Hussain et al. *Journal of Inequalities and Applications* (2022) 2022:138
https://doi.org/10.1186/s13660-022-02878-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On some new generalized fractional Bullen-type inequalities with applications

Sabir Hussain[1], Sobia Rafeeq[2], Yu-Ming Chu[3,4*], Saba Khalid[1] and Sahar Saleem[1]

*Correspondence:
chuyuming@zjhu.edu.cn
[3]Institute for Advanced Study Honoring Chen Jian Gong, Hangzhou Normal University, Hangzhou 311121, China
[4]Department of Mathematics, Huzhou University, Huzhou 313000, China
Full list of author information is available at the end of the article

Abstract

In this paper, we establish a Bullen-type generalized fractional integral identity for the differentiable functions and derive some new estimates for the Bullen-type fractional integral inequalities via the Raina fractional integrals. Further examples are given to indicate the validity of obtained results. Lastly, some error estimates for the quadrature rules are provided, and to find applications for our results to information theory, we proved a new generalization based on f-divergence measure.

MSC: 26D15; 26A51; 94A17

Keywords: Bullen's inequality; Fractional integrals; p-convex function; f-divergence measure

1 Introduction

Fractional calculus was a natural outgrowth of conventional calculus concerned with the integrals and derivatives of arbitrary order. This subject has earned significant recognition due to its applications in diverse domains of science and engineering. An eminent distinction of this field is that the researchers who proposed more effective solution of physical phenomena have tuned to new operators with strong kernels over time. There are several mathematical problems and their related real world applications where fractional derivatives occupy a significant place [1–5]. In recent years, fractional calculus has been utilized in defining the complex dynamics of real world problems from various fields of applied sciences. In the literature many applications can be found [6–8]. Fractional operators combined with the notion of convexity have been extensively used to attain new results in the theory of inequalities. Fractional integral inequalities provide uniqueness and existence of solutions for mathematical problems in terms of fractional models. Applications of integral inequalities, such as statistical problems, transform theory, numerical quadrature, and probability, are found in applied sciences. In the last few years, many researchers have contributed in establishing various types of integral inequalities by employing various different approaches. Moreover, the integral inequalities are linked with other areas such as differential equations, difference equations, mathematical analysis, mathematical physics, convexity theory, discrete fractional calculus, and fuzzy theory. The theory of convex functions has undergone a rapid progression. This is due to several factors: first, applications of convex functions are precisely involved in modern analysis; second, numerous essential inequalities are outcomes of convex functions. The Hermite–Hadamard inequality is the first fundamental result for a convex function. The classical Hermite–Hadamard inequality gives us an estimation of the mean value of a convex function $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ and $\varrho_1, \varrho_2 \in \mathbb{R}$ with $\varrho_1 < \varrho_2$

$$f\left(\frac{\varrho_1+\varrho_2}{2}\right) \leq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(x)\,dx \leq \frac{f(\varrho_1)+f(\varrho_2)}{2}. \tag{1}$$

Bullen [9] proved the following inequality, known as Bullen's inequality, giving the bound for the mean value of a convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for $\varrho_1, \varrho_2 \in I$ with $\varrho_1 < \varrho_2$, then

$$\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(x)\,dx \leq \frac{1}{2}\left[f\left(\frac{\varrho_1+\varrho_2}{2}\right) + \frac{f(\varrho_1)+f(\varrho_2)}{2}\right]. \tag{2}$$

Currently generalizations and applications of Bullen's inequality are extensively investigated in the framework of convex function [10–14]. But in much smaller extent, Bullen's inequality is also considered in the framework of numerical integration. Bullen's inequality corresponds to a convex combination of the midpoint and the trapezoidal rules having weights $\frac{1}{2}$. Hammer [15] proved an alternate version of Bullen's inequality, which says that for the convex integrands the absolute value of error in the midpoint quadrature rule is always less than the absolute value of the error in the trapezoidal rule. That is, the midpoint formula is always more accurate than the trapezoidal formula for any convex function.

Bullen's inequality is a refined form of the renowned Hermite–Hadamard inequality. These inequalities are of essential interest in numerical quadratures. We may note that Bullen's inequality should be considered as an extension of the right-hand side of the Hermite–Hadamard inequality. This follows immediately by applying the second inequality in (1) twice on the intervals $[\varrho_1, \frac{\varrho_1+\varrho_2}{2}]$ and $[\frac{\varrho_1+\varrho_2}{2}, \varrho_2]$ and adding the resulting inequalities. The primary goal of the paper is the establishment of fractional Bullen-type inequalities. This paper is organized as follows. After this Introduction, in Sect. 2 some basic concepts are discussed; in Sect. 3 some results related to the topic are discussed; in Sect. 4 examples to ensure the validity of the derived results are discussed. In Sect. 5, applications to the quadrature rules and f-divergence measure are discussed.

2 Preliminaries

This section deals with some basic definitions that are used.

Definition 1 ([16]) Let $I \subseteq (0,\infty)$ be a real interval and $p \in \mathbf{R}\backslash\{0\}$. A function $f : I \to \mathbf{R}$ is said to be p-convex if

$$f\left(\sqrt[p]{tx^p + (1-t)y^p}\right) \leq tf(x) + (1-t)f(y); \quad \forall x, y \in I, t \in [0,1].$$

Definition 2 ([17]) Let $f \in L^1([\varrho_1, \varrho_2])$. The Riemann–Liouville integrals $J^{\beta}_{\varrho_1^+} f$ and $J^{\beta}_{\varrho_2^-} f$ of order $\beta > 0$ with $\varrho_1 \geq 0$ are defined by

$$J^{\beta}_{\varrho_1^+} f(\xi) := \frac{1}{\Gamma(\beta)}\int_{\varrho_1}^{\xi} (\xi - t)^{\beta-1} f(t)\,dt, \quad \xi > \varrho_1, \tag{3}$$

and

$$J_{\varrho_2^-}^{\beta} f(\xi) := \frac{1}{\Gamma(\beta)} \int_{\xi}^{\varrho_2} (t-\xi)^{\beta-1} f(t)\,dt, \qquad \xi < \varrho_2, \tag{4}$$

respectively, where $\Gamma(\beta) = \int_0^{\infty} e^{-u} u^{\beta-1}\,du$.

Definition 3 ([18]) Raina defined a class of functions formally by

$$\mathfrak{F}_{\rho,\beta}^{\sigma}(\xi) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\beta)} \xi^k; \qquad \rho, \beta > 0, \xi \in \mathbf{R},$$

where the coefficient $\sigma(k)$ ($k \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$) is a bounded sequence of positive real numbers.

Definition 4 ([17]) The left-sided and the right-sided fractional integral operators, named Raina's fractional integrals, are defined respectively by

$$\bigl(\mathfrak{J}_{\rho,\beta,\varrho_1+;w}^{\sigma} f\bigr)(\xi) := \int_{\varrho_1}^{\xi} (\xi-t)^{\beta-1} \mathfrak{F}_{\rho,\beta}^{\sigma}\bigl[w(\xi-t)^{\rho}\bigr] f(t)\,dt, \qquad \xi > \varrho_1 > 0, \tag{5}$$

$$\bigl(\mathfrak{J}_{\rho,\beta,\varrho_2-;w}^{\sigma} f\bigr)(\xi) := \int_{\xi}^{\varrho_2} (t-\xi)^{\beta-1} \mathfrak{F}_{\rho,\beta}^{\sigma}\bigl[w(t-\xi)^{\rho}\bigr] f(t)\,dt, \qquad 0 < \xi < \varrho_2, \tag{6}$$

where $\beta, \rho > 0$, $w \in \mathbf{R}$ and $f(t)$ is such that the integrals on the right-hand side exist. For $\sigma(0) \to 1$ and $w \to 0$ in (5) and (6), respectively, then we get (3) and (4).

3 Main results

For establishing our generalized fractional Bullen-type inequalities, the following lemma plays a significant role.

Lemma 1 *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, the interior of I, $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2$; $p, \rho, \beta > 0$; let $g(x) = \sqrt[p]{x}$, $x > 0$; $u, w, \gamma \in \mathbf{R}$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\Omega(f,\varrho_1,\varrho_2;u) \\
&\quad := \bigl(\varrho_2^p - \varrho_1^p\bigr)\biggl\{(1-\lambda)^{1+\beta}(1-\gamma)\int_0^1 \bigl\{t^{\beta}\mathfrak{F}_{\rho,\beta+1}^{\sigma}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}t^{\rho}\bigr]-u\bigr\} \\
&\qquad \times \bigl[(1-t)\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+t\varrho_1^p\bigr]^{\frac{1-p}{p}} f'\Bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+t\varrho_1^p}\Bigr)\,dt \\
&\qquad + \gamma\lambda^{1+\beta}\int_0^1 \bigl\{t^{\beta}\mathfrak{F}_{\rho,\beta+1}^{\sigma}\bigl[w\bigl(\lambda\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}t^{\rho}\bigr]-u\bigr\} \\
&\qquad \times \bigl[t\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+(1-t)\varrho_2^p\bigr]^{\frac{1-p}{p}} \\
&\qquad \times f'\Bigl(\sqrt[p]{t\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+(1-t)\varrho_2^p}\Bigr)\,dt\biggr\} \\
&\quad = -p\bigl[(1-\lambda)^{\beta}(1-\gamma)\bigl[\bigl\{\mathfrak{F}_{\rho,\beta+1}^{\sigma}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}\bigr]-u\bigr\}f(\varrho_1) \\
&\qquad + uf\Bigl(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\Bigr)\bigr] + \lambda^{\beta}\gamma\bigl[\bigl\{\mathfrak{F}_{\rho,\beta+1}^{\sigma}\bigl[w\bigl(\lambda\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}\bigr]-u\bigr\}
\end{aligned}$$

$$\times f\bigl(\sqrt[p]{\lambda \varrho_1^p + (1-\lambda)\varrho_2^p}\bigr) + uf(\varrho_2)\bigr]\bigr] + \frac{p}{(\varrho_2^p - \varrho_1^p)^\beta}\bigl[(1-\gamma)$$
$$\times \bigl(\mathfrak{J}^\sigma_{\rho,\beta,\varrho_1^p+;w} f\circ g\bigr)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + \gamma\bigl(\mathfrak{J}^\sigma_{\rho,\beta,[\lambda\varrho_1^p+(1-\lambda)\varrho_2^p]+;w} f\circ g\bigr)\bigl(\varrho_2^p\bigr)\bigr]. \tag{7}$$

Proof Integrating by parts

$$\begin{aligned}
I_1 &:= \int_0^1 \bigl\{t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr)^\rho t^\rho\bigr] - u\bigr\}\bigl[(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p\bigr]^{\frac{1-p}{p}} \\
&\quad \times f'\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p}\bigr)\,dt \\
&= \left|\frac{pt^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w((1-\lambda)(\varrho_2^p - \varrho_1^p))^\rho t^\rho] - pu}{(\lambda-1)(\varrho_2^p - \varrho_1^p)} f\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p}\bigr)\right|_0^1 \\
&\quad - \int_0^1 \frac{pt^{\beta-1}\mathfrak{F}^\sigma_{\rho,\beta}[w((1-\lambda)(\varrho_2^p - \varrho_1^p))^\rho t^\rho]}{(\lambda-1)(\varrho_2^p - \varrho_1^p)} f\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p}\bigr)\,dt \\
&= \frac{p}{(\lambda-1)(\varrho_2^p - \varrho_1^p)}\bigl[\bigl\{\mathfrak{F}^\sigma_{\rho,\beta+1}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr)^\rho\bigr] - u\bigr\}f(\varrho_1) \\
&\quad + uf\bigl(\sqrt[p]{\lambda\varrho_1^p + (1-\lambda)\varrho_2^p}\bigr)\bigr] - \int_0^1 \frac{pt^{\beta-1}\mathfrak{F}^\sigma_{\rho,\beta}[w((1-\lambda)(\varrho_2^p - \varrho_1^p))^\rho t^\rho]}{(\lambda-1)(\varrho_2^p - \varrho_1^p)} \\
&\quad \times f\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p}\bigr)\,dt,
\end{aligned}$$

setting $x \to (1-t)(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p) + t\varrho_1^p$, so that $dx \to (\lambda-1)(\varrho_2^p - \varrho_1^p)\,dt$ and $0 \le t \le 1 \Leftrightarrow \lambda\varrho_1^p + (1-\lambda)\varrho_2^p \le x \le \varrho_1^p$, we have

$$\begin{aligned}
I_1 &= -\frac{p}{(1-\lambda)(\varrho_2^p - \varrho_1^p)}\bigl[\bigl\{\mathfrak{F}^\sigma_{\rho,\beta+1}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr)^\rho\bigr] - u\bigr\}f(\varrho_1) \\
&\quad + uf\bigl(\sqrt[p]{\lambda\varrho_1^p + (1-\lambda)\varrho_2^p}\bigr)\bigr] - \frac{p}{[(1-\lambda)(\varrho_2^p - \varrho_1^p)]^{1+\beta}} \\
&\quad \times \int_{\varrho_1^p}^{\lambda\varrho_1^p + (1-\lambda)\varrho_2^p} \bigl[\lambda\varrho_1^p + (1-\lambda)\varrho_2^p - x\bigr]^{\beta-1} \\
&\quad \times \mathfrak{F}^\sigma_{\rho,\beta}\bigl[w\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p - x\bigr)^\rho\bigr](f\circ g)(x)\,dx \\
\Rightarrow\ & \frac{[(1-\lambda)(\varrho_2^p - \varrho_1^p)]^{1+\beta}}{p} I_1 \\
&= -\bigl[(1-\lambda)\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr]^\beta\bigl[\bigl\{\mathfrak{F}^\sigma_{\rho,\beta+1}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr)^\rho\bigr] - u\bigr\}f(\varrho_1) \\
&\quad + uf\bigl(\sqrt[p]{\lambda\varrho_1^p + (1-\lambda)\varrho_2^p}\bigr)\bigr] + \bigl(\mathfrak{J}^\sigma_{\rho,\beta,\varrho_1^p+;w} f\circ g\bigr)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr).
\end{aligned} \tag{8}$$

Again, integrating by parts

$$\begin{aligned}
I_2 &:= \int_0^1 \bigl\{t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}\bigl[w\bigl(\lambda\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr)^\rho t^\rho\bigr] - u\bigr\}\bigl[\bigl(t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p\bigr]^{\frac{1-p}{p}} \\
&\quad \times f'\bigl(\sqrt[p]{t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p}\bigr)\,dt \\
&= \left|\frac{pt^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(\lambda(\varrho_2^p - \varrho_1^p))^\rho t^\rho] - pu}{\lambda(\varrho_1^p - \varrho_2^p)} f\bigl(\sqrt[p]{t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p}\bigr)\right|_0^1
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \frac{pt^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta}[w(\lambda(\varrho_2^p-\varrho_1^p))^{\rho}t^{\rho}]}{\lambda(\varrho_1^p-\varrho_2^p)} f\left(\sqrt[p]{t(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p)+(1-t)\varrho_2^p}\right)dt\\
&= \frac{p}{\lambda(\varrho_1^p-\varrho_2^p)}\left[\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\lambda\left(\varrho_2^p-\varrho_1^p\right)\right)^{\rho}\right]-u\right\} f\left(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\right)+uf(\varrho_2)\right]\\
&-\int_0^1 \frac{pt^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta}[w(\lambda(\varrho_2^p-\varrho_1^p))^{\rho}t^{\rho}]}{\lambda(\varrho_1^p-\varrho_2^p)} f\left(\sqrt[p]{t(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p)+(1-t)\varrho_2^p}\right)dt,
\end{aligned}
$$

setting $y \to t(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p)+(1-t)\varrho_2^p$, so that $dy \to \lambda(\varrho_1^p-\varrho_2^p)\,dt$ and $0 \le t \le 1 \Leftrightarrow \lambda\varrho_1^p+(1-\lambda)\varrho_2^p \ge y \ge \varrho_2^p$, we have

$$
\begin{aligned}
I_2 &= \frac{p}{\lambda(\varrho_1^p-\varrho_2^p)}\left[\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\lambda\left(\varrho_2^p-\varrho_1^p\right)\right)^{\rho}\right]-u\right\} f\left(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\right)+uf(\varrho_2)\right]\\
&\quad+\frac{p}{[\lambda(\varrho_2^p-\varrho_1^p)]^{1+\beta}}\int_{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}^{\varrho_2^p}\left[\varrho_2^p-y\right]^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta}\left[w\left(\varrho_2^p-y\right)^{\rho}\right](f\circ g)(y)\,dy.\\
&\Rightarrow \frac{[\lambda(\varrho_2^p-\varrho_1^p)]^{1+\beta}}{p}I_2 = -\left[\lambda\left(\varrho_2^p-\varrho_1^p\right)\right]^{\beta}\left[\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\lambda\left(\varrho_2^p-\varrho_1^p\right)\right)^{\rho}\right]-u\right\}\right.\\
&\quad\times f\left(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\right)+uf(\varrho_2)\Big]+\left(\mathfrak{J}^{\sigma}_{\rho,\beta,[\lambda\varrho_1^p+(1-\lambda)\varrho_2^p]+;w}f\circ g\right)\left(\varrho_2^p\right). \qquad (9)
\end{aligned}
$$

Multiplying (8) by $1-\gamma$ and (9) by γ and then adding the resulting equalities yields the desired identity (7). □

Proposition 1 *Let $f: I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, the interior of I, $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2$; $p, \rho, \beta > 0$; let $g(x) = \sqrt[p]{x}$, $x > 0$; $u, w \in \mathbf{R}$ and $\lambda \in [0,1]$, then*

$$
\begin{aligned}
&\lambda^{1+\beta}\left(\varrho_2^p-\varrho_1^p\right)\int_0^1\left\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\lambda\left(\varrho_2^p-\varrho_1^p\right)\right)^{\rho}t^{\rho}\right]-u\right\}\\
&\quad\times\left[t\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+(1-t)\varrho_2^p\right]^{\frac{1-p}{p}}\\
&\quad\times f'\left(\sqrt[p]{t\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+(1-t)\varrho_2^p}\right)dt\\
&= -p\lambda^{\beta}\left[\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\lambda\left(\varrho_2^p-\varrho_1^p\right)\right)^{\rho}\right]-u\right\}f\left(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\right)+uf(\varrho_2)\right]\\
&\quad+\frac{p}{(\varrho_2^p-\varrho_1^p)^{\beta}}\left(\mathfrak{J}^{\sigma}_{\rho,\beta,[\lambda\varrho_1^p+(1-\lambda)\varrho_2^p]+;w}f\circ g\right)\left(\varrho_2^p\right).
\end{aligned}
$$

Remark 1

- Lemma 1 is the generalization of [19, Lemma 1.2]. For $\gamma, \lambda, u \to \frac{1}{2}$; $p, \beta, \sigma(0) \to 1$; $w \to 0$, Lemma 1 reduces to [19, Lemma 1.2].
- Proposition 1 is the generalization of [19, Lemma 1.1]. For $p, \beta, \lambda, \sigma(0) \to 1$; $u \to \frac{1}{2}$; $w \to 0$, it reduces to [19, Lemma 1.1].

We make the following assumptions before going to our main results:

$$
A_{k,s,j} := \sqrt[s]{\frac{(\beta+\rho k+1)^{j-1}(\lambda|f'(\varrho_1)|^s+(1-\lambda)|f'(\varrho_2)|^s)+(\beta+\rho k+1)^{2-j}|f'(\varrho_j)|^s}{\beta+\rho k+2}},
$$

$$
B_{s,j} := (2-j+\lambda)\left|f'(\varrho_1)\right|^s+(j-\lambda)\left|f'(\varrho_2)\right|^s.
$$

Theorem 1 *Let $f : I \subseteq \mathbf{R}^{+} \to \mathbf{R}$ be a differentiable function on I°, the interior of I. If $f' \in L^{1}[\varrho_1, \varrho_2]$ and $|f'|$ is p-convex on $[\varrho_1, \varrho_2]$, where $\varrho_1, \varrho_2 \in I^{\circ}$ with $\varrho_1 < \varrho_2$; $p, \rho, \beta > 0$; let $g(x) = \sqrt[p]{x}$, $x > 0$; $u, w, \gamma \in \mathbf{R}$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\bigl|\Omega(f,\varrho_1,\varrho_2;u)\bigr| \\
&\leq \begin{cases}
\varrho_1^{1-p}(\varrho_2^p-\varrho_1^p)[(1-\lambda)^{1+\beta}|1-\gamma| \\
\quad\times\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,1,1}+\frac{|u|}{2}B_{1,1}\} \\
\quad+|\gamma|\lambda^{1+\beta}\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,1,2}+\frac{|u|}{2}B_{1,2}\}], \quad p\in[1,\infty); \\
\varrho_2^{1-p}(\varrho_2^p-\varrho_1^p)[(1-\lambda)^{1+\beta}|1-\gamma| \\
\quad\times\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,1,1}+\frac{|u|}{2}B_{1,1}\} \\
\quad+|\gamma|\lambda^{1+\beta}\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,1,2}+\frac{|u|}{2}B_{1,2}\}], \quad p\in(0,1).
\end{cases}
\end{aligned} \tag{10}$$

Proof By the properties of modulus and repeated application of p-convexity to Lemma 1 for the case $p \in [1,\infty)$, we have the following:

$$\begin{aligned}
&\bigl|\Omega(f,\varrho_1,\varrho_2;u)\bigr| \\
&= \Bigl|-p\bigl[(1-\lambda)^{\beta}(1-\gamma)\bigl[\bigl\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}\bigr]-u\bigr\}f(\varrho_1) \\
&\quad+uf\bigl(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\bigr)\bigr]+\lambda^{\beta}\gamma\bigl[\bigl\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\bigl[w\bigl(\lambda\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}\bigr]-u\bigr\} \\
&\quad\times f\bigl(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\bigr)+uf(\varrho_2)\bigr]\bigr]+\frac{p}{(\varrho_2^p-\varrho_1^p)^{\beta}}\bigl[(1-\gamma) \\
&\quad\times\bigl(\mathfrak{J}^{\sigma}_{\rho,\beta,\varrho_1^p+;w}f\circ g\bigr)\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+\gamma\bigl(\mathfrak{J}^{\sigma}_{\rho,\beta,[\lambda\varrho_1^p+(1-\lambda)\varrho_2^p]+;w}f\circ g\bigr)\bigl(\varrho_2^p\bigr)\bigr]\Bigr| \\
&= \Bigl|\bigl(\varrho_2^p-\varrho_1^p\bigr)\Bigl[(1-\lambda)^{1+\beta}(1-\gamma)\int_0^1\bigl\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}t^{\rho}\bigr]-u\bigr\} \\
&\quad\times\bigl[(1-t)\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+t\varrho_1^p\bigr]^{\frac{1-p}{p}} \\
&\quad\times f'\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+t\varrho_1^p}\bigr)\,dt \\
&\quad+\gamma\lambda^{1+\beta}\int_0^1\bigl\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\bigl[w\bigl(\lambda\bigl(\varrho_2^p-\varrho_1^p\bigr)\bigr)^{\rho}t^{\rho}\bigr]-u\bigr\} \\
&\quad\times\bigl[t\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+(1-t)\varrho_2^p\bigr]^{\frac{1-p}{p}} \\
&\quad\times f'\bigl(\sqrt[p]{t\bigl(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\bigr)+(1-t)\varrho_2^p}\bigr)\,dt\Bigr]\Bigr| \\
&\leq \bigl(\varrho_2^p-\varrho_1^p\bigr)\Bigl[(1-\lambda)^{1+\beta}|1-\gamma|\int_0^1\Bigl|\sum_{k=0}^{\infty}\frac{\sigma(k)w^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}-u\Bigr| \\
&\quad\times\varrho_1^{1-p}\bigl\{(1-t)\bigl|f'\bigl(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\bigr)\bigr|+t\bigl|f'(\varrho_1)\bigr|\bigr\}\,dt \\
&\quad+|\gamma|\lambda^{1+\beta}\int_0^1\Bigl|\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}-u\Bigr|
\end{aligned}$$

$$
\begin{aligned}
&\times \varrho_1^{1-p}\left\{t\left|f'\left(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\right)\right|+(1-t)\left|f'(\varrho_2)\right|\right\}dt\Bigg] \\
\le{}& \left(\varrho_2^p-\varrho_1^p\right)\varrho_1^{1-p}\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \int_0^1 t^{\beta+\rho k}\left\{(1-t)\left(\lambda\left|f'(\varrho_1)\right|+(1-\lambda)\left|f'(\varrho_2)\right|\right)+t\left|f'(\varrho_1)\right|\right\}dt \\
&+|u|\int_0^1\left\{(1-t)\left(\lambda\left|f'(\varrho_1)\right|+(1-\lambda)\left|f'(\varrho_2)\right|\right)+t\left|f'(\varrho_1)\right|\right\}dt\Bigg\} \\
&+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \int_0^1 t^{\beta+\rho k}\left\{t\left(\lambda\left|f'(\varrho_1)\right|+(1-\lambda)\left|f'(\varrho_2)\right|\right)+(1-t)\left|f'(\varrho_2)\right|\right\}dt \\
&+|u|\int_0^1\left\{t\left(\lambda\left|f'(\varrho_1)\right|+(1-\lambda)\left|f'(\varrho_2)\right|\right)+(1-t)\left|f'(\varrho_2)\right|\right\}dt\Bigg\}\Bigg] \\
={}& \left(\varrho_2^p-\varrho_1^p\right)\varrho_1^{1-p}\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times\left(\frac{\lambda|f'(\varrho_1)|+(1-\lambda)|f'(\varrho_2)|}{(\beta+\rho k+1)(\beta+\rho k+2)}+\frac{|f'(\varrho_1)|}{\beta+\rho k+2}\right) \\
&+|u|\left(\frac{\lambda|f'(\varrho_1)|+(1-\lambda)|f'(\varrho_2)|}{2}+\frac{|f'(\varrho_1)|}{2}\right)\Bigg\} \\
&+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times\left(\frac{\lambda|f'(\varrho_1)|+(1-\lambda)|f'(\varrho_2)|}{\beta+\rho k+2}+\frac{|f'(\varrho_2)|}{(\beta+\rho k+1)(\beta+\rho k+2)}\right) \\
&+|u|\left(\frac{\lambda|f'(\varrho_1)|+(1-\lambda)|f'(\varrho_2)|}{2}+\frac{|f'(\varrho_2)|}{2}\right)\Bigg\}\Bigg] \\
={}& \left(\varrho_2^p-\varrho_1^p\right)\varrho_1^{1-p}\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times\left(\frac{\lambda|f'(\varrho_1)|+(1-\lambda)|f'(\varrho_2)|+(\beta+\rho k+1)|f'(\varrho_1)|}{(\beta+\rho k+1)(\beta+\rho k+2)}\right) \\
&+\frac{|u|}{2}\left((1+\lambda)\left|f'(\varrho_1)\right|+(1-\lambda)\left|f'(\varrho_2)\right|\right)\Bigg\} \\
&+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times\left(\frac{(\beta+\rho k+1)(\lambda|f'(\varrho_1)|+(1-\lambda)|f'(\varrho_2)|)+|f'(\varrho_2)|}{(\beta+\rho k+1)(\beta+\rho k+2)}\right) \\
&+\frac{|u|}{2}\left(\lambda\left|f'(\varrho_1)\right|+(2-\lambda)\left|f'(\varrho_2)\right|\right)\Bigg\}\Bigg].
\end{aligned}
$$

Now, for the case $p \in (0,1)$, using the fact

$$\begin{aligned}&\big[(1-t)\big(\lambda \varrho_1^p + (1-\lambda)\varrho_2^p\big) + t\varrho_1^p\big]^{\frac{1-p}{p}}\\&\quad\le \big[(1-t)\big(\lambda \varrho_2^p + (1-\lambda)\varrho_2^p\big) + t\varrho_2^p\big]^{\frac{1-p}{p}} \le \varrho_2^{1-p}\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\big[t\big(\lambda \varrho_1^p + (1-\lambda)\varrho_2^p\big) + (1-t)\varrho_2^p\big]^{\frac{1-p}{p}}\\&\quad\le \big[t\big(\lambda \varrho_2^p + (1-\lambda)\varrho_2^p\big) + (1-t)\varrho_2^p\big]^{\frac{1-p}{p}} \le \varrho_2^{1-p}\end{aligned} \tag{12}$$

and applying the same procedure like $p \in [1,\infty)$, we obtain the desired result. □

Corollary 1 *Under the conditions of Theorem 1, for* $\sigma(0), p, \beta \to 1$, $\lambda, \gamma, u \to \frac{1}{2}$, *and* $w \to 0$, *we have*

$$\begin{aligned}&\left|\frac{1}{2}\left\{\frac{f(\varrho_1)+f(\varrho_2)}{2} + f\left(\frac{\varrho_1+\varrho_2}{2}\right)\right\} - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(x)\,dx\right|\\&\quad\le \frac{\varrho_2-\varrho_1}{48}\big[13\big|f'(\varrho_1)\big| + 11\big|f'(\varrho_2)\big|\big]\\&\quad\le \frac{(\varrho_2-\varrho_1)(|f'(\varrho_1)| + |f'(\varrho_2)|)}{2}.\end{aligned}$$

Theorem 2 *Let* $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ *be a differentiable function on* I°, *the interior of* I. *If* $f' \in L^1[\varrho_1,\varrho_2]$ *and* $|f'|^s$ *is p-convex on* $[\varrho_1,\varrho_2]$, *where* $\varrho_1, \varrho_2 \in I^\circ$ *with* $\varrho_1 < \varrho_2$; $p, \rho, \beta > 0$; $s \ge 1$; *let* $g(x) = \sqrt[p]{x}$, $x > 0$; $u, w, \gamma \in \mathbf{R}$ *and* $\lambda \in [0,1]$, *then*

$$\begin{aligned}&\big|\Omega(f,\varrho_1,\varrho_2;u)\big|\\&\quad\le \begin{cases}\varrho_1^{1-p}(\varrho_2^p-\varrho_1^p)[(1-\lambda)^{1+\beta}|1-\gamma|\\\quad\times\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,s,1} + |u|(\frac{B_{s,1}}{2})^{\frac{1}{s}}\}\\\quad+\lambda^{1+\beta}|\gamma|\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,s,2} + |u|(\frac{B_{s,2}}{2})^{\frac{1}{s}}\}], & p \in [1,\infty);\\\varrho_2^{1-p}(\varrho_2^p-\varrho_1^p)[(1-\lambda)^{1+\beta}|1-\gamma|\\\quad\times\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,s,1} + |u|(\frac{B_{s,1}}{2})^{\frac{1}{s}}\}\\\quad+\lambda^{1+\beta}|\gamma|\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}A_{k,s,2} + |u|(\frac{B_{s,2}}{2})^{\frac{1}{s}}\}], & p \in (0,1).\end{cases}\end{aligned}$$

Proof By the properties of modulus, power-mean inequality, and repeated application of p-convexity to Lemma 1 for the case $p \in [1,\infty)$, we obtain the following:

$$\begin{aligned}&\big|\Omega(f,\varrho_1,\varrho_2;u)\big|\\&\quad= \Big|-p\big[(1-\lambda)^\beta(1-\gamma)\big[\big\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\big[w\big((1-\lambda)\big(\varrho_2^p-\varrho_1^p\big)\big)^\rho\big] - u\big\}f(\varrho_1)\\&\qquad + uf\big(\sqrt[p]{\lambda\varrho_1^p + (1-\lambda)\varrho_2^p}\big)\big] + \lambda^\beta\gamma\big[\big\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\big[w\big(\lambda\big(\varrho_2^p-\varrho_1^p\big)\big)^\rho\big] - u\big\}\end{aligned}$$

$$
\begin{aligned}
&\times f\bigl(\sqrt[p]{\lambda \varrho_1^p + (1-\lambda)\varrho_2^p}\bigr) + uf(\varrho_2)\bigr]\bigr] + \frac{p}{(\varrho_2^p - \varrho_1^p)^\beta}\bigl[(1-\gamma) \\
&\times \bigl(\mathfrak{J}^\sigma_{\rho,\beta,\varrho_1^p+;w} f\circ g\bigr)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + \gamma\bigl(\mathfrak{J}^\sigma_{\rho,\beta,[\lambda\varrho_1^p+(1-\lambda)\varrho_2^p]+;w} f\circ g\bigr)\bigl(\varrho_2^p\bigr)\bigr]\Bigr| \\
= &\Bigl|\bigl(\varrho_2^p - \varrho_1^p\bigr)\Bigl[(1-\lambda)^{1+\beta}(1-\gamma) \\
&\times \int_0^1 \bigl\{t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}\bigl[w\bigl((1-\lambda)\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr)^\rho t^\rho\bigr] - u\bigr\} \\
&\times \bigl[(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p\bigr]^{\frac{1-p}{p}} \\
&\times f'\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p}\bigr)\,dt \\
&+ \gamma\lambda^{1+\beta}\int_0^1 \bigl\{t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}\bigl[w\bigl(\lambda\bigl(\varrho_2^p - \varrho_1^p\bigr)\bigr)^\rho t^\rho\bigr] - u\bigr\} \\
&\times \bigl[t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p\bigr]^{\frac{1-p}{p}} \\
&\times f'\bigl(\sqrt[p]{t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p}\bigr)\,dt\Bigr]\Bigr| \\
\le &\bigl(\varrho_2^p - \varrho_1^p\bigr)\Biggl[(1-\lambda)^{1+\beta}|1-\gamma|\Biggl\{\sum_{k=0}^\infty \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \int_0^1 t^{\beta+\rho k}\bigl[(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p\bigr]^{\frac{1-p}{p}} \\
&\times \bigl|f'\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p}\bigr)\bigr|\,dt \\
&+ |u|\int_0^1 \bigl[(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p\bigr]^{\frac{1-p}{p}} \\
&\times \bigl|f'\bigl(\sqrt[p]{(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p}\bigr)\bigr|\,dt\Biggr\} \\
&+ |\gamma|\lambda^{1+\beta}\Biggl\{\sum_{k=0}^\infty \frac{\sigma(k)|w|^k(\lambda(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \int_0^1 t^{\beta+\rho k}\bigl[t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p\bigr]^{\frac{1-p}{p}} \\
&\times \bigl|f'\bigl(\sqrt[p]{t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p}\bigr)\bigr|\,dt \\
&+ |u|\int_0^1 \bigl[t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p\bigr]^{\frac{1-p}{p}} \\
&\times \bigl|f'\bigl(\sqrt[p]{t\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + (1-t)\varrho_2^p}\bigr)\bigr|\,dt\Biggr\}\Biggr] \\
\le &\bigl(\varrho_2^p - \varrho_1^p\bigr)\Biggl[(1-\lambda)^{1+\beta}|1-\gamma|\Biggl\{\sum_{k=0}^\infty \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \biggl(\int_0^1 t^{\beta+\rho k}\bigl[(1-t)\bigl(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\bigr) + t\varrho_1^p\bigr]^{\frac{1-p}{p}}\,dt\biggr)^{1-\frac{1}{s}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\int_0^1 t^{\beta+\rho k}\big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}}\right.\\
&\left.\times \big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|^s\,dt\right]^{\frac{1}{s}}\\
&+|u|\left(\int_0^1 \big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}}\,dt\right)^{1-\frac{1}{s}}\\
&\times \left[\int_0^1 \big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}}\right.\\
&\left.\left.\times \big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|^s\,dt\right]^{\frac{1}{s}}\right\}\\
&+|\gamma|\lambda^{1+\beta}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\right.\\
&\times \left(\int_0^1 t^{\beta+\rho k}\big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}}\,dt\right)^{1-\frac{1}{s}}\\
&\times \left[\int_0^1 t^{\beta+\rho k}\big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}}\right.\\
&\left.\times \big|f'\big(\sqrt[p]{t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p}\big)\big|^s\,dt\right]^{\frac{1}{s}}\\
&+|u|\left(\int_0^1 \big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}}\,dt\right)^{1-\frac{1}{s}}\\
&\times \left(\int_0^1 \big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}}\right.\\
&\left.\left.\left.\times \big|f'\big(\sqrt[p]{t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p}\big)\big|^s\,dt\right]^{\frac{1}{s}}\right\}\right]\\
\le{}& \varrho_1^{1-p}\big(\varrho_2^p-\varrho_1^p\big)\left[(1-\lambda)^{1+\beta}|1-\gamma|\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\right.\right.\\
&\times \left[\int_0^1 t^{\beta+\rho k}\,dt\right]^{1-\frac{1}{s}}\\
&\times \left[\int_0^1 t^{\beta+\rho k}\big\{(1-t)\big(\lambda\big|f'(\varrho_1)\big|^s+(1-\lambda)\big|f'(\varrho_2)\big|^s\big)+t\big|f'(\varrho_1)\big|^s\big\}\,dt\right]^{\frac{1}{s}}\\
&\left.+|u|\left(\int_0^1 \big\{(1-t)\big(\lambda\big|f'(\varrho_1)\big|^s+(1-\lambda)\big|f'(\varrho_2)\big|^s\big)+t\big|f'(\varrho_1)\big|^s\big\}\,dt\right)^{\frac{1}{s}}\right\}\\
&+|\gamma|\lambda^{1+\beta}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\right.\\
&\times \left[\int_0^1 t^{\beta+\rho k}\,dt\right]^{1-\frac{1}{s}}
\end{aligned}
$$

$$\times \left[\int_0^1 t^{\beta+\rho k}\left\{t\left(\lambda\left|f'(\varrho_1)\right|^s + (1-\lambda)\left|f'(\varrho_2)\right|^s\right) + (1-t)\left|f'(\varrho_2)\right|^s\right\} dt\right]^{\frac{1}{s}}$$

$$+ |u|\left(\int_0^1 \left\{t\left(\lambda\left|f'(\varrho_1)\right|^s + (1-\lambda)\left|f'(\varrho_2)\right|^s\right) + (1-t)\left|f'(\varrho_2)\right|^s\right\} dt\right)^{\frac{1}{s}}\Bigg\}\Bigg]$$

$$= \varrho_1^{1-p}\left(\varrho_2^p - \varrho_1^p\right)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)(\beta+\rho k+1)^{1-\frac{1}{s}}}$$

$$\times \left(\frac{\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s}{(\beta+\rho k+1)(\beta+\rho k+2)} + \frac{|f'(\varrho_1)|^s}{\beta+\rho k+2}\right)^{\frac{1}{s}}$$

$$+ \frac{|u|}{2^{\frac{1}{s}}}\left\{\lambda\left|f'(\varrho_1)\right|^s + (1-\lambda)\left|f'(\varrho_2)\right|^s + \left|f'(\varrho_1)\right|^s\right\}^{\frac{1}{s}}\Bigg\}$$

$$+ |\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)(\beta+\rho k+1)^{1-\frac{1}{s}}}$$

$$\times \left\{\frac{\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s}{\beta+\rho k+2} + \frac{|f'(\varrho_2)|^s}{(\beta+\rho k+1)(\beta+\rho k+2)}\right\}^{\frac{1}{s}}$$

$$+ \frac{|u|}{2^{\frac{1}{s}}}\left\{\lambda\left|f'(\varrho_1)\right|^s + (1-\lambda)\left|f'(\varrho_2)\right|^s + \left|f'(\varrho_2)\right|^s\right\}^{\frac{1}{s}}\Bigg\}\Bigg]$$

$$= \varrho_1^{1-p}\left(\varrho_2^p - \varrho_1^p\right)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}$$

$$\times \left(\frac{\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s + (\beta+\rho k+1)|f'(\varrho_1)|^s}{\beta+\rho k+2}\right)^{\frac{1}{s}}$$

$$+ \frac{|u|}{2^{\frac{1}{s}}}\left\{(1+\lambda)\left|f'(\varrho_1)\right|^s + (1-\lambda)\left|f'(\varrho_2)\right|^s\right\}^{\frac{1}{s}}\Bigg\}$$

$$+ |\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}$$

$$\times \left(\frac{(\beta+\rho k+1)(\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s) + |f'(\varrho_2)|^s}{\beta+\rho k+2}\right)^{\frac{1}{s}}$$

$$+ \frac{|u|}{2^{\frac{1}{s}}}\left\{\lambda\left|f'(\varrho_1)\right|^s + (2-\lambda)\left|f'(\varrho_2)\right|^s\right\}^{\frac{1}{s}}\Bigg\}\Bigg].$$

Now, for the case $p \in (0,1)$, using inequalities (11) and (12), and repeating the same procedure like $p \in [1,\infty)$, we obtain the desired result. □

Corollary 2 *Under the conditions of Theorem 2 for* $\beta, u, s, p, \sigma(0) \to 1$, $\lambda \to \frac{1}{2}$, *and* $w, \gamma \to 0$, *we have*

$$\left|f\left(\frac{\varrho_1+\varrho_2}{2}\right) - \frac{2}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\frac{\varrho_1+\varrho_2}{2}} f(x)\,dx\right| \le \frac{\varrho_2-\varrho_1}{12}\left(7\left|f'(\varrho_1)\right| + 2\left|f'(\varrho_2)\right|\right).$$

Theorem 3 *Let* $f: I \subseteq \mathbf{R}^+ \to \mathbf{R}$ *be a differentiable function on* I°, *the interior of* I. *If* $f' \in L^1[\varrho_1,\varrho_2]$ *and* $|f'|^s$ *is p-convex on* $[\varrho_1,\varrho_2]$, *where* $\varrho_1, \varrho_2 \in I^\circ$ *with* $\varrho_1 < \varrho_2$; $p, \rho, \beta > 0$; $r, s > 1$

such that $\frac{1}{r}+\frac{1}{s}=1$; *let* $g(x)=\sqrt[p]{x}$, $x>0$; $u,w,\gamma\in\mathbf{R}$ *and* $\lambda\in[0,1]$, *then*

$$\begin{aligned}
&\left|\Omega(f,\varrho_1,\varrho_2;u)\right| \\
&\leq \begin{cases}
\frac{\varrho_1^{1-p}(\varrho_2^p-\varrho_1^p)}{\sqrt[s]{2}}[(1-\lambda)^{1+\beta}|1-\gamma|B_{s,1}^{\frac{1}{s}} \\
\quad\times\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)^{\frac{1}{r}}\Gamma(\rho k+\beta+1)}+|u|\} \\
\quad+|\gamma|\lambda^{1+\beta}B_{s,2}^{\frac{1}{s}}\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)^{\frac{1}{r}}\Gamma(\rho k+\beta+1)}+|u|\}], \quad p\in[1,\infty); \\
\frac{\varrho_2^{1-p}(\varrho_2^p-\varrho_1^p)}{\sqrt[s]{2}}[(1-\lambda)^{1+\beta}|1-\gamma|B_{s,1}^{\frac{1}{s}} \\
\quad\times\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)^{\frac{1}{r}}\Gamma(\rho k+\beta+1)}+|u|\} \\
\quad+|\gamma|\lambda^{1+\beta}B_{s,2}^{\frac{1}{s}}\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)^{\frac{1}{r}}\Gamma(\rho k+\beta+1)}+|u|\}], \quad p\in(0,1).
\end{cases}
\end{aligned}$$

Proof By the properties of modulus, Holder's inequality, and repeated application of p-convexity to Lemma 1 for the case $p\in[1,\infty)$, we have the following:

$$\begin{aligned}
&\left|\Omega(f,\varrho_1,\varrho_2;u)\right| \\
&=\Bigg|-p\big[(1-\lambda)^{\beta}(1-\gamma)\big[\big\{\mathfrak{F}_{\rho,\beta+1}^{\sigma}\big[w\big((1-\lambda)\big(\varrho_2^p-\varrho_1^p\big)\big)^{\rho}\big]-u\big\}f(\varrho_1) \\
&\quad+uf\big(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\big)\big]+\lambda^{\beta}\gamma\big[\big\{\mathfrak{F}_{\rho,\beta+1}^{\sigma}\big[w\big(\lambda\big(\varrho_2^p-\varrho_1^p\big)\big)^{\rho}\big]-u\big\} \\
&\quad\times f\big(\sqrt[p]{\lambda\varrho_1^p+(1-\lambda)\varrho_2^p}\big)+uf(\varrho_2)\big]\big]+\frac{p}{(\varrho_2^p-\varrho_1^p)^{\beta}}\big[(1-\gamma) \\
&\quad\times\big(\mathfrak{J}_{\rho,\beta,\varrho_1^p+;w}^{\sigma}f\circ g\big)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+\gamma\big(\mathfrak{J}_{\rho,\beta,[\lambda\varrho_1^p+(1-\lambda)\varrho_2^p]+;w}^{\sigma}f\circ g\big)\big(\varrho_2^p\big)\big]\Bigg| \\
&\leq\big(\varrho_2^p-\varrho_1^p\big)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\quad\times\int_0^1 t^{\beta+\rho k}\big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}} \\
&\quad\times\big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|\,dt \\
&\quad+|u|\int_0^1\big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}} \\
&\quad\times\big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|\,dt\Bigg\} \\
&\quad+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\quad\times\int_0^1 t^{\beta+\rho k}\big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}} \\
&\quad\times\big|f'\big(\sqrt[p]{t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p}\big)\big|\,dt \\
&\quad+|u|\int_0^1\big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left|f'\left(\sqrt[p]{t\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+(1-t)\varrho_2^p}\right)\right| dt\Bigg\}\Bigg]\\
&\le \left(\varrho_2^p-\varrho_1^p\right)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\\
&\quad\times\left(\int_0^1\left(t^{\beta+\rho k}\left[(1-t)\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+t\varrho_1^p\right]^{\frac{1-p}{p}}\right)^r dt\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_0^1\left|f'\left(\sqrt[p]{(1-t)\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+t\varrho_1^p}\right)\right|^s dt\right)^{\frac{1}{s}}\\
&\quad+|u|\left(\int_0^1\left[(1-t)\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+t\varrho_1^p\right]^{\frac{r(1-p)}{p}} dt\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_0^1\left|f'\left(\sqrt[p]{(1-t)\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+t\varrho_1^p}\right)\right|^s dt\right)^{\frac{1}{s}}\Bigg\}\\
&\quad+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\\
&\quad\times\left(\int_0^1 t^{\beta+\rho k}\left[t\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+(1-t)\varrho_2^p\right]^{\frac{r(1-p)}{p}} dt\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_0^1\left|f'\left(\sqrt[p]{t\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+(1-t)\varrho_2^p}\right)\right|^s dt\right)^{\frac{1}{s}}\\
&\quad+|u|\left(\int_0^1\left[t\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+(1-t)\varrho_2^p\right]^{\frac{r(1-p)}{p}} dt\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_0^1\left|f'\left(\sqrt[p]{t\left(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\right)+(1-t)\varrho_2^p}\right)\right|^s dt\right)^{\frac{1}{s}}\Bigg\}\Bigg]\\
&\le \varrho_1^{1-p}\left(\varrho_2^p-\varrho_1^p\right)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\\
&\quad\times\left[\int_0^1 t^{r(\beta+\rho k)}\,dt\right]^{\frac{1}{r}}\left[\int_0^1\left[(1-t)\left\{\lambda\left|f'(\varrho_1)\right|^s+(1-\lambda)\left|f'(\varrho_2)\right|^s\right\}+t\left|f'(\varrho_1)\right|^s\right]dt\right]^{\frac{1}{s}}\\
&\quad+|u|\left(\int_0^1\left[(1-t)\left\{\lambda\left|f'(\varrho_1)\right|^s+(1-\lambda)\left|f'(\varrho_2)\right|^s\right\}+t\left|f'(\varrho_1)\right|^s\right]dt\right)^{\frac{1}{s}}\Bigg\}\\
&\quad+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\left[\int_0^1 t^{r(\beta+\rho k)}\,dt\right]^{\frac{1}{r}}\\
&\quad\times\left[\int_0^1\left[t\left\{\lambda\left|f'(\varrho_1)\right|^s+(1-\lambda)\left|f'(\varrho_2)\right|^s\right\}+(1-t)\left|f'(\varrho_2)\right|^s\right]dt\right]^{\frac{1}{s}}\\
&\quad+|u|\left(\int_0^1\left[t\left\{\lambda\left|f'(\varrho_1)\right|^s+(1-\lambda)\left|f'(\varrho_2)\right|^s\right\}+(1-t)\left|f'(\varrho_2)\right|^s\right]dt\right)^{\frac{1}{s}}\Bigg\}\Bigg]\\
&=\frac{1}{2^{\frac{1}{s}}}\varrho_1^{1-p}\left(\varrho_2^p-\varrho_1^p\right)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\left((1+\lambda)\left|f'(\varrho_1)\right|^s+(1-\lambda)\left|f'(\varrho_2)\right|^s\right)^{\frac{1}{s}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p - \varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)^{\frac{1}{r}}\Gamma(\rho k+\beta+1)} + |u| \right\} \\
&\times + |\gamma|\lambda^{1+\beta}\left(\lambda\left|f'(\varrho_1)\right|^s + (2-\lambda)\left|f'(\varrho_2)\right|^s\right)^{\frac{1}{s}} \\
&\times \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(\varrho_2^p - \varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)^{\frac{1}{r}}\Gamma(\rho k+\beta+1)} + |u| \right\}\Bigg].
\end{aligned}
$$

Now, for the case $p \in (0,1)$, using inequalities (11) and (12), and repeating the same procedure like $p \in [1,\infty)$, we obtain the desired result. □

Corollary 3 *Under the conditions of Theorem 3 for $p, \beta, \lambda, \gamma, \sigma(0) \to 1$, $s \to 2$, $u \to \frac{1}{2}$, and $w \to 0$, we have*

$$
\left| \frac{f(\varrho_1)+f(\varrho_2)}{2} - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(x)\,dx \right| \le \frac{\varrho_2-\varrho_1}{2\sqrt{6}}(2+\sqrt{3})\sqrt{\left|f'(\varrho_1)\right|^2 + \left|f'(\varrho_2)\right|^2}.
$$

Theorem 4 *Let $f: I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, the interior of I. If $f' \in L^1[\varrho_1,\varrho_2]$ and $|f'|^s$ is p-convex on $[\varrho_1,\varrho_2]$, where $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2$; $p, \rho, \beta > 0$; $r, s > 1$ such that $\frac{1}{r} + \frac{1}{s} = 1$; let $g(x) = \sqrt[p]{x}$, $x > 0$; $u, w, \gamma \in \mathbf{R}$ and $\lambda \in [0,1]$, then*

$$
\begin{aligned}
&\left|\Omega(f,\varrho_1,\varrho_2;u)\right| \\
&\le \begin{cases}
\frac{\varrho_2^p-\varrho_1^p}{2rs}[(1-\lambda)^{1+\beta}|1-\gamma|\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)\Gamma(\rho k+\beta+1)} \\
\quad \times (2s\varrho_1^{r(1-p)} + r(r(\beta+\rho k)+1)B_{s,1}) + |u|(2s\varrho_1^{r(1-p)} + rB_{s,1})\} \\
\quad + |\gamma|\lambda^{1+\beta}\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}(2s\varrho_1^{r(1-p)}+r(r(\beta+\rho k)+1)B_{s,2})}{(r(\beta+\rho k)+1)\Gamma(\rho k+\beta+1)} \\
\quad + |u|(2s\varrho_1^{r(1-p)} + rB_{s,2})\}], & p \in [1,\infty); \\
\frac{\varrho_2^p-\varrho_1^p}{2rs}[(1-\lambda)^{1+\beta}|1-\gamma|\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{(r(\beta+\rho k)+1)\Gamma(\rho k+\beta+1)} \\
\quad \times (2s\varrho_2^{r(1-p)} + r(r(\beta+\rho k)+1)B_{s,1}) + |u|(2s\varrho_2^{r(1-p)} + rB_{s,1})\} \\
\quad + |\gamma|\lambda^{1+\beta}\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}(2s\varrho_2^{r(1-p)}+r(r(\beta+\rho k)+1)B_{s,2})}{(r(\beta+\rho k)+1)\Gamma(\rho k+\beta+1)} \\
\quad + |u|(2s\varrho_2^{r(1-p)} + rB_{s,2})\}], & p \in (0,1).
\end{cases} \qquad (13)
\end{aligned}
$$

Proof By the properties of modulus, Young's inequality, and repeated application of p-convexity to Lemma 1 for the case $p \in [1,\infty)$, we have the following:

$$
\begin{aligned}
&\left|\Omega(f,\varrho_1,\varrho_2;u)\right| \\
&= \Bigg| -p\big[(1-\lambda)^\beta(1-\gamma)\big[\big\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\big[w\big((1-\lambda)\big(\varrho_2^p-\varrho_1^p\big)\big)^\rho\big] - u\big\} f(\varrho_1) \\
&\quad + uf\big(\sqrt[p]{\lambda\varrho_1^p + (1-\lambda)\varrho_2^p}\big)\big] + \lambda^\beta\gamma\big[\big\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\big[w\big(\lambda\big(\varrho_2^p-\varrho_1^p\big)\big)^\rho\big] - u\big\} \\
&\quad \times f\big(\sqrt[p]{\lambda\varrho_1^p + (1-\lambda)\varrho_2^p}\big) + uf(\varrho_2)\big]\big] + \frac{p}{(\varrho_2^p-\varrho_1^p)^\beta}\big[(1-\gamma) \\
&\quad \times \big(\mathfrak{J}^{\sigma}_{\rho,\beta,\varrho_1^p+;w} f\circ g\big)\big(\lambda\varrho_1^p + (1-\lambda)\varrho_2^p\big) + \gamma\big(\mathfrak{J}^{\sigma}_{\rho,\beta,[\lambda\varrho_1^p+(1-\lambda)\varrho_2^p]+;w} f\circ g\big)\big(\varrho_2^p\big)\big]\Bigg| \\
&\le \big(\varrho_2^p-\varrho_1^p\big)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\left\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^1 t^{\beta+\rho k}\big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}} \\
&\times \big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|\,dt \\
&+|u|\int_0^1 \big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}} \\
&\times \big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|\,dt\bigg\} \\
&+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \int_0^1 t^{\beta+\rho k}\big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}} \\
&\times \big|f'\big(\sqrt[p]{t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p}\big)\big|\,dt \\
&+|u|\int_0^1 \big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}} \\
&\times \big|f'\big(\sqrt[p]{t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p}\big)\big|\,dt\bigg\}\Bigg] \\
\le{}& \big(\varrho_2^p-\varrho_1^p\big)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \bigg\{\frac{1}{r}\int_0^1 \big(t^{\beta+\rho k}\big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{1-p}{p}}\big)^r\,dt \\
&+\frac{1}{s}\int_0^1 \big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|^s\,dt\bigg\} \\
&+|u|\bigg\{\frac{1}{r}\int_0^1 \big[(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p\big]^{\frac{r(1-p)}{p}}\,dt \\
&+\frac{1}{s}\int_0^1 \big|f'\big(\sqrt[p]{(1-t)\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+t\varrho_1^p}\big)\big|^s\,dt\bigg\}\Bigg\} \\
&+|\gamma|\lambda^{1+\beta}\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)} \\
&\times \bigg\{\frac{1}{r}\int_0^1 \big(t^{\beta+\rho k}\big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{1-p}{p}}\big)^r\,dt \\
&+\frac{1}{s}\int_0^1 \big|f'\big(\sqrt[p]{t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p}\big)\big|^s\,dt\bigg\} \\
&+|u|\bigg\{\frac{1}{r}\int_0^1 \big[t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p\big]^{\frac{r(1-p)}{p}}\,dt \\
&+\frac{1}{s}\int_0^1 \big|f'\big(\sqrt[p]{t\big(\lambda\varrho_1^p+(1-\lambda)\varrho_2^p\big)+(1-t)\varrho_2^p}\big)\big|^s\,dt\bigg\}\Bigg\}\Bigg] \\
\le{}& \big(\varrho_2^p-\varrho_1^p\big)\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(\varrho_2^p-\varrho_1^p))^{\rho k}}{\Gamma(\rho k+\beta+1)}
\end{aligned}
$$

$$\times \left\{ \frac{\varrho_1^{r(1-p)}}{r} \int_0^1 t^{r(\beta+\rho k)}\,dt \right.$$

$$\left. + \frac{1}{s} \int_0^1 \left[(1-t)\left\{ \lambda \left|f'(\varrho_1)\right|^s + (1-\lambda)\left|f'(\varrho_2)\right|^s \right\} + t\left|f'(\varrho_1)\right|^s \right] dt \right\}$$

$$+ |u| \left\{ \frac{\varrho_1^{r(1-p)}}{r} + \frac{1}{s} \int_0^1 \left[(1-t)\left\{ \lambda \left|f'(\varrho_1)\right|^s \right.\right.\right.$$

$$\left.\left.\left. + (1-\lambda)\left|f'(\varrho_2)\right|^s \right\} + t\left|f'(\varrho_1)\right|^s \right] dt \right\} \Bigg\}$$

$$+ |\gamma| \lambda^{1+\beta} \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (\lambda(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k + \beta + 1)} \left\{ \frac{\varrho_1^{r(1-p)}}{r} \int_0^1 t^{r(\beta+\rho k)}\,dt \right.\right.$$

$$\left. + \frac{1}{s} \int_0^1 \left[t\left\{ \lambda \left|f'(\varrho_1)\right|^s + (1-\lambda)\left|f'(\varrho_2)\right|^s \right\} + (1-t)\left|f'(\varrho_2)\right|^s \right] dt \right\}$$

$$+ |u| \left\{ \frac{\varrho_1^{r(1-p)}}{r} + \frac{1}{s} \int_0^1 \left[t\left\{ \lambda \left|f'(\varrho_1)\right|^s \right.\right.\right.$$

$$\left.\left.\left.\left. + (1-\lambda)\left|f'(\varrho_2)\right|^s \right\} + (1-t)\left|f'(\varrho_2)\right|^s \right] dt \right\} \right\} \Bigg]$$

$$= \left(\varrho_2^p - \varrho_1^p\right) [(1-\lambda)^{1+\beta} |1-\gamma| \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k ((1-\lambda)(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k + \beta + 1)} \right.$$

$$\times \left\{ \frac{\varrho_1^{r(1-p)}}{r(r(\beta+\rho k)+1)} + \frac{\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s + |f'(\varrho_1)|^s}{2s} \right\}$$

$$\left. + |u| \left\{ \frac{\varrho_1^{r(1-p)}}{r} + \frac{\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s + |f'(\varrho_1)|^s}{2s} \right\} \right\}$$

$$+ |\gamma| \lambda^{1+\beta} \{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (\lambda(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k + \beta + 1)} \{ \frac{\varrho_1^{r(1-p)}}{r(r(\beta+\rho k)+1)}$$

$$+ \frac{\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s + |f'(\varrho_2)|^s}{2s} \}$$

$$+ |u| \left\{ \frac{\varrho_1^{r(1-p)}}{r} + \frac{\lambda|f'(\varrho_1)|^s + (1-\lambda)|f'(\varrho_2)|^s + |f'(\varrho_2)|^s}{2s} \right\} \}]$$

$$= \left(\varrho_2^p - \varrho_1^p\right) \left[(1-\lambda)^{1+\beta} |1-\gamma| \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k ((1-\lambda)(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k + \beta + 1)} \right.\right.$$

$$\left. \times \left(\frac{2s\varrho_1^{r(1-p)} + r(r(\beta+\rho k)+1)B_{s,1}}{2rs(r(\beta+\rho k)+1)} \right) + |u| \left(\frac{2s\varrho_1^{r(1-p)} + rB_{s,1}}{2rs} \right) \right\}$$

$$+ |\gamma| \lambda^{1+\beta} \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (\lambda(\varrho_2^p - \varrho_1^p))^{\rho k}}{\Gamma(\rho k + \beta + 1)} \left(\frac{2s\varrho_1^{r(1-p)} + r(r(\beta+\rho k)+1)B_{s,2}}{2rs(r(\beta+\rho k)+1)} \right) \right.$$

$$\left.\left. + |u| \left(\frac{2s\varrho_1^{r(1-p)} + rB_{s,2}}{2rs} \right) \right\} \right].$$

Now, for the case $p \in (0,1)$, using inequalities (11) and (12), and repeating the same procedure like $p \in [1,\infty)$, we obtain the desired result. □

Table 1 Validity of Theorem 3 through numerical approach

ϱ_1	ϱ_2	u	γ	λ	LHS of (13)	RHS of (13)
0.2	3	−2	−3	0	5.8320e+03	2.4448e+09
5	7	−3	11	0.2	2.6291e+05	7.5952e+05
0.2	0.9	3	11	0.5	7.7838e−04	16.7265
7	1000	−0.2	−0.3	0.7	1.1859e+15	4.8801e+25
2	13	3	−11	0.8	1.3682e+07	1.0246e+11
4	25	10	80	0.99	1.7487e+11	9.2853e+14
9	11	−17	−18	1	3.7948e+08	7.2025e+08

Table 2 Validity of Theorem 1 through numerical approach

ϱ_1	ϱ_2	u	γ	λ	LHS of (10)	RHS of (10)
1	704	−180	19	0	8.2727e+04	1.1389e+06
2	404	−18	19	0.2	416.0355	3.7142e+03
0.2	0.9	14	130	0.3	12.5349	21.6725
11	1001	−201	500	0.5	1.3898	7.3137e+03
11	795	102	103	0.8	1.7962e+04	7.1857e+04
15	21	21	−20	0.999	294.4720	321.4211
7	80	−19	−10	1	1.1967e+03	2.6212e+03

Corollary 4 *Under the conditions of Theorem 4 for $\lambda, w \to 0$, $\gamma, p, u \to \frac{1}{2}$, $s \to 2$ and $\beta, \sigma(0) \to 1$, we have*

$$\begin{aligned} &\left|\frac{f(\varrho_1)+f(\varrho_2)}{2} - \frac{1}{\sqrt{\varrho_2}-\sqrt{\varrho_1}} \int_{\sqrt{\varrho_1}}^{\sqrt{\varrho_2}} f(\sqrt{x})\,dx\right| \\ &\quad \leq \frac{\sqrt{\varrho_2}-\sqrt{\varrho_1}}{12}\left\{10\varrho_2 + 9\left(\left|f'(\varrho_1)\right|^2 + \left|f'(\varrho_2)\right|^2\right)\right\}. \end{aligned}$$

4 Examples

Example 1 Let $f(x) = \frac{x^6}{6}$ such that $x \in (0,\infty)$, $p = 6$, $w = 0$, $\sigma(0) = 1$, $t = 10$, $r = 7$, $s = \frac{7}{6}$. We compute the values from result (13) of Theorem 3. In our calculation, we find out values separately for the right-hand side and the left-hand side of (13). We easily find from Table 1 that the numerical solution agrees with the analytical solution.

Example 2 Let $f(x) = 2\sqrt{x}$ such that $x \in (0,\infty)$, $p = \frac{1}{2}$, $w = 0$, $\sigma(0) = 1$, $t = 11$. We compute the values from result (10) of Theorem 1. In our calculation, we find out values separately for the right-hand side and the left-hand side of (10).

We easily find from Table 2 that the numerical solution agrees with the analytical solution.

5 Applications

5.1 Application to quadrature rules

Let I_t be a partition of the interval $[\varrho_1, \varrho_2]$ such that: $(\varrho_1 =)x_0 < x_1 < \cdots < x_t(= \varrho_2)$ and $l_j = x_{j+1} - x_j$, $0 \leq j \leq t-1$. Consider the following trapezoidal formula [20]:

$$T(f, I_t) := \sum_{j=0}^{t-1} \frac{f(x_j) + f(x_{j+1})}{2} l_j; \tag{14}$$

and the approximate error of $\int_{\varrho_1}^{\varrho_2} f(x)\,dx$ by $T(f,I_t)$ is defined by

$$E(f,I_t) := \int_{\varrho_1}^{\varrho_2} f(x)\,dx - T(f,I_t).$$

Let $S_\gamma(f,I_t,x_j,x_{j+1};u)$ be the extended quadrature formula, $R_\gamma(f,I_t,x_j,x_{j+1};u)$ be the associated error of $I_\gamma(f\circ g,I_t,x_j,x_{j+1};u)$ by $S_\gamma(f,I_t,x_j,x_{j+1};u)$, so that

$$R_\gamma(f,I_t,x_j,x_{j+1};u) = I_\gamma(f\circ g,I_t,x_j,x_{j+1};u) - S_\gamma(f,I_t,x_j,x_{j+1};u)$$

provided that

$$\begin{aligned}
&S_\gamma(f,I_t,x_j,x_{j+1};u)\\
&\quad := \sum_{j=0}^{t-1}\left(x_{j+1}^p - x_j^p\right)^\beta\big[(1-\lambda)^\beta(1-\gamma)\\
&\qquad \times\Big[\big\{\mathfrak{F}^\sigma_{\rho,\beta+1}\big[w\big((1-\lambda)\big(x_{j+1}^p - x_j^p\big)\big)^\rho\big] - u\big\}f(x_j) + uf\Big(\sqrt[p]{\lambda x_j^p + (1-\lambda)x_{j+1}^p}\Big)\Big]\\
&\qquad + \lambda^\beta\gamma\Big[\big\{\mathfrak{F}^\sigma_{\rho,\beta+1}\big[w\big(\lambda\big(x_{j+1}^p - x_j^p\big)\big)^\rho\big] - u\big\}f\Big(\sqrt[p]{\lambda x_j^p + (1-\lambda)x_{j+1}^p}\Big) + uf(x_{j+1})\Big]\big],\\
&I_\gamma(f\circ g,I_t,x_j,x_{j+1};u) \qquad\qquad (15)\\
&\quad := \sum_{j=0}^{t-1}\Big[(1-\gamma)\big(\mathfrak{I}^\sigma_{\rho,\beta,x_j^p+;w} f\circ g\big)\big(\lambda x_j^p + (1-\lambda)x_{j+1}^p\big)\\
&\qquad + \gamma\big(\mathfrak{I}^\sigma_{\rho,\beta,[\lambda x_j^p+(1-\lambda)x_{j+1}^p]+;w} f\circ g\big)\big(x_{j+1}^p\big)\Big],\\
&R_\gamma(f,I_t,x_j,x_{j+1};u)\\
&\quad := \sum_{j=0}^{t-1}\left(x_{j+1}^p - x_j^p\right)^\beta\big[(1-\lambda)^\beta(1-\gamma)\\
&\qquad \times\Big[\big\{\mathfrak{F}^\sigma_{\rho,\beta+1}\big[w\big((1-\lambda)\big(x_{j+1}^p - x_j^p\big)\big)^\rho\big] - u\big\}f(x_j) + uf\Big(\sqrt[p]{\lambda x_j^p + (1-\lambda)x_{j+1}^p}\Big)\Big]\\
&\qquad + \lambda^\beta\gamma\Big[\big\{\mathfrak{F}^\sigma_{\rho,\beta+1}\big[w\big(\lambda\big(x_{j+1}^p - x_j^p\big)\big)^\rho\big] - u\big\}f\Big(\sqrt[p]{\lambda x_j^p + (1-\lambda)x_{j+1}^p}\Big) + uf(x_{j+1})\Big]\big]\\
&\qquad - \sum_{j=0}^{t-1}\Big[(1-\gamma)\big(\mathfrak{I}^\sigma_{\rho,\beta,x_j^p+;w} f\circ g\big)\big(\lambda x_j^p + (1-\lambda)x_{j+1}^p\big)\\
&\qquad + \gamma\big(\mathfrak{I}^\sigma_{\rho,\beta,[\lambda x_j^p+(1-\lambda)x_{j+1}^p]+;w} f\circ g\big)\big(x_{j+1}^p\big)\Big].
\end{aligned}$$

For $\gamma \to 1$, identity (15) reduces to the following formula:

- *The fractional trapezoidal formula*

$$\begin{aligned}
S_1(f,I_t,x_j,x_{j+1};u) &:= \sum_{j=0}^{t-1}\big(\big(x_{j+1}^p - x_j^p\big)\lambda\big)^\beta\Big[\big\{\mathfrak{F}^\sigma_{\rho,\beta+1}\big[w\big(\lambda\big(x_{j+1}^p - x_j^p\big)\big)^\rho\big] - u\big\}\\
&\quad \times f\Big(\sqrt[p]{\lambda x_j^p + (1-\lambda)x_{j+1}^p}\Big) + uf(x_{j+1})\Big]. \qquad (16)
\end{aligned}$$

Moreover, for $\omega \to 0$ and $p,\beta,\lambda,2u,\sigma(0) \to 1$, identity (16) reduces to (14).

Proposition 2 *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, the interior of I. If $f' \in L^1[\varrho_1, \varrho_2]$ and $|f'|^s$ is p-convex on $[\varrho_1, \varrho_2]$, where $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2$; $p, \rho, \beta > 0$; $s \geq 1$; let $g(x) = \sqrt[p]{x}$, $x > 0$; $u, w, \gamma \in \mathbf{R}$ and $\lambda \in [0,1]$, then*

$$
\begin{aligned}
&\left|R_\gamma(f, I_t, x_j, x_{j+1}; u)\right| \\
&\leq \begin{cases}
\sum_{j=0}^{t-1} \frac{x_j^{1-p}(x_{j+1}^p - x_j^p)^{1+\beta}}{p}[(1-\lambda)^{1+\beta}|1-\gamma|\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(x_{j+1}^p - x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)} \\
\quad \times (\frac{(\lambda+\beta+\rho k+1)|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s}{\beta+\rho k+2})^{\frac{1}{s}} \\
\quad + |u|(\frac{(1+\lambda)|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s}{2})^{\frac{1}{s}}\} \\
\quad + \lambda^{1+\beta}|\gamma|\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(x_{j+1}^p - x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)} \\
\quad \times (\frac{(\beta+\rho k+1)(\lambda|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s) + |f'(x_{j+1})|^s}{\beta+\rho k+2})^{\frac{1}{s}} \\
\quad + |u|(\frac{\lambda|f'(x_j)|^s + (2-\lambda)|f'(x_{j+1})|^s}{2})^{\frac{1}{s}}\}], & p \in [1,\infty); \\
\sum_{j=0}^{t-1} \frac{x_{j+1}^{1-p}(x_{j+1}^p - x_j^p)^{1+\beta}}{p}[(1-\lambda)^{1+\beta}|1-\gamma|\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(x_{j+1}^p - x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)} \\
\quad \times (\frac{(\lambda+\beta+\rho k+1)|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s}{\beta+\rho k+2})^{\frac{1}{s}} \\
\quad + |u|(\frac{(1+\lambda)|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s}{2})^{\frac{1}{s}}\} \\
\quad + \lambda^{1+\beta}|\gamma|\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(x_{j+1}^p - x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)} \\
\quad \times (\frac{(\beta+\rho k+1)(\lambda|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s) + |f'(x_{j+1})|^s}{\beta+\rho k+2})^{\frac{1}{s}} \\
\quad + |u|(\frac{\lambda|f'(x_j)|^s + (2-\lambda)|f'(x_{j+1})|^s}{2})^{\frac{1}{s}}\}], & p \in (0,1);
\end{cases} \\
&\leq \begin{cases}
\frac{\max\{|f'(\varrho_1)|^s, |f'(\varrho_2)|^s\}}{p} \sum_{j=0}^{t-1} x_j^{1-p}(x_{j+1}^p - x_j^p)^{\beta+1}, & p \in [1,\infty); \\
\frac{\max\{|f'(\varrho_1)|^s, |f'(\varrho_2)|^s\}}{p} \sum_{j=0}^{t-1} x_{j+1}^{1-p}(x_{j+1}^p - x_j^p)^{\beta+1}, & p \in (0,1).
\end{cases}
\end{aligned}
$$

Proof Application of Theorem 2 for the case $p \in [1,\infty)$ on the subinterval $[x_j, x_{j+1}]$, $0 \leq j \leq t-1$, yields the following:

$$
\begin{aligned}
&\Big|-\big(x_{j+1}^p - x_j^p\big)^\beta\Big[(1-\lambda)^\beta(1-\gamma)\Big[\Big\{\mathfrak{F}_{\rho,\beta+1}^\sigma\Big[w\big((1-\lambda)\big(x_{j+1}^p - x_j^p\big)\big)^\rho\Big] - u\Big\}f(x_j) \\
&\quad + uf\Big(\sqrt[p]{\lambda x_j^p + (1-\lambda)x_{j+1}^p}\Big)\Big] + \lambda^\beta\gamma\Big[\Big\{\mathfrak{F}_{\rho,\beta+1}^\sigma\Big[w\big(\lambda\big(x_{j+1}^p - x_j^p\big)\big)^\rho\Big] - u\Big\} \\
&\quad \times f\Big(\sqrt[p]{\lambda x_j^p + (1-\lambda)x_{j+1}^p}\Big) + uf(x_{j+1})\Big]\Big] + \Big[(1-\gamma) \\
&\quad \times \Big(\mathfrak{J}_{\rho,\beta,x_j^p+;w}^\sigma f\circ g\Big)\big(\lambda x_j^p + (1-\lambda)x_{j+1}^p\big) + \gamma\Big(\mathfrak{J}_{\rho,\beta,[\lambda x_j^p + (1-\lambda)x_{j+1}^p]+;w}^\sigma f\circ g\Big)\big(x_{j+1}^p\big)\Big]\Big| \\
&\leq \frac{x_j^{1-p}(x_{j+1}^p - x_j^p)^{1+\beta}}{p}\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k((1-\lambda)(x_{j+1}^p - x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)} \\
&\quad \times \left(\frac{(\lambda+\beta+\rho k+1)|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s}{\beta+\rho k+2}\right)^{\frac{1}{s}} \\
&\quad + |u|\left(\frac{(1+\lambda)|f'(x_j)|^s + (1-\lambda)|f'(x_{j+1})|^s}{2}\right)^{\frac{1}{s}}\Bigg\} \\
&\quad + \lambda^{1+\beta}|\gamma|\Bigg\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(\lambda(x_{j+1}^p - x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}
\end{aligned}
$$

$$\times\left(\frac{(\beta+\rho k+1)(\lambda|f'(x_j)|^s+(1-\lambda)|f'(x_{j+1})|^s)+|f'(x_{j+1})|^s}{\beta+\rho k+2}\right)^{\frac{1}{s}}$$
$$+|u|\left(\frac{\lambda|f'(x_j)|^s+(2-\lambda)|f'(x_{j+1})|^s}{2}\right)^{\frac{1}{s}}\Bigg\}\Bigg].$$

Finally, summing over j from 0 to $t-1$ and taking into account that $|f'|^s$ is p-convex, we deduce, by the triangular inequality, the following:

$$\begin{aligned}
&\bigl|I_\gamma(f\circ g,I_t,x_j,x_{j+1};u)-S_\gamma(f,I_t,x_j,x_{j+1};u)\bigr|\\
&\leq\sum_{j=0}^{t-1}\frac{x_j^{1-p}(x_{j+1}^p-x_j^p)^{1+\beta}}{p}\Bigg[(1-\lambda)^{1+\beta}|1-\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k((1-\lambda)(x_{j+1}^p-x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}\\
&\quad\times\left(\frac{(\lambda+\beta+\rho k+1)|f'(x_j)|^s+(1-\lambda)|f'(x_{j+1})|^s}{\beta+\rho k+2}\right)^{\frac{1}{s}}\\
&\quad+|u|\left(\frac{(1+\lambda)|f'(x_j)|^s+(1-\lambda)|f'(x_{j+1})|^s}{2}\right)^{\frac{1}{s}}\Bigg\}\\
&\quad+\lambda^{1+\beta}|\gamma|\Bigg\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(\lambda(x_{j+1}^p-x_j^p))^{\rho k}}{\Gamma(\rho k+\beta+2)}\\
&\quad\times\left(\frac{(\beta+\rho k+1)(\lambda|f'(x_j)|^s+(1-\lambda)|f'(x_{j+1})|^s)+|f'(x_{j+1})|^s}{\beta+\rho k+2}\right)^{\frac{1}{s}}\\
&\quad+|u|\left(\frac{\lambda|f'(x_j)|^s+(2-\lambda)|f'(x_{j+1})|^s}{2}\right)^{\frac{1}{s}}\Bigg\}\Bigg]\\
&\leq\frac{\max\{|f'(\varrho_1)|^s,|f'(\varrho_2)|^s\}}{p}\sum_{j=0}^{t-1}x_j^{1-p}\bigl(x_{j+1}^p-x_j^p\bigr)^{\beta+1}.
\end{aligned}$$

Now, for the case $p\in(0,1)$, repeating the same procedure like $p\in[1,\infty)$, we obtain the desired result. □

Remark 2

- For $\gamma\to 1$, Proposition 2 provides an approximate error for the fractional trapezoidal formula.
- For $\gamma\to 0$, Proposition 2 provides an approximate error for the fractional midpoint formula.
- For $\gamma\to\frac{1}{2}$, Proposition 2 provides a generalized fractional Bullen-type inequality.

5.2 *f*-divergence measure

Let ϕ be the set and μ be the given σ finite measure, and let the set of all probability densities on μ be defined as $\Omega:=\{\chi|\chi:\phi\to\mathbb{R},\chi(\xi)>0,\int_\phi\chi(\xi)\,d\mu(\xi)=1\}$. Let $f:(0,\infty)\to\mathbb{R}$ be the given mapping and consider $D_f(\chi,\psi)$ defined as

$$D_f(\chi,\psi)=\int_\phi\chi(\xi)f\left(\frac{\psi(\xi)}{\chi(\xi)}\right)d\mu(\xi),\qquad\chi,\psi\in\Omega.\tag{17}$$

If f is convex, then (17) is known as *Csisźar f*-divergence. The Hermite–Hadamard (HH) divergence is defined as

$$D^f_{HH}(\chi,\psi)=\int_\phi \chi(\xi)\frac{\int_1^{\frac{\psi(\xi)}{\chi(\xi)}} f(t)\,dt}{\frac{\psi(\xi)}{\chi(\xi)}-1}\,d\mu(\xi),\quad \chi,\psi\in\Omega,$$

where f is convex on $(0,\infty)$ with $f(1)=0$. Note that $D^f_{HH}(\chi,\psi)\geq 0$ with the equality holds if and only if $\chi=\psi$.

Proposition 3 *Let $f: I\subseteq\mathbb{R}^+\to\mathbb{R}$ be a differentiable function on I°, the interior of I, $\varrho_1,\varrho_2\in I^\circ$ with $\varrho_1<\varrho_2$ such that $|f'|$ is convex and $f(1)=0$, then*

$$\begin{aligned}&\left|\frac{1}{4}\left[D_f(\chi,\psi)+2\int_\phi \chi(\xi)f\left(\frac{\chi(\xi)+\psi(\xi)}{2\chi(\xi)}\right)d\mu(\xi)\right]-D^f_{HH}(\chi,\psi)\right|\\&\quad\leq\frac{1}{2}\left[\left|f'(1)\right|\int_\phi\left|\psi(\xi)-\chi(\xi)\right|d\mu(\xi)\right.\\&\qquad\left.+\int_\phi\left|\psi(\xi)-\chi(\xi)\right|\left|f'\left(\frac{\psi(\xi)}{\chi(\xi)}\right)\right|d\mu(\xi)\right].\end{aligned}$$

Proof Let $\phi_1:=\{\xi\in\phi:\psi(\xi)>\chi(\xi)\}$; $\phi_2:=\{\xi\in\phi:\psi(\xi)<\chi(\xi)\}$ and $\phi_3:=\{\xi\in\phi:\psi(\xi)=\chi(\xi)\}$. If $\xi\in\phi_3$, then clearly the equality holds. Now if $\xi\in\phi_1$, then for $\varrho_1\to 1$ and $\varrho_2\to\frac{\psi(\xi)}{\chi(\xi)}$ in Corollary 1, multiplying the obtained result on both sides by $\chi(\xi)$ and integrating over ϕ_1, we have

$$\begin{aligned}&\left|\frac{1}{4}\left[\int_{\phi_1}\chi(\xi)f\left(\frac{\psi(\xi)}{\chi(\xi)}\right)d\mu(\xi)+2\int_{\phi_1}\chi(\xi)f\left(\frac{\chi(\xi)+\psi(\xi)}{2\chi(\xi)}\right)d\mu(\xi)\right]\right.\\&\qquad\left.-\int_{\phi_1}\chi(\xi)\frac{\int_1^{\frac{\psi(\xi)}{\chi(\xi)}}f(t)\,dt}{\frac{\psi(\xi)}{\chi(\xi)}-1}\,d\mu(\xi)\right|\\&\quad\leq\frac{1}{2}\int_{\phi_1}\bigl(\psi(\xi)-\chi(\xi)\bigr)\left[\left(\left|f'(1)\right|+\left|f'\left(\frac{\psi(\xi)}{\chi(\xi)}\right)\right|\right)\right]d\mu(\xi).\end{aligned}\tag{18}$$

Now if $\xi\in\phi_2$, then for $\varrho_2\to 1$ and $\varrho_1\to\frac{\psi(\xi)}{\chi(\xi)}$ in Corollary 1, multiplying the obtained result on both sides by $\chi(\xi)$ and integrating over ϕ_2, we have

$$\begin{aligned}&\left|\frac{1}{4}\left[\int_{\phi_2}\chi(\xi)f\left(\frac{\psi(\xi)}{\chi(\xi)}\right)d\mu(\xi)+2\int_{\phi_2}\chi(\xi)f\left(\frac{\chi(\xi)+\psi(\xi)}{2\chi(\xi)}\right)d\mu(\xi)\right]\right.\\&\qquad\left.-\int_{\phi_2}\chi(\xi)\frac{\int_1^{\frac{\psi(\xi)}{\chi(\xi)}}f(t)\,dt}{\frac{\psi(\xi)}{\chi(\xi)}-1}\,d\mu(\xi)\right|\\&\quad\leq\frac{1}{2}\left[\left|f'(1)\right|\int_{\phi_2}\bigl(\chi(\xi)-\psi(\xi)\bigr)\,d\mu(\xi)+\int_{\phi_2}\bigl(\chi(\xi)-\psi(\xi)\bigr)\left|f'\left(\frac{\psi(\xi)}{\chi(\xi)}\right)\right|d\mu(\xi)\right].\end{aligned}\tag{19}$$

Adding inequalities (18) and (19) and utilizing the triangular inequality, we have obtained the intended result. □

6 Conclusion

New estimates for the generalized fractional Bullen-type functionals have been derived to provide some error estimates for quadrature rules and of inequalities relating to f-divergence measures.

Acknowledgements
The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding
The work was supported by the Natural Science Foundation of China (Grant No. 11971142).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
SH has made the major analysis and gave the idea of original draft preparation. SR has contributed significantly in the preparation of main results and their numerical validity. Y-MC has a role in methodology, investigation, reviewing, editing and provision of the study resources. SK and SS have contributed in the writing of this paper and deriving applications of the results. All the authors have read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of Engineering and Technology, 54890 Lahore, Pakistan. [2]Department of Mathematics and Statistics, The University of Lahore, 54590 Lahore, Pakistan. [3]Institute for Advanced Study Honoring Chen Jian Gong, Hangzhou Normal University, Hangzhou 311121, China. [4]Department of Mathematics, Huzhou University, Huzhou 313000, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 March 2022 Accepted: 27 October 2022 Published online: 10 November 2022

References

1. Adjabi, Y., Jarad, F., Baleanu, D., Abdeljawad, T.: On Cauchy problems with Caputo Hadamard fractional derivatives. J. Comput. Anal. Appl. **21**(4), 661–681 (2016)
2. Tan, W., Jiang, F.L., Huang, C.X., Zhou, L.: Synchronization for a class of fractional-order hyperchaotic system and its application. J. Appl. Math. **2012**, Article ID 974639 (2012)
3. Zhou, X., Huang, C., Hu, H., Liu, L.: Inequality estimates for the boundedness of multilinear singular and fractional integral operators. J. Inequal. Appl. **2013**, Article ID 303 (2013)
4. Mohammed, P.O., Machado, J.A.T., Guirao, J.L.G., Agarwal, R.P.: Adomian decomposition and fractional power series solution of a class of nonlinear fractional differential equations. Mathematics **2021**, Article ID 9091070 (2021)
5. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
6. Cai, Z., Huang, J., Huang, L.: Periodic orbit analysis for the delayed Filippov system. Proc. Am. Math. Soc. **146**(11), 4667–4682 (2018)
7. Chen, T., Huang, L., Yu, P., Huang, W.: Bifurcation of limit cycles at infinity in piecewise polynomial systems. Nonlinear Anal., Real World Appl. **41**, 82–106 (2018)
8. Wang, J., Chen, X., Huang, L.: The number and stability of limit cycles for planar piecewise linear systems of node-saddle type. J. Math. Anal. Appl. **469**(1), 405–427 (2019)
9. Bullen, P.S.: Error estimates for some elementary quadrature rules. Publ. Elektroteh. Fak. Univ. Beogr., Ser. Mat. Fiz. **602**(633), 97–103 (1978)
10. Cakmak, M.: Refinements of Bullen-type inequalities for s-convex functions via Riemann–Liouville fractional integrals involving Gauss hypergeometric function. J. Interdiscip. Math. **22**(6), 975–989 (2019)
11. Cakmak, M.: On some Bullen-type inequalities via conformable fractional integrals. J. Sci. Perspect. **3**(4), 285–298 (2019)
12. Du, T.S., Luo, C.Y., Cao, Z.J.: On the Bullen-type inequalities via generalized fractional integrals and their applications. Fractals **29**(7), Article ID 2150188 (2021)
13. Erden, S., Sarikaya, M.Z.: Generalized Bullen type inequalities for local fractional integrals and its applications. Palest. J. Math. **9**(2), 945–956 (2020)
14. Set, E., Korkut, N., Uygun, N.: Bullen-type and Simpson-type inequalities for fractional integrals with applications. AIP Conf. Proc. **1833**, Article ID 020053 (2017)

15. Hammer, P.C.: The midpoint method of numerical integration. Math. Mag. **31**(4), 193–195 (1958)
16. Iscan, I.: Ostrowski type inequalities for p-convex functions. New Trends Math. Sci. **4**(3), 140–150 (2016)
17. Agarwal, R.P., Luo, M.-J., Raina, R.K.: On Ostrowski type inequalities. Fasc. Math. **56**(1), 5–27 (2016)
18. Raina, R.K.: On generalized Wright's hypergeometric functions and fractional calculus operators. East Asian Math. J. **21**(2), 191–203 (2005)
19. Iscan, I., Toplu, T., Yetgin, F.: Some new inequalities on generalization of Hermite–Hadamard and Bullen type inequalities, applications to trapezoidal and midpoint formula. Kragujev. J. Math. **45**(4), 647–657 (2021)
20. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998)