N}$.

Finally, the above last result and relation (2.31) lead us to $jB_j + (j + 2)B_{j+2} \geq 2(j + 1)B_{j+1}$, for all $j \in \mathbb{N}$, and the proof is complete. □

Remark 2.3 1. As we can see in Fig. 4, the assumptions $\sigma \geq r$, with $r \in (0, +\infty)$, and (2.30) are not contradictory: the points of the region colored with "grey" colour satisfy all these conditions, or, for example, $r = 1$ and $\sigma = 100$ satisfies both of these assumptions.

2. As it is shown in the above-mentioned figure, we could presume that for $r, \sigma > 0$, we have

$$\bigl\{(r, \sigma) \in \mathbb{R}^2 : 32\sigma^2 + 32(1 - 2r)\sigma + 9r^2 - 64r \geq 0, r, \sigma > 0\bigr\}$$

$$\subset \{(r,\sigma) \in \mathbb{R}^2 : \sigma \geq r, r, \sigma > 0\},$$

hence we shall try to prove the following implication:

If $r, \sigma > 0$, then

$$32\sigma^2 + 32(1-2r)\sigma + 9r^2 - 64r \geq 0 \quad \Rightarrow \quad \sigma \geq r,$$

or equivalently

$$\sigma \in (0,r), \quad r > 0 \quad \Rightarrow \quad \widetilde{U}(\sigma) := 32\sigma^2 + 32(1-2r)\sigma + 9r^2 - 64r < 0. \tag{2.33}$$

Since

$$\lim_{\sigma \to 0} \widetilde{U}(\sigma) = r(9r - 64) =: \widetilde{V}(r),$$

and

$$\widetilde{V}(r) \not< 0, \quad \text{for all } r \in (0, +\infty),$$

implication (2.33) is not true, hence

$$\{(r,\sigma) \in \mathbb{R}^2 : 32\sigma^2 + 32(1-2r)\sigma + 9r^2 - 64r \geq 0, r, \sigma > 0\}$$
$$\not\subset \{(r,\sigma) \in \mathbb{R}^2 : \sigma \geq r, r, \sigma > 0\}.$$

Remark 2.4 Theorem 2.5 and Theorem 2.6 give us sufficient conditions for the convexity of the function $\frac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$. According to Theorem 2.5, it is necessary to assume that $\sigma \geq 2r$, with $r \in (0, +\infty)$, while Theorem 2.6 requirements are $\sigma \geq r$, with $r \in (0, +\infty)$, and inequality (2.30).

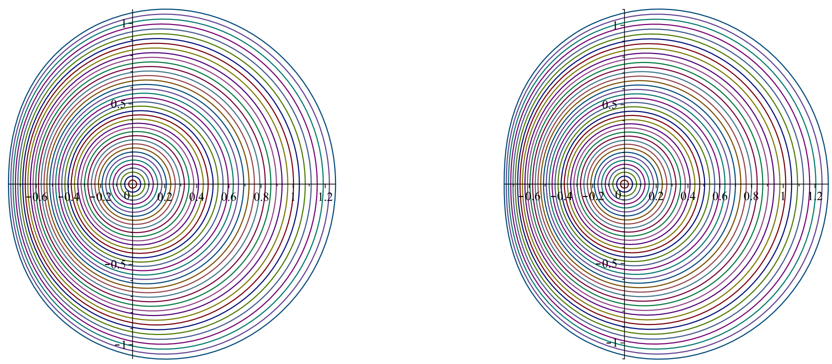
Since for $r \in (0, +\infty)$ the assumption $\sigma \geq r$ is weaker than $\sigma \geq 2r$, the next two figures obtained with MAPLE™ computer software show graphically the convexity of this function for $\sigma = 5$, $r = 2.45$ (Fig. 5(A), using Theorem 2.5), and $\sigma = 4$, $r = 2.1$, (Fig. 5(B), according to Theorem 2.6), respectively. We remark that for the second pair of the above values, we cannot use Theorem 2.5 to prove the convexity, but Theorem 2.6 could be applied successfully.

Theorem 2.7 *If* $\sigma \geq \frac{r}{4}$ *with* $r \in (0, +\infty)$, *then* $(z \cos \sqrt{z}) * \mathrm{U}_{\sigma,r}(z)$ *is a close-to-convex function in* $\mathbb{U}$ *with respect to* $-\log(1-z)$.

Proof To prove that $(z \cos \sqrt{z}) * \mathrm{U}_{\sigma,r}(z)$ is a close-to-convex function in $\mathbb{U}$ with respect to $-\log(1-z)$, we will use Lemma 1.3.

The function $(z \cos \sqrt{z}) * \mathrm{U}_{\sigma,r}(z)$ has the power series expansion of the form

$$(z \cos \sqrt{z}) * \mathrm{U}_{\sigma,r}(z) = \sum_{j=1}^{\infty} C_j z^j, \quad z \in \mathbb{U},$$



(A) The image of $\mathbb{U}$ by $\dfrac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ for $\sigma = 5$ and $r = 2.45$

(B) The image of $\mathbb{U}$ by $\dfrac{z}{1+z} * \mathrm{U}_{\sigma,r}(z)$ for $\sigma = 4$ and $r = 2.1$

Figure 5 Figures for Remark 2.4

where

$$C_j = \frac{r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}(2j-2)!}, \quad j \in \mathbb{N}.$$

We proceed to prove that $\{jC_j\}_{j\in\mathbb{N}}$ is a decreasing sequence with $jC_j > 0$, $j \in \mathbb{N}$. Nothing that

$$\begin{aligned} jC_j - (j+1)C_{j+1} &= \frac{r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}(2j-2)!}\left[j - \frac{(j+1)r}{4j(\sigma+j-1)(2j)(2j-1)}\right] \\ &= \frac{r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}(2j-2)!} \cdot \frac{8j^3(2j-1)(\sigma+j-1)-(j+1)r}{4j(\sigma+j)(2j)(2j-1)} \\ &= \frac{r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}(2j-2)!} \cdot \frac{\chi(j)}{4j(\sigma+j)(2j)(2j-1)}, \end{aligned} \tag{2.34}$$

where

$$\chi(j) := 8j^3(2j-1)(\sigma+j-1)-(j+1)r, \quad j \in \mathbb{N},$$

we will use the mathematical induction to prove the nonnegativity of $\chi(j)$ for all $j \in \mathbb{N}$.

First, for $j = 1$, according to the assumption $\sigma \geq \frac{r}{4}$, we have

$$\chi(1) = 8\sigma - 2r \geq 0.$$

Now, suppose that $\chi(m) \geq 0$ for a fixed $m \in \mathbb{N}$. It is easy to check that

$$\chi(m+1) = \chi(m) + \widetilde{\chi}(m),$$

where

$$\widetilde{\chi}(m) := 80m^4 + 64(\sigma+1)m^3 + 8(9\sigma+5)m^2 + 8(5\sigma+1)m - r + 8\sigma > 0,$$

which holds under the assumption $\sigma \geq \frac{r}{4} > \frac{r}{8}$, with $r \in (0,+\infty)$, and $m \in \mathbb{N}$.

Therefore, inequality (2.34) implies $jC_j - (j+1)C_{j+1} \geq 0$, $j \in \mathbb{N}$, under our assumptions. Also, since $1 \geq 2C_2$, it is equivalent with our assumption $\sigma \geq \frac{r}{4}$, the proof is complete. □

Theorem 2.8 *If* $\sigma \geq \frac{r}{8}$ *with* $r \in (0, +\infty)$, *then* $(\sin z) * \frac{\mathrm{U}_{\sigma,r}(z^2)}{z}$ *is a close-to-convex function in* $\mathbb{U}$ *with respect to* $\log\sqrt{\frac{1+z}{1-z}}$.

Proof From the power series expansion of the function $\sin z$ as well as (1.4), we have

$$(\sin z) * \frac{\mathrm{U}_{\sigma,r}(z^2)}{z} = z + \sum_{j=1}^{\infty} C_{2j+1} z^{2j+1}, \quad z \in \mathbb{U},$$

where

$$C_{2j+1} = \frac{r^j}{4^j (1)_j (\sigma)_j (2j+1)!}, \quad j \in \mathbb{N}.$$

To use Lemma 1.4 for our proof, it can further be shown that $\{(2j+1)C_{2j+1}\}_{j\in\mathbb{N}}$ is a decreasing nonnegative sequence with $1 \geq 3C_3$. Thus, a simple computation shows that

$$\begin{aligned}
&(2j-1)C_{2j-1} - (2j+1)C_{2j+1} \\
&\quad = \frac{(2j-1)r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}(2j-1)!} - \frac{(2j+1)r^j}{4^j(1)_j(\sigma)_j(2j+1)!} \\
&\quad = \frac{r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}(2j-1)!}\left[(2j-1) - \frac{(2j+1)r}{4j(\sigma+j-1)(2j)(2j+1)}\right] \\
&\quad = \frac{r^{j-1}}{4^{j-1}(1)_{j-1}(\sigma)_{j-1}(2j-1)!} \cdot \frac{\varkappa(j)}{4j(\sigma+j-1)(2j)(2j+1)},
\end{aligned}$$

where

$$\varkappa(j) := 8j^2(2j+1)(2j-1)(\sigma+j-1) - (2j+1)r, \quad j \in \mathbb{N}.$$

We use the mathematical induction to prove the nonnegativity of $\varkappa(j)$ for all $j \in \mathbb{N}$. For $j = 1$, according to the assumption $\sigma \geq \frac{r}{8}$, we have

$$\varkappa(1) = 24\sigma - 3r \geq 0.$$

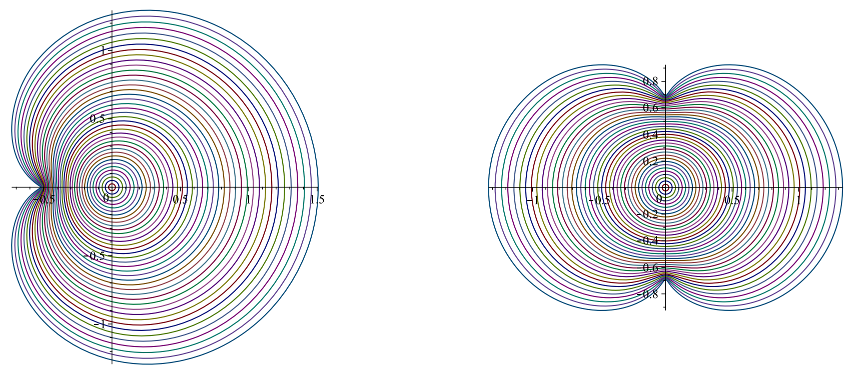
Supposing that $\varkappa(m) \geq 0$ for a fixed $m \in \mathbb{N}$, it is easy to check that

$$\varkappa(m+1) = \varkappa(m) + \widetilde{\varkappa}(m),$$

where

$$\widetilde{\varkappa}(m) := 160m^4 + 64(3\sigma+3)m^3 + 8(24\sigma+13)m^2 + 8(14\sigma+3)m - 2r + 24\sigma > 0,$$

which holds under the assumption $\sigma \geq \frac{r}{8} > \frac{r}{12}$, with $r \in (0, +\infty)$, and $m \in \mathbb{N}$. Also, since $\sigma \geq \frac{r}{8} > 0$, it is equivalent to $1 \geq 3C_3$, it follows that our result is proved. □



(A) The image of $\mathbb{U}$ by $(z\cos\sqrt{z}) * \mathrm{U}_{\sigma,r}(z)$ for $\sigma = 2.1$ and $r = 8.3$

(B) The image of $\mathbb{U}$ by $(\sin z) * \dfrac{\mathrm{U}_{\sigma,r}(z^2)}{z}$ for $\sigma = 3$ and $r = 23$

Figure 6 Figures for Remark 2.5

Remark 2.5 1. Taking in Theorem 2.7 the values $\sigma = 2.1$ and $r = 8.3$, the function $(z\cos\sqrt{z}) * \mathrm{U}_{\sigma,r}(z)$ will be close-to-convex in $\mathbb{U}$, but definitively not convex in $\mathbb{U}$, as we can see in Fig. 6(A).

2. For the particular case $\sigma = 3$ and $r = 23$, Theorem 2.8 yields that the function $(\sin z) * \frac{\mathrm{U}_{\sigma,r}(z^2)}{z}$ is close-to-convex in $\mathbb{U}$. Figure 6(B) obtained with MAPLE™ computer software shows graphically that this function is not convex in $\mathbb{U}$.

3 Concluding remarks

In the current work, we have employed a new investigation procedure. First, using an identity for the logarithmic of the gamma function, as well as an inequality for the digamma function, we established the sufficient conditions on the parameters such that $\mathrm{U}_{\sigma,r}$ is a starlike or a convex function of order α $(0 \leq \alpha \leq 1)$ in the open unit disk. Moreover, other starlikeness and convexity conditions for $\mathrm{U}_{\sigma,r}$ have been determined, where the leading concept of the proofs comes from the starlikeness of the power series $f(z) = \sum_{j=1}^{\infty} A_j z^j$, and from the classical Alexander duality theorem between the classes of starlike and convex functions. The results are followed by a simple demonstration showing that our conditions are not contradictory. Finally, simple sufficient conditions for the close-to-convexity of the functions $(z\cos\sqrt{z}) * \mathrm{U}_{p,q,r}$ and $(\sin z) * \frac{\mathrm{U}_{\sigma,r}(z^2)}{z}$ have been considered. Further investigations connected with this topic are now underway and will be reported in forthcoming papers.

Acknowledgements
The authors are grateful to the reviewers for the valuable remarks, comments, and advices that help us to improve the quality of the manuscript.

Funding
Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

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Both authors contributed equally to the writing of this paper. They read and approved the final version of the paper.

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Received: 7 October 2022 Accepted: 23 November 2022 Published online: 15 December 2022

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