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On the reciprocal products of generalized Fibonacci sequences

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Abstract

In this paper, we use the properties of error estimation and the analytic method to study the reciprocal products of the bi-periodic Fibonacci sequence, the bi-periodic Lucas sequence, and the mth-order linear recursive sequence.

Keywords: Reciprocal products; Bi-periodic Fibonacci sequence; Bi-periodic Lucas sequence; mth-order linear recursive sequence; Landau symbol; Asymptotic equivalence

1 Introduction

The so-called Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ are defined by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

and

$$L_0 = 2, \qquad L_1 = 1, \qquad L_n = L_{n-1} + L_{n-2}, \quad n \geq 2.$$

The Fibonacci and Lucas sequences have many interesting properties and applications [1]. In addition, in [2], Ohtsuka and Nakamura considered the partial infinite sums of reciprocal Fibonacci sequence and proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2,$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ is even;} \\ F_{n-1}F_n, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2,$$

where $\lfloor \cdot \rfloor$ (the floor function) denotes the greatest integer less than or equal to x.

Many authors have studied the Fibonacci and Lucas sequences by changing initial conditions or recursive relations. For instance, for any two nonzero real numbers a and b, Edson and Yayenie [3] introduced the bi-periodic Fibonacci sequence $\{f_n\}$ as:

$$f_0 = 0, \qquad f_1 = 1, \qquad f_n = \begin{cases} af_{n-1} + f_{n-2}, & \text{if } n \text{ is even;} \\ bf_{n-1} + f_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2. \tag{1}$$

For $a = b = 1$, $\{f_n\}$ reduces to the Fibonacci sequence $\{F_n\}$. If $a = b = k$, then $\{f_n\}$ becomes the k-Fibonacci sequence $\{q_n\}$ defined in [4], etc. Similarly, for any two nonzero real numbers a and b, Bilgici [5] introduced the bi-periodic Lucas sequence $\{l_n\}$ as:

$$l_0 = 2, \qquad l_1 = a, \qquad l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even;} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2. \tag{2}$$

For $a = b = 1$, $\{l_n\}$ reduces to the Lucas sequence $\{L_n\}$. If $a = b = k$, then $\{l_n\}$ becomes the k-Lucas sequence $\{p_n\}$ defined in [6]. In [7], Tan and Leung considered a generalization of Horadam sequence $\{w_n\}$, which is defined by the recurrence relation

$$w_0 = w_0, \qquad w_1 = w_1, \qquad w_n = \begin{cases} aw_{n-1} + cw_{n-2}, & \text{if } n \text{ is even;} \\ bw_{n-1} + cw_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2,$$

with arbitrary initial conditions w_0, w_1 and nonzero real numbers a, b, and c. In [8], Tan considered the sequence $\{w_n\}$ when $c = 1$. In [9], Ramírez and Sirvent introduced a q-bi-periodic Fibonacci sequence by

$$F_n^{(a,b)}(q,s) = \begin{cases} aF_{n-1}^{(a,b)}(q,s) + q^{n-2}sF_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is even;} \\ bF_{n-1}^{(a,b)}(q,s) + q^{n-2}sF_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2,$$

with initial conditions $F_0^{(a,b)}(q,s) = 0$ and $F_1^{(a,b)}(q,s) = 1$ and nonzero real numbers a, b, q and s. Motivated by [9], in [10] Tan introduced a q-bi-periodic Lucas sequence by

$$L_n^{(a,b)}(q,s) = \begin{cases} bL_{n-1}^{(a,b)}(q,s) + sL_{n-2}^{(a,b)}(q,qs), & \text{if } n \text{ is even;} \\ aL_{n-1}^{(a,b)}(q,s) + sL_{n-2}^{(a,b)}(q,qs), & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2,$$

with initial conditions $L_0^{(a,b)}(q,s) = 2$ and $L_1^{(a,b)}(q,s) = q$, and nonzero real numbers a , b ,q and s.

In [11], Holliday and Komatsu obtained the infinite sums of the reciprocal of k-Fibonacci sequence $\{q_n\}$. In [12], Basbük and Yazlik obtained the infinite sums of the reciprocal of the bi-periodic Fibonacci sequence $\{f_n\}$. Various authors studied the infinite sums of the reciprocal of the other famous sequences [13–15].

Recently, some authors studied the nearest integer of the sums of reciprocal of some linear recurrence sequences. In [16], Komatsu proved that there exists a positive integer n_1 such that:

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{q_k} \right)^{-1} \right\| = q_n - q_{n-1}, \quad n \geq n_1,$$

where $\{q_n\}$ is the k-Fibonacci sequence. $\|\cdot\|$ denotes the nearest integer. Specifically, suppose that $\|x\| = \lfloor x + \frac{1}{2} \rfloor$.

On the other hand, Wu and Zhang [17] considered an mth-order linear recursive sequence $\{u_n\}$ defined by

$$u_n = x_1 u_{n-1} + x_2 u_{n-2} + \cdots + x_m u_{n-m}, \quad n > m, \tag{3}$$

where initial values $u_i \in N$ for $0 \le i < m$, at least one of them is different from zero, and $x_1, x_2, \ldots, x_m$ are positive integers. The characteristic polynomial of the sequence $\{u_n\}$ is given by

$$\psi(y) = y^m - x_1 y^{m-1} - \cdots - x_{m-1} y - x_m.$$

For $m = 2$, $x_1 = x_2 = 1$ and initial values $u_0 = 0$, $u_1 = 1$, $\{u_n\}$ reduces to the Fibonacci sequence. If $m = 2$, $x_1 = x_2 = 1$ and initial values $u_0 = 2$, $u_1 = 1$, then $\{u_n\}$ becomes the Lucas sequence.

In addition, they proved that there exists a positive integer n_2 such that:

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \quad n \ge n_2,$$

for any positive integers $x_1 \ge x_2 \ge \cdots \ge x_m \ge 1$. For more the nearest integer of the sums of reciprocal of the recurrence sequence studies, see [18–21]. Specifically, in [19], Trojorský considered finding a sequence that is "asymptotically equivalent" to partial infinite sums and proved that

$$\left\{ \left(\sum_{k=n}^{\infty} \frac{1}{P(u_k)} \right)^{-1} \right\}_n \quad \text{and} \quad \left\{ P(u_n) - P(u_{n-1}) \right\}_n$$

are asymptotically equivalent, where $P(z) \in C[z]$ is a non-constant polynomial. Specifically, we say that two sequences $\{G_n\}$ and $\{H_n\}$ are called "asymptotically equivalent" if $\{G_n\}/\{H_n\}$ tends to 1 as $n \to \infty$.

In addition to the study of the infinite reciprocal sums of recursive sequence, we can also consider the infinite reciprocal products of recursive sequence. In 2006, Wu [22] studied the partial infinite products of $\frac{q_k^i - 1}{q_k^i}$. He used the element method and the properties of the floor function and proved that

$$\left\lfloor \left(1 - \prod_{k=n}^{\infty} \left(1 - \frac{1}{q_k} \right) \right)^{-1} \right\rfloor = q_n - q_{n-1}, \quad n \ge 2,$$

and

$$\left\lfloor \left(1 - \prod_{k=n}^{\infty} \left(1 - \frac{1}{q_k^2} \right) \right)^{-1} \right\rfloor = \begin{cases} q_n^2 - q_{n-1}^2, & \text{if } n \text{ is even;} \\ q_n^2 - q_{n-1}^2 - 1, & \text{if } n \text{ is odd,} \end{cases} \quad n \ge 2,$$

where $\{q_n\}$ is the k-Fibonacci sequence. For more the partial infinite products of the other sequences, see [23, 24].

Inspired by [19], in this paper, we apply a different research method from the previous one and use the properties of error estimation and the analytic method to study the reciprocal products of $\{f_n\}$, $\{l_n\}$ and $\{u_n\}$. We derive some sequences that are asymptotically equivalent to reciprocal products including $\{f_n\}$, $\{l_n\}$ and $\{u_n\}$. Our main results are the following:

Theorem 1 *Let $\{f_n\}$ be the bi-periodic Fibonacci sequence, and $\{l_n\}$ be the bi-periodic Lucas sequence. For positive integers a and b with $a \geq 1$, $b \geq 1$, the sequences*

$$\left\{\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{f_k}\right)\right)^{-1}\right\}_n \quad \text{and} \quad \{f_n - f_{n-1}\}_n \tag{4}$$

are asymptotically equivalent, and the sequences

$$\left\{\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{l_k}\right)\right)^{-1}\right\}_n \quad \text{and} \quad \{l_n - l_{n-1}\}_n \tag{5}$$

are asymptotically equivalent.

Corollary 1 *We obtain the infinite products of the reciprocal of the k-Fibonacci sequence q_n and k-Lucas sequence p_n, when $a = b = k$. Then, the sequences*

$$\left\{\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{q_k}\right)\right)^{-1}\right\}_n \quad \text{and} \quad \{q_n - q_{n-1}\}_n \tag{6}$$

are asymptotically equivalent, and the sequences

$$\left\{\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{p_k}\right)\right)^{-1}\right\}_n \quad \text{and} \quad \{p_n - p_{n-1}\}_n \tag{7}$$

are asymptotically equivalent.

Theorem 2 *Let $\{u_n\}$ be an* m*th-order linear recursive sequence with any positive integers $x_1 \geq x_2 \geq \cdots \geq x_m \geq 1$. Then, the sequences*

$$\left\{\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{u_k}\right)\right)^{-1}\right\}_n \quad \text{and} \quad \{u_n - u_{n-1}\}_n \tag{8}$$

are asymptotically equivalent.

2 Proof of the theorems

To complete the proof of our theorems, we need the following:

Lemma 1 ([3, 5], Generalized Binet's formula) *The terms of the bi-periodic Fibonacci sequence $\{f_n\}$, and bi-periodic Lucas sequence $\{l_n\}$ are given by*

$$f_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right),$$

and

$$l_n = \frac{a^{\zeta(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}(\alpha^n + \beta^n),$$

where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ *and* $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$*, i.e.* α *and* β *are roots of the equation* $x^2 - abx - ab = 0$*. It is obvious that* $\alpha > 1$ *and* $-1 < \beta < 0$ *with* $a \geq 1$, $b \geq 1$*. In addition,* $\zeta(n)$ *is the parity function, such that* $\zeta(n) = 0$ *if n is even and* $\zeta(n) = 1$ *if n is odd.*

Lemma 2 *Let* $\{f_n\}$ *be the bi-periodic Fibonacci sequence defined by* (1)*, and* $\{l_n\}$ *be the bi-periodic Lucas sequence defined by* (2)*. Then, we have*

$$f_n = \begin{cases} \frac{c\alpha^n}{(ab)^{\frac{n}{2}}} - \frac{c\beta^n}{(ab)^{\frac{n}{2}}}, & \text{if } n \text{ is even;} \\ \frac{d\alpha^n}{(ab)^{\frac{n-1}{2}}} - \frac{d\beta^n}{(ab)^{\frac{n-1}{2}}}, & \text{if } n \text{ is odd,} \end{cases}$$

where $c = \frac{a}{\alpha-\beta}$, $d = \frac{1}{\alpha-\beta}$*, and*

$$l_n = \begin{cases} \frac{\alpha^n}{(ab)^{\frac{n}{2}}} + \frac{\beta^n}{(ab)^{\frac{n}{2}}}, & \text{if } n \text{ is even;} \\ \frac{a\alpha^n}{(ab)^{\frac{n+1}{2}}} + \frac{a\beta^n}{(ab)^{\frac{n+1}{2}}}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof By Lemma 1, we can easily prove it. □

Lemma 3 ([17]) *Let* $\{u_n\}$ *be an* m*th-order linear recursive sequence defined by* (3)*. The coefficients of the characteristic polynomial* $\psi(y)$ *are satisfied that* $x_1 \geq x_2 \geq \cdots \geq x_m \geq 1$*. Then, the closed formula of* $\{u_n\}$ *is given by*

$$u_n = s\gamma^n + \mathcal{O}(t^{-n}), \quad (n \to \infty),$$

where $s > 0$, $t > 1$, γ *is the positive real zero of* $\psi(y)$ *for* $x_1 < \gamma < x_1 + 1$*, and "*$\mathcal{O}$*" (the Landau symbol) denotes if* $g(x) > 0$ *for all* $x \geq a$*, we write* $f(x) = \mathcal{O}(g(x))$ *to mean that the quotient* $f(x)/g(x)$ *is bounded for* $x \geq a$*.*

Lemma 4 *Let* a, b, c, d, α*, and* β *be defined by Lemma 1 or Lemma 2 and* s, γ*, and* t *be defined by Lemma 3. Then, we have*

$$\begin{aligned} &\prod_{k=n}^{\infty}\left(1 - \frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k + \mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^2}\right)^k\right)\right) \\ &\quad = 1 - \sum_{k=n}^{\infty}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k + \mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} &\prod_{k=n}^{\infty}\left(1 - \frac{1}{(ab)^{\frac{1}{2}}d}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k + \mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^2}\right)^k\right)\right) \\ &\quad = 1 - \sum_{k=n}^{\infty}\frac{1}{(ab)^{\frac{1}{2}}d}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k + \mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right), \end{aligned} \tag{10}$$

$$\prod_{k=n}^{\infty}\left(1 - \frac{1}{s\gamma^k} + \mathcal{O}(\gamma^{-2k}t^{-k})\right) = 1 - \sum_{k=n}^{\infty}\frac{1}{s\gamma^k} + \mathcal{O}(\gamma^{-2n}). \tag{11}$$

Proof We shall prove only (6) in Lemma 4, and other identities are proved similarly. The identity $ab = -\alpha\beta$ now yield $|\beta| < (ab)^{\frac{1}{2}} = (-\alpha\beta)^{\frac{1}{2}} < \alpha$, where $\alpha > 1$ and $-1 < \beta < 0$. First, we prove the following equation

$$\begin{aligned}
&\prod_{k=n}^{n+m}\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{k}\right)\right)\\
&\quad=1-\sum_{k=n}^{n+m}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right),
\end{aligned} \tag{12}$$

We prove (9) by mathematical induction. When $m = 1$,

$$\begin{aligned}
&\prod_{k=n}^{n+1}\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{k}\right)\right)\\
&\quad=\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n}\right)\right)\\
&\qquad\times\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+1}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n+1}\right)\right)\\
&\quad=1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n}-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+1}+\frac{1}{c^{2}}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n+1}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n}\right)\\
&\quad=1-\sum_{k=n}^{n+1}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n}\right)\\
&\quad=1-\sum_{k=n}^{n+1}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right).
\end{aligned}$$

When $m = 2$,

$$\begin{aligned}
&\prod_{k=n}^{n+2}\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{k}\right)\right)\\
&\quad=\left(1-\sum_{k=n}^{n+1}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)\right)\\
&\qquad\times\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+2}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n+2}\right)\right)\\
&\quad=1-\sum_{k=n}^{n+1}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+2}+\frac{1}{c^{2}}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+2}\left(\sum_{k=n}^{n+1}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}\right)\\
&\qquad+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n}\right)\\
&\quad=1-\sum_{k=n}^{n+2}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n}\right)\\
&\quad=1-\sum_{k=n}^{n+1}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right).
\end{aligned}$$

That is, (9) is true for $m = 1$ or $m = 2$. Suppose that for any integer m, we have

$$\begin{aligned}&\prod_{k=n}^{n+m}\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{k}\right)\right)\\&\quad=1-\sum_{k=n}^{n+m}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right).\end{aligned}\tag{13}$$

Then, for $m + 1$, we have

$$\begin{aligned}&\prod_{k=n}^{n+m+1}\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{k}\right)\right)\\&\quad=\left(1-\sum_{k=n}^{n+m}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)\right)\\&\qquad\times\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+m+1}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n+m+1}\right)\right)\\&\quad=1-\sum_{k=n}^{n+m}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+m+1}+\frac{1}{c^{2}}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{n+m+1}\left(\sum_{k=n}^{n+m}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}\right)\\&\qquad+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n}\right)\\&\quad=1-\sum_{k=n}^{n+m+1}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{n}\right)\\&\quad=1-\sum_{k=n}^{n+m+1}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right).\end{aligned}$$

Taking $m \to \infty$, we have

$$\begin{aligned}&\prod_{k=n}^{\infty}\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^{2}}\right)^{k}\right)\right)\\&\quad=1-\sum_{k=n}^{\infty}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{k}+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right),\end{aligned}$$

which completes the proof. □

Proof of Theorem 1 We shall prove only (4) in Theorem 1, and the identity (5) is proved similarly. From the geometric series as $\epsilon \to 0$, we find

$$\frac{1}{1\pm\epsilon}=1+\mathcal{O}(\epsilon).$$

If n is even, with $n \geq 2$. Using Lemma 2, we have

$$\frac{1}{f_k}=\frac{1}{\frac{c\alpha^{k}}{(ab)^{\frac{k}{2}}}-\frac{c\beta^{k}}{(ab)^{\frac{k}{2}}}}=\frac{1}{\frac{c\alpha^{k}}{(ab)^{\frac{k}{2}}}(1-(\frac{\beta}{\alpha})^{k})}=\frac{(ab)^{\frac{k}{2}}}{c\alpha^{k}}\left(1+\mathcal{O}\left(\frac{\beta}{\alpha}\right)^{k}\right).$$

By Lemma 4, we obtain

$$\begin{aligned}\prod_{k=n}^{\infty}\left(1-\frac{1}{f_k}\right)&=\prod_{k=n}^{\infty}\left(1-\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^2}\right)^k\right)\right)\\&=1-\sum_{k=n}^{\infty}\frac{1}{c}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)\\&=1-\frac{(ab)^{\frac{n}{2}}}{c\alpha^n}\left(\frac{\alpha}{\alpha-(ab)^{\frac{1}{2}}}\right)+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right).\end{aligned}$$

Taking the reciprocal of this expression yields

$$\begin{aligned}\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{f_k}\right)\right)^{-1}&=\frac{1}{\frac{(ab)^{\frac{n}{2}}}{c\alpha^n}(\frac{\alpha}{\alpha-(ab)^{\frac{1}{2}}})+\mathcal{O}((\frac{(ab)^{\frac{1}{2}}}{\alpha})^{2n})}\\&=\frac{1}{\frac{(ab)^{\frac{n}{2}}}{c\alpha^n}(\frac{\alpha}{\alpha-(ab)^{\frac{1}{2}}})(1+\mathcal{O}(\frac{(ab)^{\frac{1}{2}}}{\alpha})^n)}\\&=\frac{c\alpha^n}{(ab)^{\frac{n}{2}}}\left(\frac{\alpha-(ab)^{\frac{1}{2}}}{\alpha}\right)\left(1+\mathcal{O}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^n\right)\\&=\left(f_n-f_{n-1}+\frac{c\beta^n}{(ab)^{\frac{n}{2}}}-\frac{c\beta^{n-1}}{(ab)^{\frac{n-1}{2}}}\right)\left(1+\mathcal{O}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^n\right),\end{aligned}$$

where $|\beta|<(ab)^{\frac{1}{2}}$ yields

$$\left(f_n-f_{n-1}+\frac{c\beta^n}{(ab)^{\frac{n}{2}}}-\frac{c\beta^{n-1}}{(ab)^{\frac{n-1}{2}}}\right)\quad\text{tends to }(f_n-f_{n-1}),$$

as $n\to\infty$. In addition, as $(ab)^{\frac{1}{2}}<\alpha$, we obtain

$$\frac{(1-\prod_{k=n}^{\infty}(1-\frac{1}{f_k}))^{-1}}{(f_n-f_{n-1})}\quad\text{tends to 1},$$

as $n\to\infty$.

If n is odd, with $n\geq 1$. Using Lemma 2, we have

$$\frac{1}{f_k}=\frac{1}{\frac{d\alpha^k}{(ab)^{\frac{k-1}{2}}}-\frac{d\beta^k}{(ab)^{\frac{k-1}{2}}}}=\frac{1}{\frac{d\alpha^k}{(ab)^{\frac{k-1}{2}}}(1-(\frac{\beta}{\alpha})^k)}=\frac{(ab)^{\frac{k-1}{2}}}{d\alpha^k}\left(1+\mathcal{O}\left(\frac{\beta}{\alpha}\right)^k\right).$$

By Lemma 4, we obtain

$$\begin{aligned}\prod_{k=n}^{\infty}\left(1-\frac{1}{f_k}\right)&=\prod_{k=n}^{\infty}\left(1-\frac{1}{(ab)^{\frac{1}{2}}d}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}\beta}{\alpha^2}\right)^k\right)\right)\\&=1-\sum_{k=n}^{\infty}\frac{1}{(ab)^{\frac{1}{2}}d}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^k+\mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right)\end{aligned}$$

$$= 1 - \frac{(ab)^{\frac{n-1}{2}}}{d\alpha^n}\left(\frac{\alpha}{\alpha - (ab)^{\frac{1}{2}}}\right) + \mathcal{O}\left(\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^{2n}\right).$$

Taking the reciprocal of this expression yields

$$\begin{aligned}\left(1 - \prod_{k=n}^{\infty}\left(1 - \frac{1}{f_k}\right)\right)^{-1} &= \frac{1}{\frac{(ab)^{\frac{n-1}{2}}}{d\alpha^n}(\frac{\alpha}{\alpha-(ab)^{\frac{1}{2}}}) + \mathcal{O}((\frac{(ab)^{\frac{1}{2}}}{\alpha})^{2n})} \\ &= \frac{1}{\frac{(ab)^{\frac{n-1}{2}}}{d\alpha^n}(\frac{\alpha}{\alpha-(ab)^{\frac{1}{2}}})(1 + \mathcal{O}(\frac{(ab)^{\frac{1}{2}}}{\alpha})^n)} \\ &= \frac{d\alpha^n}{(ab)^{\frac{n-1}{2}}}\left(\frac{\alpha - (ab)^{\frac{1}{2}}}{\alpha}\right)\left(1 + \mathcal{O}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^n\right) \\ &= \left(f_n - f_{n-1} + \frac{d\beta^n}{(ab)^{\frac{n}{2}}} - \frac{d\beta^{n-1}}{(ab)^{\frac{n-1}{2}}}\right)\left(1 + \mathcal{O}\left(\frac{(ab)^{\frac{1}{2}}}{\alpha}\right)^n\right),\end{aligned}$$

where $|\beta| < (ab)^{\frac{1}{2}}$ yields

$$\left(f_n - f_{n-1} + \frac{c\beta^n}{(ab)^{\frac{n}{2}}} - \frac{c\beta^{n-1}}{(ab)^{\frac{n-1}{2}}}\right) \quad \text{tends to } (f_n - f_{n-1}),$$

as $n \to \infty$. In addition, as $(ab)^{\frac{1}{2}} < \alpha$, we obtain

$$\frac{(1 - \prod_{k=n}^{\infty}(1 - \frac{1}{f_k}))^{-1}}{(f_n - f_{n-1})} \quad \text{tends to } 1,$$

as $n \to \infty$, which completes the proof. □

Proof of Theorem 2 Using Lemma 3, we have

$$\frac{1}{u_k} = \frac{1}{s\gamma^k + \mathcal{O}(t^{-k})} = \frac{1}{s\gamma^k(1 + \mathcal{O}(\gamma^{-k}t^{-k}))} = \frac{1}{s\gamma^k}\left(1 + \mathcal{O}\left(\gamma^{-k}t^{-k}\right)\right).$$

By Lemma 4, we obtain

$$\begin{aligned}\prod_{k=n}^{\infty}\left(1 - \frac{1}{u_k}\right) &= \prod_{k=n}^{\infty}\left(1 - \frac{1}{s\gamma^k} + \mathcal{O}\left(\gamma^{-2k}t^{-k}\right)\right) \\ &= 1 - \sum_{k=n}^{\infty}\frac{1}{s\gamma^k} + \mathcal{O}\left(\gamma^{-2n}\right) \\ &= 1 - \frac{\gamma}{s\gamma^n(\gamma - 1)} + \mathcal{O}\left(\gamma^{-2n}\right).\end{aligned}$$

Taking the reciprocal of this expression yields

$$\left(1 - \prod_{k=n}^{\infty}\left(1 - \frac{1}{u_k}\right)\right)^{-1} = \frac{1}{\frac{\gamma}{s\gamma^n(\gamma-1)} + \mathcal{O}(\gamma^{-2n})}$$

$$
\begin{aligned}
&= \frac{1}{\frac{\gamma}{s\gamma^n(\gamma-1)}(1+\mathcal{O}(\gamma^{-n}))} \\
&= \frac{s\gamma^n(\gamma-1)}{\gamma}\bigl(1+\mathcal{O}\bigl(\gamma^{-n}\bigr)\bigr) \\
&= (u_n - u_{n-1})\bigl(1+\mathcal{O}\bigl(\gamma^{-n}\bigr)\bigr),
\end{aligned}
$$

which yields

$$
\frac{(1-\prod_{k=n}^{\infty}(1-\frac{1}{u_k}))^{-1}}{(u_n-u_{n-1})} \quad \text{tends to 1,}
$$

as $n \to \infty$, which completes the proof. □

3 Discussion

In this paper, we obtain the sequences that are asymptotically equivalent to reciprocal products of $\frac{f_k-1}{f_k}$, $\frac{l_k-1}{l_k}$ and $\frac{u_k-1}{u_k}$, where $\{f_n\}$ denotes the bi-periodic Fibonacci sequence, $\{l_n\}$ denotes the bi-periodic Lucas sequence, and $\{u_n\}$ denotes an mth-order linear recursive sequence. For any positive integers j, an open problem is whether there exists the similar identities for the infinity products of $\frac{f_k^j-1}{f_k^j}$, $\frac{l_k^j-1}{l_k^j}$ and $\frac{u_k^j-1}{u_k^j}$.

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Declarations

Competing interests
The authors declare no competing interests.

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References

1. Koshy, T.: Fibonacci and Lucas Numbers with Applications. Wiley, New York (2001)
2. Ohtsuka, H., Nakamura, S.: On the sum of reciprocal Fibonacci numbers. Fibonacci Q. **46–47**, 153–159 (2008)
3. Edson, M., Yayenie, O.: A new generalization of Fibonacci sequence and extended Binet's formula. Integers **9**, 639–654 (2009)
4. Falcon, S.: On the Fibonacci k-numbers. Chaos Solitons Fractals **32**, 1615–1624 (2007)
5. Bilgici, G.: Two generalizations of Lucas sequence. Appl. Math. Comput. **245**, 526–538 (2014)
6. Falcon, S.: On the k-Lucas numbers. Int. J. Contemp. Math. Sci. **6**, 1039–1050 (2011)
7. Tan, E., Leung, H.-H.: Some basic properties of the generalized bi-periodic Fibonacci and Lucas sequences. Adv. Differ. Equ. **2020**, 26 (2020)
8. Tan, E.: Some properties of bi-periodic Horadam sequences. Notes Number Theory Discrete Math. **23**(4), 56–65 (2017)
9. Ramírez, J.L., Sirvent, V.F.: A q-analoque of the biperiodic Fibonacci sequence. J. Integer Seq. **19**(2), 3 (2016)
10. Tan, E.: A Q-analog of the BI-periodic Lucas sequence. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **67**(2), 220–228 (2018)

11. Holliday, S., Komatsu, T.: On the sum of reciprocal generalized Fibonacci numbers. Integers **11**, 441–455 (2011)
12. Basbük, M., Yazlik, Y.: On the sum of reciprocal of generalized bi-periodic Fibonacci numbers. Miskolc Math. Notes **17**, 35–41 (2016)
13. Zhang, W., Wang, T.: The infinite sum of reciprocal Pell numbers. Appl. Math. Comput. **218**, 6164–6167 (2012)
14. Choi, G., Choo, Y.: On the reciprocal sums of products of Fibonacci and Lucas numbers. Filomat **32**, 2911–2920 (2018)
15. Choi, G., Choo, Y.: On the reciprocal sums of square of generalized bi-periodic Fibonacci numbers. Miskolc Math. Notes **19**, 201–209 (2018)
16. Komatsu, T.: On the nearest integer of the sum of reciprocal Fibonacci numbers. Aport. Mat. Investig. **20**, 171–184 (2011)
17. Wu, Z., Han, Z.: On the reciprocal sums of higher-order sequences. Adv. Differ. Equ. **2013**, 189 (2013)
18. Wu, Z., Zhang, J.: On the higher power sums of reciprocal higher-order sequences. Sci. World J. **2014**, 521358 (2014)
19. Trojovský, P.: On the sum of reciprocal of polynomial applied to higher order recurrences. Mathematics **7**(7), 638 (2019)
20. Zhang, H., Wu, Z.: On the reciprocal sums of the generalized Fibonacci sequences. Adv. Differ. Equ. **2013**, 377 (2013)
21. Kiliç, E., Arikan, T.: More on the infinite sum of reciprocal Fibonacci, Pell and higher order recurrences. Appl. Math. Comput. **219**, 7783–7788 (2013)
22. Wu, Z.: Several identities relating to reciprocal products of generalized Fibonacci numbers. J. Northwest Univ. Nat. Sci. **46**(3), 317–320 (2016)
23. Wu, Z.: On the study of some identities related to Riemann zeta function. J. Shaanxi Normal Univ. Nat. Sci. Ed. **46**(2), 26–29 (2018)
24. Jiang, Y., Wang, T.: Some identities involving the reciprocal products of the Pell numbers. J. Shaanxi Normal Univ. Nat. Sci. Ed. **45**(4), 23–27 (2017)