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Norm inequalities for maximal operators

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Abstract

In this paper, we introduce a family of one-dimensional maximal operators $\mathscr{M}_{\kappa,m}$, $\kappa \geq 0$ and $m \in \mathbb{N} \setminus \{0\}$, which includes the Hardy–Littlewood maximal operator as a special case ($\kappa = 0$, $m = 1$). We establish the weak type $(1, 1)$ and the strong type (p, p) inequalities for $\mathscr{M}_{\kappa,m}$, $p > 1$. To do so, we prove a technical covering lemma for a finite collection of intervals.

MSC: 42B25

Keywords: Covering lemma; Generalized Fourier transform; Maximal operators; Weak and strong type inequalities

1 Introduction

Maximal operators have proved to be tools of great importance in the theory of differentiation of functions, complex and harmonic analysis, ergodic theory, and also in index theory. In general, one considers a certain collection of sets $\mathscr{C}$ in $\mathbb{R}^n$ and then, given any locally integrable function f, at every x one measures the maximal average value of f with respect to the collection $\mathscr{C}$, translated by x. Then it is of fundamental importance to obtain certain regularity properties of these operators such as weak type inequalities and L^p boundedness.

The simplest example of such a maximal operator is the centered operator defined by

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| \, dy$$

for every $f \in L^1(\mathbb{R})$. This operator was introduced by Hardy and Littlewood in the 1930s, and its higher dimensional version was first used by Wiener in 1939. Since then the operator has been widely studied and used. It is well known that the Hardy–Littlewood maximal operator plays a major role in several places of analysis. It is a classical mean operator, and it is frequently used to majorize other important operators in harmonic analysis. For instance, from the boundedness of the Hardy–Littlewood maximal operator, one can give a quick proof of Lebesgue's differentiation theorem; see, e.g., [25, 26]. The generalization of the differentiation theorem to averages over a variety of families of sets leads to the definition of several variants of the Hardy–Littlewood maximal operator. The purpose of this article is to contribute to this endeavor by studying a variant of the Hardy–Littlewood maximal operator.

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To be more precise, given an integer $m \geq 1$ and a parameter κ such that $2\kappa > 1 - (2/m)$, we consider the maximal function

$$\mathscr{M}_{\kappa,m}f(x) = \sup_{r>0} \frac{1}{\mu_{\kappa,m}(]-r,r[)} \left| \int_{\mathbb{R}} f(y)\tau_x(\chi_r)\big((-1)^m y\big)\, d\mu_{\kappa,m}(y) \right|,$$

where χ_r is the characteristic function of the interval $]-r,r[$, $\tau_x = \tau_x^{\kappa,m}$ is a generalized translation operator introduced and studied in [10], and $d\mu_{\kappa,m}$ is the measure given by

$$d\mu_{\kappa,m}(y) = |y|^{2\kappa+\frac{2}{m}-2}\, dy.$$

It is of fundamental importance to mention that the translation operator τ_x is associated with the generalized Fourier transform $\mathscr{F}_{\kappa,m}$ built in [7] as follows:

$$\begin{aligned}\mathscr{F}_{\kappa,m}f(\lambda) &= 2^{-1}\left(\frac{2}{m}\right)^{-(\kappa m-\frac{m}{2})} \\ &\quad\times \int_{\mathbb{R}} f(x) \underbrace{\left(\widetilde{J}_{\kappa m-\frac{m}{2}}\big(m|xy|^{\frac{1}{m}}\big) + \left(\frac{m}{2i}\right)^m xy\widetilde{J}_{\kappa m+\frac{m}{2}}\big(m|xy|^{\frac{1}{m}}\big)\right)}_{:=E_{\kappa,m}(x,y)} d\mu_{\kappa,m}(x). \quad (1.1)\end{aligned}$$

Here

$$\widetilde{J}_\nu(w) := \left(\frac{w}{2}\right)^{-\nu} J_\nu(w) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell w^{2\ell}}{2^{2\ell}\ell!\Gamma(\nu+\ell+1)},$$

where J_ν is the modified Bessel function of the first kind. In particular,

$$\mathscr{F}_{\kappa,m}(\tau_x f)(y) = E_{\kappa,m}\big((-1)^m x, y\big)\mathscr{F}_{\kappa,m}f(y).$$

Some notable special cases include, up to a scalar:

- $m = 1$ and $\kappa = 0$: Then $\mathscr{F}_{\kappa,m}$ and τ_x are the Euclidean Fourier transform and the usual translation, respectively, see, e.g., [27];
- $m = 1$, $\kappa = 0$ and f is an even function: We recover the Hankel transform and the corresponding translation operator, see, e.g., [16, 18];
- $m = 1$ and $\kappa > 0$: We recover the Dunkl transform and the Dunkl translation operator, see, e.g., [14, 23, 28];
- $m = 2$ and $\kappa = 0$: Then $\mathscr{F}_{\kappa,m}$ and τ_x are the generalized Hankel transform and the corresponding translation operator, see, e.g., [3, 20];
- $m = 2$, $\kappa = 0$ and f is an even function: We recover the Hankel–Clifford transform and the corresponding translation operator, see, e.g., [17, 22];
- $m = 2$ and $\kappa > 0$: We recover the κ-Hankel transform and the corresponding translation operator, see, e.g., [3–5].

This transform, which started with the paper [7], was developed extensively afterwards and continues to receive considerable attention (see, e.g., [2, 4, 11–13, 15, 19, 21]).

The main result of the paper is to prove that the maximal operator $\mathscr{M}_{\kappa,m}$ is of weak type $(1,1)$ and strong type (p,p) for all $p > 1$. One of the major technical obstacles in the investigation of $\mathscr{M}_{\kappa,m}$ is a lack of known deeper properties of the translation operator τ_x. Therefore we introduce the uncentered maximal function

$$\mathbb{M}_{\kappa,m}f(x) = \sup_{r>0} \frac{1}{\mu_{\kappa,m}(I(x,r))} \int_{|y|\in I(x,r)} \left|f(y)\right| d\mu_{\kappa,m}(y),$$

where

$$I(x,r) := \left]\left(|x|^{\frac{1}{m}} - r^{\frac{1}{m}}\right)_+^m, \left(|x|^{\frac{1}{m}} + r^{\frac{1}{m}}\right)^m\right[$$

(here $(|x|^{\frac{1}{m}} - r^{\frac{1}{m}})_+ = \max\{0, |x|^{\frac{1}{m}} - r^{\frac{1}{m}}\}$). We pin down that $I(x,r)$ is linked to the support of the kernel that appears in the integral representation of τ_x. In particular, if $y \notin I(x,r)$, then $\tau_x(\chi_r)(y) = 0$. We prove that $\mathscr{M}_{\kappa,m}f \lesssim \mathbb{M}_{\kappa,m}f$ holds pointwise. To prove the weak $(1,1)$ and the strong (p,p) estimates for $\mathbb{M}_{\kappa,m}$, for all $p > 1$, it is enough by Marcinkiewicz interpolation to prove the first kind of estimate as the strong (∞,∞) estimate is trivial. Now, to show the weak $(1,1)$ estimate, we prove a covering lemma to a finite collection of intervals $I(x_j, r_j)_j$. This result, which is far from being obvious in our setting, states that if two such intervals overlap, then the smaller one is contained in some dilate of the larger. This is a fairly excepted result in our setting as $\mu_{\kappa,m}(I(x,2r)) = O(\mu_{\kappa,m}(I(x,r)))$ (Lemma 3.1). However, the intervals $I(x,r)$ are geometrically complicated objects.

Throughout the paper we use a fairly standard notation. We write $L^p(\mu_{\kappa,m})$ and $L^{1,\infty}(\mu_{\kappa,m})$ to denote the weighted L^p and the weighted weak L^1 spaces that consist of all functions f on $\mathbb{R}$ for which

$$\|f\|_{L^p_\kappa} = \left(\int_{\mathbb{R}} \left|f(x)\right|^p d\mu_{\kappa,m}(x)\right)^{1/p} < \infty$$

or

$$\|f\|_{L^{1,\infty}_\kappa} = \sup_{\lambda>0}\left(\lambda \int_{\{|f|>\lambda\}} d\mu_{\kappa,m}(x)\right) < \infty,$$

respectively. The main result is the following.

Theorem *The maximal operator $\mathscr{M}_{\kappa,m}$ is bounded from $L^1(\mu_{\kappa,m})$ to the Lorentz space $L^{1,\infty}(\mu_{\kappa,m})$, and from $L^p(\mu_{\kappa,m})$ to $L^p(\mu_{\kappa,m})$ for all $1 < p \leq \infty$.*

As a direct application of the above result, we give a quick proof of a Lebesgue differentiation type theorem that for almost every point, the value of an integrable function with respect to $\mu_{\kappa,m}$ is the limit of infinitesimal "averages" taken about the point.

The structure of the paper is as follows: In Sect. 2 we briefly recall some fundamental properties of the generalized Fourier transform $\mathscr{F}_{\kappa,m}$ and the translation operator τ_x. Section 3 is devoted to the covering lemma for the intervals $I(x,r)$ (Theorem 3.2). In Sect. 4 we provide a sharp estimate for $\tau_x(\chi_r)(y)$ (Corollary 4.5). This estimate will be the key tool to prove that $\mathscr{M}_{\kappa,m}f \leq \mathbb{M}_{\kappa,m}f$. Finally, in Sect. 5 we prove that the maximal operator $\mathbb{M}_{\kappa,m}$, and therefore $\mathscr{M}_{\kappa,m}$, is of weak type $(1,1)$ and of strong type (p,p) for all $1 < p \leq \infty$ (Theorem 5.1).

Throughout this paper, the notation $U \lesssim V$ stands for $U \leq cV$ for some constant $c > 0$.

2 Background

Recall from the previous section the integral transform $\mathscr{F}_{\kappa,m}$. We refer the reader to [7] (or [6]) for a detailed study. Further, it shares many of the important properties with usual integral transforms, part of which are listed as follows.

Theorem 2.1 *Let $m \in \mathbb{N}_{\geq 1}$ be given and assume $2\kappa > 1 - (2/m)$.*

1) (Plancherel formula) *The transform $\mathscr{F}_{\kappa,m}$ is a unitary map of $L^2(\mu_{\kappa,m})$ onto itself.*
2) (Inversion formula) *Let $f \in L^1(\mu_{\kappa,m})$ and suppose that $\mathscr{F}_{\kappa,m}f \in L^1(\mu_{\kappa,m})$. Then $\mathscr{F}^{-1}_{\kappa,m}f(x) = \mathscr{F}_{\kappa,m}f((-1)^m x)$, almost everywhere.*

Let $x \in \mathbb{R}$ be given. In [10] the authors introduced a translation operator τ_x on $C_b(\mathbb{R})$ defined, up to a constant, by

$$\tau_x f(y) = \tau_x^{\kappa,m} f(y) = \int_{\mathbb{R}} f(z) K_{\kappa,m}(x,y;z)\, d\mu_{\kappa,m}(z)$$

for some kernel $K_{\kappa,m}$ (see [10] for the explicit formula) satisfying

$$\mathscr{F}_{\kappa,m}(\tau_x f)(y) = E_{\kappa,m}\big((-1)^m x, y\big)\mathscr{F}_{\kappa,m}f(y).$$

It is important to mention that, for given $x, y \in \mathbb{R}$, the support of $K_{\kappa,m}(x,y;z) = K_{\kappa,m}(y,x;z)$ is included in

$$\Big\{z \in \mathbb{R} : \big||x|^{\frac{1}{m}} - |y|^{\frac{1}{m}}\big| < |z|^{\frac{1}{m}} < |x|^{\frac{1}{m}} + |y|^{\frac{1}{m}}\Big\}. \tag{2.1}$$

Moreover, it was shown that the convolution product of suitable functions f and g defined by

$$f \circledast g(x) = 2^{-1}\left(\frac{2}{m}\right)^{-(\kappa m - \frac{m}{2})} \int_{\mathbb{R}} f(y)\tau_x g\big((-1)^m y\big)\, d\mu_{\kappa,m}(y)$$

satisfies $\mathscr{F}_{\kappa,m}(f \circledast g)(x) = \mathscr{F}_{\kappa,m}f(x)\mathscr{F}_{\kappa,m}g(x)$, and $f \circledast g = g \circledast f$.

The main properties of the translation operator and convolution are collected below [10].

Theorem 2.2 *Let $m \in \mathbb{N}_{\geq 1}$ be given and now assume that $2\kappa > 1 - (1/m)$.*

1) *For every $f \in L^p(\mu_{\kappa,m})$, $1 \leq p \leq \infty$, we have*

$$\|\tau_x f\|_{L^p_\kappa} \leq 4\Gamma\left(\kappa m - \frac{m}{2} + 1\right)^{-1} \|f\|_{L^p_\kappa}.$$

2) (Young's inequality) *Let $p, q, r \geq 1$ with $1/r = 1/p + 1/q - 1$. For $f \in L^p(\mu_{\kappa,m})$ and $g \in L^q(\mu_{\kappa,m})$, we have*

$$\|f \circledast g\|_{L^r_\kappa} \leq 2\left(\frac{2}{m}\right)^{-(km - \frac{m}{2})} \Gamma\left(\kappa m - \frac{m}{2} + 1\right)^{-1} \|f\|_{L^p_\kappa}\|g\|_{L^q_\kappa}.$$

3) *Let $1 \leq p, q, r \leq 2$ with $1/r = 1/p + 1/q - 1$. For $f \in L^p(\mu_{\kappa,m})$ and $g \in L^q(\mu_{\kappa,m})$, we have $\mathscr{F}_{\kappa,m}(f \circledast g) = \mathscr{F}_{\kappa,m}(f)\mathscr{F}_{\kappa,m}(g)$.*

3 A covering lemma

The classical Vitali covering lemma is one of the fundamental tools of modern analysis and geometric measure theory. Its one-dimensional version states that there exists a constant $c > 0$ such that, given a finite collection of intervals $\{I_j\}$ in $\mathbb{R}$, there exists a disjoint subcollection $\{\tilde{I}_j\} \subset \{I_j\}$ such that $|\bigcup \tilde{I}_j| \geq c|\bigcup I_j|$. In this section we develop a covering lemma for the intervals

$$I(x,r) := \big]\big(|x|^{\frac{1}{m}} - r^{\frac{1}{m}}\big)_+^m, \big(|x|^{\frac{1}{m}} + r^{\frac{1}{m}}\big)^m\big[\tag{3.1}$$

for $x \in \mathbb{R}$ and $r > 0$. Here $(|x|^{\frac{1}{m}} - r^{\frac{1}{m}})_+ = \max\{0, |x|^{\frac{1}{m}} - r^{\frac{1}{m}}\}$. This result will lead us to the weak-$(1,1)$ boundedness of the maximal operator $\mathscr{M}_{\kappa,m}$. Observe that the intervals $I(x,r)$ are very close related to the support of the translation operator τ_x in (2.1).

The covering theorem requires the following structure of $\mu_{\kappa,m}$.

Lemma 3.1 *The measure $\mu_{k,m}$ is doubling, i.e.,*

$$0 < \mu_{\kappa,m}\big(I(x,2r)\big) \lesssim \mu_{\kappa,m}\big(I(x,r)\big) < \infty \tag{3.2}$$

for all $x \in \mathbb{R}$ and $r > 0$.

By iterating the doubling condition, we conclude that, for all $\lambda > 0$, we have $\mu_{\kappa,m}(I(x, \lambda r)) \lesssim \mu_{\kappa,m}(I(x,r))$ for all $x \in \mathbb{R}$ and $r > 0$.

Proof of Lemma 3.1 There are three possibilities to encounter.

1) The case $|x|^{\frac{1}{m}} \leq r^{\frac{1}{m}}$, i.e., $I(x,2r) =]0, (|x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}})^m[$ and $I(x,r) =]0, (|x|^{\frac{1}{m}} + r^{\frac{1}{m}})^m[$. In this case,

$$\begin{aligned}\mu_{\kappa,m}\big(I(x,2r)\big) &= \int_0^{(|x|^{\frac{1}{m}}+(2r)^{\frac{1}{m}})^m} y^{2\kappa+\frac{2}{m}-2}\,dy \\ &= \frac{(|x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}})^{2\kappa m+2-m}}{2\kappa + (2/m) - 1} \leq \frac{(|2x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}})^{2\kappa m+2-m}}{2\kappa + (2/m) - 1} \\ &= 2^{2\kappa+\frac{2}{m}-1}\mu_{\kappa,m}\big(I(x,r)\big).\end{aligned}$$

2) The case $r^{\frac{1}{m}} \leq |x|^{\frac{1}{m}} \leq (2r)^{\frac{1}{m}}$. Therefore we have $I(x,2r) =]0, (|x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}})^m[$ and $I(x,r) =](|x|^{\frac{1}{m}} - r^{\frac{1}{m}})^m, (|x|^{\frac{1}{m}} + r^{\frac{1}{m}})^m[$. Thus,

$$\mu_{\kappa,m}\big(I(x,2r)\big) \leq \int_0^{2^{m+1}r} y^{2\kappa+\frac{2}{m}-2}\,dy = \frac{2^{(m+1)(2\kappa+\frac{2}{m}-1)}}{2\kappa + (2/m) - 1} r^{2\kappa+\frac{2}{m}-1}. \tag{3.3}$$

On the other hand,

$$\begin{aligned}\mu_{\kappa,n}\big(I(x,r)\big) &= \int_{(|x|^{\frac{1}{m}}-r^{\frac{1}{m}})^m}^{(|x|^{\frac{1}{m}}+r^{\frac{1}{m}})^m} y^{2\kappa+\frac{2}{m}-2}\,dy \geq \int_{(2^{\frac{1}{m}}-1)^m r}^{2^m r} y^{2\kappa+\frac{2}{m}-2}\,dy \\ &= \frac{2^{2\kappa m+2-m} - (2^{\frac{1}{m}} - 1)^{2\kappa m+2-m}}{2\kappa + (2/m) - 1} r^{2\kappa+\frac{2}{m}-1}.\end{aligned} \tag{3.4}$$

Now, by putting together (3.3) and (3.4), we deduce the doubling property in this case.

3) The case $|x|^{\frac{1}{m}} \geq (2r)^{\frac{1}{m}}$. That is, $I(x,2r) =](|x|^{\frac{1}{m}} - (2r)^{\frac{1}{m}})^m, (|x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}})^m[$ and $I(x,r) =](|x|^{\frac{1}{m}} - r^{\frac{1}{m}})^m, (|x|^{\frac{1}{m}} + r^{\frac{1}{m}})^m[$. By making a change of variable, we get

$$\mu_{\kappa,m}\big(I(x,2r)\big) = m\int_{|x|^{\frac{1}{m}}-(2r)^{\frac{1}{m}}}^{|x|^{\frac{1}{m}}+(2r)^{\frac{1}{m}}} u^{2\kappa m-m+1}\,du \leq 2^{\frac{1}{m}+1} m r^{\frac{1}{m}}\big(|x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}}\big)^{2\kappa m-m+1}.$$

Similarly, we have

$$\mu_{\kappa,m}\big(I(x,r)\big) = m\int_{|x|^{\frac{1}{m}}-r^{\frac{1}{m}}}^{|x|^{\frac{1}{m}}+r^{\frac{1}{m}}} u^{2\kappa m-m+1}\,du \geq 2 m r^{\frac{1}{m}}\big(|x|^{\frac{1}{m}} - r^{\frac{1}{m}}\big)^{2\kappa m-m+1}.$$

In order to obtain the doubling inequality (3.2), it is enough to find a constant α such that $(|x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}}) \leq \alpha(|x|^{\frac{1}{m}} - r^{\frac{1}{m}})$. Obviously, for every α, $\alpha(|x|^{\frac{1}{m}} - r^{\frac{1}{m}}) = |x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}} + R$ with $R = (\alpha-1)|x|^{\frac{1}{m}} - (2r)^{\frac{1}{m}} - \alpha r^{\frac{1}{m}}$. Since $|x|^{\frac{1}{m}} \geq (2r)^{\frac{1}{m}}$, then for every $\alpha \geq 1$, we have $R \geq (\alpha-2)(2r)^{\frac{1}{m}} - \alpha r^{\frac{1}{m}}$. Thus, it is enough to choose $\alpha = 2^{1+\frac{1}{m}}/(2^{\frac{1}{m}} - 1)$ (which is ≥ 1) to get $R \geq 0$, and therefore $\alpha(|x|^{\frac{1}{m}} - r^{\frac{1}{m}}) \geq |x|^{\frac{1}{m}} + (2r)^{\frac{1}{m}}$. This finishes the proof of Lemma 3.1. □

Now we are ready to prove the main result of this section.

Theorem 3.2 (Finite version of Vitali type covering lemma) *Consider the finite collection* $\mathscr{I} = \{I(x_1,r_1),\dots,I(x_N,r_N)\}$. *Then there exists a disjoint subcollection* $I(x_{j_1},r_{j_1}),\dots,I(x_{j_\ell},r_{j_\ell})$ *of* $\mathscr{I}$ *such that*

$$\mu_{\kappa,m}\Bigg(\bigcup_{i=1}^{N} I(x_i,r_i)\Bigg) \lesssim \sum_{s=1}^{\ell}\mu_{\kappa,m}\big(I(x_{j_s},r_{j_s})\big). \tag{3.5}$$

Proof The argument relies on the following observation: Suppose that $I(x,r)$ and $I(y,r')$ are a pair of intervals that intersect, with the diameter of $I(y,r')$ being not greater than that of $I(x,r)$. Then $I(y,r')$ is contained in the interval $I(x,cr)$ for some $c \geq 1$.

As a first step, we pick an interval $I(x_{j_1},r_{j_1})$ in $\mathscr{I}$ with the largest diameter, and then delete from $\mathscr{I}$ the interval $I(x_{j_1},r_{j_1})$ as well as any intervals that intersect $I(x_{j_1},r_{j_1})$. The remaining intervals yield a new collection $\mathscr{I}'$, for which we repeat the procedure. We pick $I(x_{j_2},r_{j_2})$ and any interval that intersects $I(x_{j_2},r_{j_2})$. Continuing this way, we find, after at most N steps, a collection of disjoint intervals $I(x_{j_1},r_{j_1}),\dots,I(x_{j_\ell},r_{j_\ell})$.

To prove that this disjoint collection of intervals satisfies the inequality in the theorem, we will use the doubling structure of $\mu_{\kappa,m}$ (Lemma 3.1) to prove that every removed interval $I(x_i,r_i)$ is included in a certain interval $I(x_{j_s},cr_{j_s})$, $1 \leq s \leq \ell$, for some constant $c \geq 1$.

Consider a deleted interval $I(x_i,r_i)$. From the above algorithm, there exists smallest s, $1 \leq s \leq \ell$, such that $I(x_i,r_i) \cap I(x_{j_s},r_{j_s}) \neq \varnothing$ with $\operatorname{diam} I(x_i,r_i) \leq \operatorname{diam} I(x_{j_s},r_{j_s})$. We shall prove that there exists $c \geq 1$ such that $I(x_i,r_i) \subset I(x_{j_s},cr_{j_s})$. To do so, we will distinguish two cases.

1) The case where $I(x_i,r_i) =]0,(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}})^m[$.

1-a) Presume that $I(x_{j_s},r_{j_s}) =]0,(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}})^m[$. As $\operatorname{diam} I(x_i,r_i) \leq \operatorname{diam} I(x_{j_s},r_{j_s})$, clearly we have $I(x_i,r_i) \subset I(x_{j_s},r_{j_s})$.

1-b) Presume that $I(x_{j_s},r_{j_s}) =](|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}})^m,(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}})^m[$. The fact that $\operatorname{diam} I(x_i,r_i) \leq \operatorname{diam} I(x_{j_s},r_{j_s})$ implies

$$\big(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\big)^m \leq \big(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\big)^m \leq \big(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\big)^m - \big(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\big)^m.$$

Using the binomial formula, we get

$$\sum_{\ell \text{even}} \binom{m}{\ell} |x_{j_s}|^{\frac{m-\ell}{m}} r_{j_s}^{\frac{\ell}{m}} - \sum_{\ell \text{ odd}} \binom{m}{\ell} |x_{j_s}|^{\frac{m-\ell}{m}} r_{j_s}^{\frac{\ell}{m}} \leq 2 \sum_{\ell \text{ odd}} \binom{m}{\ell} |x_{j_s}|^{\frac{m-\ell}{m}} r_{j_s}^{\frac{\ell}{m}}.$$

That is,

$$\sum_{\ell} \binom{m}{2\ell} |x_{j_s}|^{\frac{m-2\ell}{m}} r_{j_s}^{\frac{2\ell}{m}} \leq 3 \sum_{\ell} \binom{m}{2\ell+1} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell+1}{m}}.$$

Using the well-known facts that $\binom{m}{2\ell} = \binom{m-1}{2\ell} + \binom{m-1}{2\ell-1}$ together with $\binom{m}{2\ell+1} \leq m\binom{m-1}{2\ell}$, we deduce that

$$\begin{aligned} &\sum_{\ell} \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell}{m}} r_{j_s}^{\frac{2\ell}{m}} + \sum_{\ell} \binom{m-1}{2\ell-1} |x_{j_s}|^{\frac{m-2\ell}{m}} r_{j_s}^{\frac{2\ell}{m}} \\ &\quad \leq 3mr_{j_s}^{\frac{1}{m}} \sum_{\ell} \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}}, \end{aligned}$$

which implies

$$\bigl(|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}}\bigr) \sum_{\ell} \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}} + \sum_{\ell} \binom{m-1}{2\ell-1} |x_{j_s}|^{\frac{m-2\ell}{m}} r_{j_s}^{\frac{2\ell}{m}} \leq 0.$$

That is,

$$|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}} \leq 0.$$

In these circumstances, the interval $I(x_{j_s}, (3m)^m r_{j_s}) =]0, |x_{j_s}|^{\frac{1}{m}} + 3mr_{j_s}^{\frac{1}{m}}[$, which contains the interval $I(x_i, r_i)$ as

$$\bigl(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\bigr)^m \leq \bigl(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\bigr)^m \leq \bigl(|x_{j_s}|^{\frac{1}{m}} + 3mr_{j_s}^{\frac{1}{m}}\bigr)^m.$$

2) The case where $I(x_i, r_i) =](|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}})^m, (|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}})^m[$. In other words, here we assume $|x_i|^{\frac{1}{m}} \geq r_i^{\frac{1}{m}}$.

2-a) Presume that $I(x_{j_s}, r_{j_s}) =]0, (|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}})^m[$. That is, $|x_{j_s}|^{\frac{1}{m}} \leq r_{j_s}^{\frac{1}{m}}$.

2-a-i) Let $(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}})^m \leq (|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}})^m$. Then, clearly, we have $I(x_i, r_i) \subset I(x_{j_s}, r_{j_s})$.

2-a-ii) Let $(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}})^m \geq (|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}})^m$. Since $I(x_i, r_i) \cap I(x_{j_s}, r_{j_s}) \neq \varnothing$, it follows

$$\bigl(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\bigr)^m \leq \bigl(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\bigr)^m \leq \bigl(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\bigr)^m.$$

Using the facts that $\operatorname{diam} I(x_i, r_i) \leq \operatorname{diam} I(x_{j_s}, r_{j_s})$ and $|x_{j_s}|^{\frac{1}{m}} \leq r_{j_s}^{\frac{1}{m}}$, we deduce that

$$\begin{aligned} \bigl(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\bigr)^m &\leq \bigl(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\bigr)^m + \bigl(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\bigr)^m \\ &\leq \bigl(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\bigr)^m + \bigl(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\bigr)^m \end{aligned}$$

$$\begin{aligned}
&\le \left(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\right)^m + 2^m r_{j_s} \\
&= \sum_{\ell=0}^{m-1} \binom{m}{\ell} |x_{j_s}|^{\frac{m-\ell}{m}} r_{j_s}^{\frac{\ell}{m}} + \left(2^m + 1\right) r_{j_s} \\
&\le \left(|x_{j_s}|^{\frac{1}{m}} + \left(2^m + 1\right)^{\frac{1}{m}} r_{j_s}^{\frac{1}{m}}\right)^m. \qquad (3.6)
\end{aligned}$$

On the other hand, the fact that $|x_{j_s}|^{\frac{1}{m}} \le r_{j_s}^{\frac{1}{m}} \le ((2^m+1)r_{j_s})^{\frac{1}{m}}$ implies $I(x_{j_s}, (2^m+1)r_{j_s}) =]0, (|x_{j_s}|^{\frac{1}{m}} + ((2^m+1)r_{j_s})^{\frac{1}{m}})^m[$. This fact together with inequality (3.6) yields the inclusion $I(x_i, r_i) \subset I(x_{j_s}, (2^m+1)r_{j_s})$.

2-b) Presume that $I(x_{j_s}, r_{j_s}) =](|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}})^m, (|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}})^m[$. That is, $|x_{j_s}|^{\frac{1}{m}} \ge r_{j_s}^{\frac{1}{m}}$. Based on the fact that $I(x_i, r_i) \cap I(x_{j_s}, r_{j_s}) \neq \varnothing$, there will be three possible cases to be discussed.

2-b-i) The first possibility is

$$\left(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\right)^m \le \left(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\right)^m \le \left(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\right)^m \le \left(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\right)^m.$$

Then, obviously, we have $I(x_i, r_i) \subset I(x_{j_s}, r_{j_s})$.

2-b-ii) The second possibility is

$$\left(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\right)^m \le \left(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\right)^m \le \left(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\right)^m \le \left(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\right)^m.$$

Since $\operatorname{diam} I(x_i, r_i) \le \operatorname{diam} I(x_{j_s}, r_{j_s})$, we have

$$\left(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\right)^m \le 2\left(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\right)^m - \left(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\right)^m.$$

In other words,

$$\begin{aligned}
\left(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\right)^m &\le \sum_{\ell=0}^{m} \binom{m}{\ell} \left(2 - (-1)^\ell\right) |x_{j_s}|^{\frac{m-\ell}{m}} r_{j_s}^{\frac{\ell}{m}} \\
&\le |x_{j_s}| + \sum_{\ell=1}^{m} \binom{m}{\ell} 3^\ell |x_{j_s}|^{\frac{m-\ell}{m}} r_{j_s}^{\frac{\ell}{m}} = \left(|x_{j_s}|^{\frac{1}{m}} + 3 r_{j_s}^{\frac{1}{m}}\right)^m. \qquad (3.7)
\end{aligned}$$

Now, if $|x_{j_s}|^{\frac{1}{m}} - 3r_{j_s}^{\frac{1}{m}} \le 0$, then $I(x_{j_s}, 3^m r_{j_s}) =]0, (|x_{j_s}|^{\frac{1}{m}} + 3r_{j_s}^{\frac{1}{m}})^m[$, and hence $I(x_i, r_i) \subset I(x_{j_s}, 3^m r_{j_s})$.

If $|x_{j_s}|^{\frac{1}{m}} - 3r_{j_s}^{\frac{1}{m}} \ge 0$, then $I(x_{j_s}, 3^m r_{j_s}) =](|x_{j_s}|^{\frac{1}{m}} - 3r_{j_s}^{\frac{1}{m}})^m, (|x_{j_s}|^{\frac{1}{m}} + 3r_{j_s}^{\frac{1}{m}})^m[$. Using (3.7), we deduce that

$$\begin{aligned}
\left(|x_{j_s}|^{\frac{1}{m}} - 3r_{j_s}^{\frac{1}{m}}\right)^m &\le \left(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\right)^m \le \left(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\right)^m \\
&\le \left(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\right)^m \le \left(|x_{j_s}|^{\frac{1}{m}} + 3r_{j_s}^{\frac{1}{m}}\right)^m,
\end{aligned}$$

and therefore $I(x_i, r_i) \subset I(x_{j_s}, 3^m r_{j_s})$.

2-b-iii) The third possibility is

$$\big(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\big)^m \le \big(|x_{j_s}|^{\frac{1}{n}} - r_{j_s}^{\frac{1}{m}}\big)^m \le \big(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\big)^m \le \big(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\big)^m. \tag{3.8}$$

Since $\operatorname{diam} I(x_i, r_i) \le \operatorname{diam} I(x_{j_s}, r_{j_s})$, the recurrence relation $\binom{m}{\ell} = \binom{m-1}{\ell} + \binom{m-1}{\ell-1}$ implies

$$\begin{aligned}
\big(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\big)^m &= \big(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\big)^m - \big\{\big(|x_i|^{\frac{1}{m}} + r_i^{\frac{1}{m}}\big)^m - \big(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\big)^m\big\} \\
&\ge \big(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\big)^m - \big\{\big(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\big)^m - \big(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\big)^m\big\} \\
&= \sum_\ell \binom{m}{2\ell} |x_{j_s}|^{\frac{m-2\ell}{m}} r_{j_s}^{\frac{2\ell}{m}} - 3\sum_\ell \binom{m}{2\ell+1} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell+1}{m}} \\
&= |x_{j_s}|^{\frac{1}{m}} \sum_\ell \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}} + |x_{j_s}|^{\frac{1}{m}} \sum_\ell \binom{m-1}{2\ell-1} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}} \\
&\quad - 3r_{j_s}^{\frac{1}{m}} \sum_\ell \frac{m}{2\ell+1} \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}} \\
&\ge \big(|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}}\big) \sum_\ell \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}} \\
&\quad + |x_{j_s}|^{\frac{1}{m}} \sum_\ell \binom{m-1}{2\ell-1} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}}.
\end{aligned} \tag{3.9}$$

Now, if $|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}} \le 0$, then $I(x_{j_s}, (3m)^m r_{j_s}) =]0, (|x_{j_s}|^{\frac{1}{m}} + 3mr_{j_s}^{\frac{1}{m}})^m[$, and by (3.8), clearly, we have $I(x_i, r_i) \subset I(x_{j_s}, (3m)^m r_{j_s})$.

If $|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}} \ge 0$, then $I(x_{j_s}, (3m)^m r_{j_s}) =]|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}})^m, (|x_{j_s}|^{\frac{1}{m}} + 3mr_{j_s}^{\frac{1}{m}})^m[$. Recall from (3.8) that

$$\big(|x_i|^{\frac{1}{m}} + r_j^{\frac{1}{m}}\big)^m \le \big(|x_{j_s}|^{\frac{1}{m}} + r_{j_s}^{\frac{1}{m}}\big)^m \le \big(|x_{j_s}|^{\frac{1}{m}} + 3mr_{j_s}^{\frac{1}{m}}\big)^m.$$

Further, by inequality (3.9), we have

$$\begin{aligned}
&\big(|x_i|^{\frac{1}{m}} - r_i^{\frac{1}{m}}\big)^m \\
&\ge \big(|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}}\big) \sum_\ell \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}} \\
&\ge \big(|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}}\big) \left\{ \sum_\ell \binom{m-1}{2\ell} |x_{j_s}|^{\frac{m-2\ell-1}{m}} r_{j_s}^{\frac{2\ell}{m}} - \sum_\ell \binom{m-1}{2\ell+1} |x_{j_s}|^{\frac{m-1-2\ell-1}{m}} r_{j_s}^{\frac{2\ell+1}{m}} \right\} \\
&= \big(|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}}\big)\big(|x_{j_s}|^{\frac{1}{m}} - r_{j_s}^{\frac{1}{m}}\big)^{m-1} \\
&\ge \big(|x_{j_s}|^{\frac{1}{m}} - 3mr_{j_s}^{\frac{1}{m}}\big)^m.
\end{aligned}$$

In conclusion, $I(x_i, r_i) \subset I(x_{j_s}, (3m)^m r_{j_s})$.

In summary, we have proved that every interval $I(x_i, r_i)$ is either of type $I(x_{j_s}, r_{j_s})$ or included in a certain interval of type $I(x_{j_s}, cr_{j_s})$ for some constant $c \ge 1$. Thus, using the doubling property of the measure $\mu_{\kappa,m}$ (see Lemma 3.1), we conclude that

$$\mu_{\kappa,m}\left(\bigcup_{i=1}^{N} I(x_i,r_i)\right) \le \mu_{\kappa,m}\left(\bigcup_{s=1}^{\ell} I(x_{j_s},cr_{j_s})\right) \le \sum_{s=1}^{\ell} \mu_{\kappa,m}\big(I(x_{j_s},cr_{j_s})\big)$$
$$\lesssim \sum_{s=1}^{\ell} \mu_{\kappa,m}\big(I(x_{j_s},r_{j_s})\big).$$

This finishes the proof of Theorem 3.2. □

4 A sharp estimate

The main goal of this section is to obtain a sharp estimate for $\tau_x(\chi_r)$, where χ_r denotes the characteristic function of the interval $]-r,r[$ with $r>0$ (see Corollary 4.5 below). This result will play a key role in the proof of the weak type $(1,1)$ estimates for the maximal operator $\mathscr{M}_{k,m}$. To obtain the sharp estimates, some preliminary lemmas are needed.

Lemma 4.1 *The kernel $E_{\kappa,m}$ given in (1.1) satisfies*

$$\big|E_{\kappa,m}(x,y)\big| \lesssim |xy|^{-\kappa+\frac{1}{2}-\frac{1}{2m}} \tag{4.1}$$

for all $x,y\in\mathbb{R}\setminus\{0\}$.

Proof From the definition of $E_{\kappa,m}$ we have

$$\big|E_{\kappa,m}(x,y)\big| \le \left(\frac{m}{2}\right)^{-\kappa m+\frac{m}{2}} |\lambda x|^{-\kappa+\frac{1}{2}}\left(\big|J_{\kappa m-\frac{m}{2}}\big(m|\lambda x|^{\frac{1}{m}}\big)\big| + \big|J_{\kappa m+\frac{m}{2}}\big(m|\lambda x|^{\frac{1}{m}}\big)\big|\right).$$

The lemma follows from the bound $J_\alpha(u)=O(u^{-1/2})$ for every $\alpha>-1$ and $u\ge 0$ [1, p. 238]. □

Lemma 4.2 *For all $x\in\mathbb{R}$, we have*

$$\mathscr{F}_{\kappa,m}(\chi_r)(x) \lesssim r^{2\kappa+\frac{2}{m}-1}, \tag{4.2}$$

while if $x\neq 0$, then we have

$$\mathscr{F}_{\kappa,m}(\chi_r)(x) \lesssim \frac{r^{\kappa-\frac{1}{2}+\frac{1}{2m}}}{|x|^{\kappa-\frac{1}{2}+\frac{3}{2m}}}. \tag{4.3}$$

Proof Inequality (4.2) follows immediately from the fact that $|E_{\kappa,m}(x,y)|\le 1$ for all $x,y\in\mathbb{R}$ (see [19, Lemma 2.9]). To prove (4.3), observe that

$$\mathscr{F}_{\kappa,m}(\chi_r)(y) = \left(\frac{2}{m}\right)^{-(km-\frac{m}{2})} \int_0^r \widetilde{J}_{\kappa m-\frac{m}{2}}\big(m|x|^{\frac{1}{m}}y^{\frac{1}{m}}\big)y^{2\kappa+\frac{2}{m}-2}\,dy$$
$$= m|x|^{-\kappa+\frac{1}{2}}r^{\kappa+\frac{2}{m}-\frac{1}{2}} \int_0^1 J_{\kappa m-\frac{m}{2}}\big(m|x|^{\frac{1}{m}}r^{\frac{1}{m}}u\big)u^{\kappa m-\frac{m}{2}+1}\,du.$$

By Sonine's formula [29, §12.11 (1)], we get

$$\mathscr{F}_{\kappa,m}(\chi_r)(x) = \left(\frac{r}{|x|}\right)^{\kappa-\frac{1}{2}+\frac{1}{m}} J_{\kappa m-\frac{m}{2}+1}\big(m|x|^{\frac{1}{m}} r^{\frac{1}{m}}\big).$$

Using again the boundedness $J_\alpha(u) = O(u^{-1/2})$ for all $\alpha > -1$ and $u \geq 0$, we deduce the upper bound in (4.3) □

One more lemma is now needed. It concerns the function

$$h_t(x) = m^{-1}\Gamma\left(\kappa m - \frac{m}{2}+1\right)^{-1} t^{-\kappa m+\frac{m}{2}-1} e^{-\frac{|x|^{\frac{2}{m}}}{t}},$$

which solves the heat equation $|x|^{2-\frac{2}{m}}\Delta_k^x u(x,t) - \frac{4}{m^2}\partial_t u(x,t) = 0$ on $\mathbb{R}\times\mathbb{R}_{>0}$. It is the analogue of the fundamental solution for the classical heat equation $\Delta^x u(x,t) - \partial_t u(x,t) = 0$, which is given by $h_t^0(x) = t^{-1/2} e^{\frac{-|x|^2}{4t}}$, up to a normalization constant.

Lemma 4.3 *The heat function h_t satisfies $\|h_t\|_{L^1_\kappa} = 1$, and*

$$\mathscr{F}_{\kappa,m}(h_t)(x) = 2^{-1}\left(\frac{m}{2}\right)^{km-\frac{m}{2}}\Gamma\left(\kappa m - \frac{m}{2}+1\right)^{-1} e^{-\frac{m^2}{4}t|x|^{\frac{2}{m}}}. \tag{4.4}$$

Proof The first statement follows immediately from the following known identity:

$$|y|^{-a} = \Gamma\left(\frac{a}{2}\right)^{-1}\int_0^\infty u^{\frac{a}{2}} e^{-u|y|^2}\frac{du}{u}. \tag{4.5}$$

For the second part of the statement, using the fact that h_t is an even function and Weber's first exponential integral

$$\int_0^\infty J_\nu(au)u^{\nu+1}e^{-p^2u^2}\,du = \frac{a^\nu}{(2p^2)^{\nu+1}} e^{-\frac{a^2}{4p^2}}, \tag{4.6}$$

we deduce that

$$\begin{aligned}
&\mathscr{F}_{\kappa,m}(h_t)(x)\\
&= \left(\frac{2}{m}\right)^{-(km-\frac{m}{2})}\int_0^\infty h_t(y)\widetilde{J}_{\kappa m-\frac{m}{2}}\big(m|x|^{\frac{1}{m}}y^{\frac{1}{m}}\big)y^{2\kappa+\frac{2}{m}-2}\,dy\\
&= \Gamma\left(\kappa m-\frac{m}{2}+1\right)^{-1} t^{-\kappa m+\frac{m}{2}-1}|x|^{-k+\frac{1}{2}}\int_0^\infty e^{-\frac{u^2}{t}} J_{\kappa m-\frac{m}{2}}\big(m|x|^{\frac{1}{m}}u\big)u^{\kappa m-\frac{m}{2}+1}\,du\\
&= 2^{-1}\left(\frac{m}{2}\right)^{\kappa m-\frac{m}{2}}\Gamma\left(\kappa m-\frac{m}{2}+1\right)^{-1} e^{-\frac{m^2}{4}t|x|^{\frac{2}{m}}}.
\end{aligned}$$

□

These lemmas will give considerable help in establishing the following key step towards the main result of this section.

Theorem 4.4 *For all $x, y \in \mathbb{R}^*$, we have*

$$\left|\tau_x(\chi_r)(y)\right| \lesssim \left(\frac{r}{|x|}\right)^{2\kappa+\frac{1}{m}-1}.$$

Proof We shall distinguish two cases:

1) Assume first that $|x|^{\frac{1}{m}} \le mr^{\frac{1}{m}}$. According to Theorem 2.2, we have

$$\left|\tau_x(\chi_r)(y)\right| \le \left\|\tau_x(\chi_r)\right\|_{L^\infty_\kappa} \lesssim \|\chi_r\|_{L^\infty_\kappa} \le 1 \lesssim \left(\frac{r}{|x|}\right)^{2\kappa+\frac{1}{m}-1}.$$

2) Next, assume that $|x|^{\frac{1}{m}} \ge mr^{\frac{1}{m}}$. We may choose y such that $(|x|^{\frac{1}{m}} - |y|^{\frac{1}{m}})^m \le r$; otherwise, in view of the support of the translation operator τ_x, we have $\tau_x(\chi_r)(y) = 0$. We claim that $\tau_x(\chi_r \circledast h_t)$ and $\mathscr{F}_{\kappa,m}(\tau^\kappa_x(\chi_r \circledast h_t))$ belong to $L^1(\mu_{k,m})$. Therefore, by the inversion formula for $\mathscr{F}_{k,m}$ and Lemma 4.3, we obtain

$$\begin{aligned}
\tau_x(\chi_r \circledast h_t)(y) &= 2^{-2}\left(\frac{m}{2}\right)^{2\kappa m-m}\Gamma\left(\kappa m - \frac{m}{2}+1\right)^{-1} \\
&\quad \times \int_{\mathbb{R}} E_{\kappa,m}\big((-1)^m x, z\big)E_{\kappa,m}\big((-1)^m y, z\big)\mathscr{F}_{k,m}(\chi_r)(z)e^{-\frac{m^2}{4}t|z|^{\frac{2}{m}}}\,d\mu_{\kappa,m}(z) \\
&= \underbrace{\int_{\{z\in\mathbb{R}:|z|\le\frac{1}{r}\}}\cdots}_{:=I_1} + \underbrace{\int_{\{z\in\mathbb{R}:|z|\ge\frac{1}{r}\}}\cdots}_{:=I_2}.
\end{aligned} \tag{4.7}$$

Now, let us prove the above claim. By Theorem 2.2, clearly the function $\chi_r \circledast h_t$ belongs to $L^1(\mu_{\kappa,m})$. Then, for all $x \in \mathbb{R}$, $\tau_x(\chi_r \circledast h_t) \in L^1(\mu_{\kappa,m})$ as $\|\tau_x(\chi_r \circledast h_t)\|_{L^1_\kappa} \le 4\Gamma(\kappa m - \frac{m}{2} + 1)^{-1}\|\chi_r \circledast h_t\|_{L^1_\kappa}$. Furthermore,

$$\begin{aligned}
\left|\mathscr{F}_{\kappa,m}\big(\tau_x(\chi_r \circledast h_t)\big)(y)\right| &= \left|E_{\kappa,m}\big((-1)^m x, y\big)\right|\left|\mathscr{F}_{\kappa,m}(\chi_r \circledast h_t)(y)\right| \\
&\le \left|\mathscr{F}_{\kappa,m}(\chi_r \circledast h_t)(y)\right|.
\end{aligned} \tag{4.8}$$

Above we have used $|E_{\kappa,m}(x,y)| \le 1$ for all $x, y \in \mathbb{R}$ (see [19, Lemma 2.9]). On the other hand, Hölder's inequality and the Plancherel formula imply

$$\begin{aligned}
\left\|\mathscr{F}_{\kappa,m}(\chi_r \circledast h_t)\right\|_{L^1_\kappa} &= \left\|\mathscr{F}_{\kappa,m}(\chi_r)\mathscr{F}_{\kappa,m}(h_t)\right\|_{L^1_\kappa} \le \left\|\mathscr{F}_{\kappa,m}(\chi_r)\right\|_{L^2_\kappa}\left\|\mathscr{F}_{\kappa,m}(h_t)\right\|_{L^2_\kappa} \\
&= \|\chi_r\|_{L^2}\|h_t\|_{L^2_\kappa} < \infty.
\end{aligned}$$

Thus, $\mathscr{F}_{\kappa,m}(\chi_r \circledast h_t) \in L^1(\mu_{\kappa,m})$. It follows from (4.8) that $\mathscr{F}_{\kappa,m}(\tau^\kappa_x(\chi_r \circledast h_t)) \in L^1(\mu_{\kappa,m})$. This finishes the proof of the above claim.

Next let us estimate the integrals I_1 and I_2 in (4.7). From Lemma 4.1 and relation (4.2), we find

$$\begin{aligned}
|I_1| &\le 2^{-1}\left(\frac{m}{2}\right)^{2\kappa m-m}\Gamma\left(\kappa m - \frac{m}{2}+1\right)^{-1}\frac{r^{2\kappa+\frac{2}{m}-1}}{|xy|^{\kappa-\frac{1}{2}+\frac{1}{2m}}}\int_0^{\frac{1}{r}} z^{\frac{1}{m}-1}\,dz \\
&= \left(\frac{m}{2}\right)^{2\kappa m-m+1}\Gamma\left(\kappa m - \frac{m}{2}+1\right)^{-1}\frac{r^{2\kappa+\frac{1}{m}-1}}{|xy|^{\kappa-\frac{1}{2}+\frac{1}{2m}}}.
\end{aligned}$$

The assumptions $|x|^{\frac{1}{m}} - |y|^{\frac{1}{m}} \leq r^{\frac{1}{m}}$ and $|x|^{\frac{1}{m}} \geq mr^{\frac{1}{m}}$ lied to $|y|^{\frac{1}{m}} \geq \frac{m-1}{m}|x|^{\frac{1}{m}}$. It follows

$$|I_1| \lesssim \frac{r^{2\kappa+\frac{1}{m}-1}}{|x|^{2\kappa+\frac{1}{m}-1}}. \tag{4.9}$$

We claim that $|I_2|$ satisfies the same boundedness as in (4.9). Indeed, using Lemma 4.1 and relation (4.3), we get

$$\begin{aligned} |I_2| &\leq 2^{-1}\left(\frac{m}{2}\right)^{2\kappa m-m}\Gamma\left(\kappa m-\frac{m}{2}+1\right)^{-1}\frac{r^{\kappa+\frac{1}{2m}-\frac{1}{2}}}{|xy|^{\kappa+\frac{1}{2m}-\frac{1}{2}}}\int_{\frac{1}{r}}^{\infty} z^{-\kappa-\frac{1}{2m}-\frac{1}{2}}\,dz \\ &= \left(\frac{m}{2}\right)^{2\kappa m-m+1}\Gamma\left(\kappa m-\frac{m}{2}+1\right)^{-1}\frac{2}{2\kappa m+1-m}\frac{r^{2\kappa+\frac{1}{m}-1}}{|xy|^{\kappa+\frac{1}{2m}-\frac{1}{2}}}. \end{aligned} \tag{4.10}$$

As a consequence of (4.9) and (4.10), we obtain, for all $t > 0$ and $y \in \mathbb{R}$,

$$\left|\tau_x(\chi_r \circledast h_t)(y)\right| \lesssim \left(\frac{r}{|x|}\right)^{2\kappa+\frac{1}{m}-1}. \tag{4.11}$$

Finally, by the Plancherel formula for $\mathscr{F}_{\kappa,m}$, the convolution product $\chi_r \circledast h_t$ goes to χ_r in $L^2(\mu_{\kappa,m})$ as $t \to 0$. Further, the L^2-boundedness of τ_x for every $x \in \mathbb{R}$ implies $\tau_x(\chi_r \circledast h_t) \to \tau_x(\chi_r)$ as $t \to 0$ in $L^2(\mu_{\kappa,m})$. Therefore, by a standard argument [9], inequality (4.11) leads to

$$\left|\tau_x(\chi_r)(y)\right| \lesssim \left(\frac{r}{|x|}\right)^{2\kappa+\frac{1}{m}-1}.$$

This finishes the proof of Theorem 4.4. □

Now we are ready to state the main result of this section, i.e., a sharp estimate of $|\tau_x(\chi_r)(y)|$.

Corollary 4.5 *For every $x, y \in \mathbb{R}^*$, we have*

$$\left|\tau_x(\chi_r)(y)\right| \lesssim \frac{\mu_{\kappa,m}(]-r,r[)}{\mu_{\kappa,m}(I(x,r))}.$$

Proof 1) Assume that $|x|^{\frac{1}{m}} \leq r^{\frac{1}{m}}$. Then $I(x,r) =]0,(|x|^{\frac{1}{m}} + r^{\frac{1}{m}})^m[$, and

$$\begin{aligned} \mu_{\kappa,m}\big(I(x,r)\big) &= \int_0^{(|x|^{\frac{1}{m}}+r^{\frac{1}{m}})^m} d\mu_{\kappa,m}(z) \\ &\leq \int_0^{2^m r} z^{2\kappa+\frac{2}{m}-2}\,dz \\ &= \frac{2^{2\kappa m+2-m}m}{2\kappa m+2-m} r^{2\kappa+\frac{2}{m}-1} = 2^{2\kappa m+1-m}\mu_{\kappa,m}\big(]-r,r[\big). \end{aligned}$$

On the other hand,

$$\left|\tau_x(\chi_r)(y)\right| \leq \left\|\tau_x(\chi_r)\right\|_{L^\infty_\kappa} \lesssim \|\chi_r\|_{L^\infty_\kappa} = 1 \lesssim \frac{\mu_{\kappa,m}(]-r,r[)}{\mu_{\kappa,m}(I(x,r))}.$$

2) Assume that $|x|^{\frac{1}{m}} \geq r^{\frac{1}{m}}$. Then $I(x,r) =](|x|^{\frac{1}{m}} - r^{\frac{1}{m}})^m, (|x|^{\frac{1}{m}} + r^{\frac{1}{m}})^m[$, and

$$\begin{aligned}
\mu_{\kappa,n}\big(I(x,r)\big) &= \int_{(|x|^{\frac{1}{m}}-r^{\frac{1}{m}})^m}^{(|x|^{\frac{1}{m}}+r^{\frac{1}{m}})^m} d\mu_{\kappa,m}(z) \\
&= m\int_{|x|^{\frac{1}{m}}-r^{\frac{1}{m}}}^{|x|^{\frac{1}{m}}+r^{\frac{1}{m}}} u^{2\kappa m-m+1}\,du \\
&\leq 2mr^{\frac{1}{m}}\big(|x|^{\frac{1}{m}}+r^{\frac{1}{m}}\big)^{2\kappa m-m+1} \\
&\leq 2^{2\kappa m-m+2} m r^{2\kappa-1+\frac{2}{m}}\left(\frac{|x|}{r}\right)^{2\kappa-1+\frac{1}{m}} \\
&= 2^{2\kappa m-m+1}(2km+2-m)\left(\frac{|x|}{r}\right)^{2\kappa-1+\frac{1}{m}}\mu_{k,m}\big(]-r,r[\big).
\end{aligned}$$

Thus

$$\left(\frac{r}{|x|}\right)^{2\kappa-1+\frac{1}{m}} \lesssim \frac{\mu_{\kappa,m}(]-r,r[)}{\mu_{\kappa,m}(I(x,r))}.$$

Now, Theorem 4.4 finishes the proof in this case. □

5 Weak and strong type estimates

For a locally integrable function f on $\mathbb{R}$ with respect to the measure $\mu_{\kappa,m}$, we introduce the maximal function

$$\mathbb{M}_{\kappa,m}f(x) = \sup_{r>0}\frac{1}{\mu_{\kappa,m}(I(x,r))}\int_{|y|\in I(x,r)}\big|f(y)\big|\,d\mu_{\kappa,m}(y).$$

We claim that

$$\mathscr{M}_{\kappa,m}f(x) \lesssim \mathbb{M}_{\kappa,m}f(x) \tag{5.1}$$

holds pointwise. Indeed, if $x = 0$, then $\tau_0 f((-1)^m y) = \Gamma(\kappa m - \frac{m}{2} + 1)^{-1} f((-1)^m y)$ and $I(0,r) =]0,r[$. Thus, (5.1) holds when $x = 0$. Next, let us assume $x \neq 0$. Since $|y| \notin I(x,r)$ implies $\tau_x(\chi_r)((-1)^m y) = 0$, it follows from Corollary 4.5

$$\left|\int_{\mathbb{R}} f(y)\tau_x(\chi_r)\big((-1)^m y\big)\,d\mu_{\kappa,m}(y)\right| \lesssim \frac{\mu_{\kappa,m}(]-r,r[)}{\mu_{\kappa,m}(I(x,r))}\int_{\{y\in\mathbb{R}:|y|\in I(x,r)\}}\big|f(y)\big|\,d\mu_{\kappa,m}(y).$$

This finishes the proof of inequality (5.1).

Theorem 5.1 *We have:*

1) *The maximal operator* $\mathbb{M}_{\kappa,m}$ *is weak type* $(1,1)$. *That is, for every* $f \in L^1(\mu_{\kappa,m})$ *and* $\lambda > 0$,

$$\mu_{\kappa,m}\big(\big\{x : \big|\mathbb{M}_{\kappa,m}f(x)\big| > \lambda\big\}\big) \lesssim \frac{\|f\|_{L^1_\kappa}}{\lambda}.$$

2) *For every* $1 < p \leq \infty$, *the maximal operator* $\mathbb{M}_{\kappa,m}$ *is strong type* (p,p). *That is, for every* $f \in L^p(\mu_{\kappa,m})$,

$$\|\mathbb{M}_{\kappa,m}f\|_{L^p_\kappa} \lesssim \|f\|_{L^p_\kappa}.$$

Proof It is obvious that $\mathbb{M}_{\kappa,m}$ is bounded on $L^\infty(\mu_{\kappa,m})$ (indeed, it is a contraction on this space). To prove that $\mathbb{M}_{\kappa,m}$ is strong type (p,p) for $1 < p < \infty$, it suffices by Marcinkiewicz's interpolation theorem [24, p. 21] to prove the weak type $(1,1)$ inequality. Thus, the proof of the theorem reduces to the proof of the weak type $(1,1)$ inequality.

For $\lambda > 0$, consider the sets $\mathbb{R}^+_\lambda = \{x \in \mathbb{R}_{\geq 0} : \mathbb{M}_{\kappa,m}f(x) > \lambda\}$ and $\mathbb{R}^-_\lambda = \{x \in \mathbb{R}_{\leq 0} : \mathbb{M}_{\kappa,m}f(x) > \lambda\}$. Then we have

$$\mu_{\kappa,m}\big(\{x \in \mathbb{R} : \mathbb{M}_{\kappa,m}f(x) > \lambda\}\big) \leq \mu_{\kappa,m}\big(\mathbb{R}^+_\lambda\big) + \mu_{\kappa,m}\big(\mathbb{R}^-_\lambda\big).$$

Since $\mathbb{M}_{\kappa,m}f(-x) = \mathbb{M}_{\kappa,m}f(x)$, we get

$$\mu_{\kappa,m}\big(\{x \in \mathbb{R} : \mathbb{M}_{\kappa,m}f(x) > \lambda\}\big) \leq 2\mu_{\kappa,m}\big(\mathbb{R}^+_\lambda\big).$$

To prove that $\mathbb{M}_{\kappa,n}$ is weak type $(1,1)$, it is enough to prove

$$\mu_{\kappa,m}\big(\mathbb{R}^+_\lambda\big) \lesssim \frac{\|f\|_{L^1_\kappa}}{\lambda}.$$

Fix f and λ. By the inner regularity of the weighted Lebesgue measure $\mu_{\kappa,m}$, it suffices to show that for all compact $K \subset \mathbb{R}^+_\lambda$, we have

$$\mu_{\kappa,m}(K) \lesssim \frac{1}{\lambda}\|f\|_{L^1_\kappa},$$

and then to take supremum over K. Hence, let K be arbitrary. By the definition of K, for each $x \in K$, there exists $r_x > 0$ such that

$$\frac{1}{\lambda}\int_{|y|\in I(x,r_x)} \big|f(y)\big|\,d\mu_{\kappa,m}(y) > \mu_{\kappa,m}\big(I(x,r_x)\big). \tag{5.2}$$

Construct a collection $I(x,r_x)_{x\in K}$ of such intervals with $K \subset \cup_{x\in K} I(x,r_x)$. In other words, $I(x,r_x)_{x\in K}$ is an open cover of K. By compactness, we then must be able to find a finite subcover $I(x_j,r_j)_{1\leq j\leq N}$ of $I(x,r_x)_{x\in K}$. Using the Vitali covering Lemma 3.2, there is a disjoint subcover $I(x_{j_s},r_{j_s})_{1\leq s\leq \ell}$ of $I(x_j,r_j)_{1\leq j\leq N}$, which still covers all of the K, such that

$$\mu_{\kappa,m}(K) \lesssim \sum_{s=1}^{\ell} \mu_{\kappa,m}\big(I(x_{j_s},r_{j_s})\big).$$

Applying inequality (5.2) for $x = x_{j_s}$, we get

$$\mu_{\kappa,m}(K) \lesssim \frac{1}{\lambda}\sum_{s=1}^{\ell}\int_{|y|\in I(x_{j_s},r_{j_s})} \big|f(y)\big|\,d\mu_{\kappa,m}(y).$$

Therefore

$$\mu_{\kappa,m}(K) \lesssim \frac{1}{\lambda} \int_{|y|\in\bigcup_{s=1}^{\ell} I(x_{j_s},r_{j_s})} |f(y)|\, d\mu_{\kappa,m}(y)$$
$$\lesssim \frac{1}{\lambda} \|f\|_{L^1_\kappa}.$$

This finishes the proof of the weak type (1, 1) inequality, and by consequence the proof of Theorem 5.1. □

As an immediate consequence of the main result, we deduce the following Lebesgue differentiation type theorem.

Corollary 5.2 *If $f \in L^1_{\mathrm{loc}}(\mu_{\kappa,m})$, then*

$$\lim_{r\to 0} \frac{1}{\mu_{\kappa,m}(]-r,r[)} \int_{\mathbb{R}} f(y)\tau_x(\chi_r)\big((-1)^m y\big)\, d\mu_{\kappa,m} = f(x) \quad a.e.$$

Proof Since the statement is local in nature, we can assume that $f \in L^1(\mu_{\kappa,m})$. Let $F_r(x) = \frac{1}{\mu_{\kappa,m}(]-r,r[)} \int_{\mathbb{R}} f(y)\tau_x(\chi_r)((-1)^m y)\, d\mu_{\kappa,m}$ and define

$$\Omega f(x) = \Big|\limsup_{r\to 0} F_r(x) - \liminf_{r\to 0} F_r(x)\Big|.$$

It suffices to prove that $\Omega f = 0$ a.e. and that $F_r \to f$ in $L^1(\mu_{\kappa,m})$. Indeed, the first property means that F_r converges a.e. to a measurable function g, while the second one implies that for a subsequence $F_{r_i} \to f$ a.e. and hence $g = f$ a.e.

We have $\Omega f \le 2\mathscr{M}_{\kappa,m} f$, and hence for any $\varepsilon > 0$, the weak (1, 1) estimate yields

$$\mu_{\kappa,m}\big(\{x : \Omega f(x) > \varepsilon\}\big) \lesssim \frac{\|f\|_{L^1_\kappa}}{\varepsilon}.$$

Let h be a compactly supported continuous function such that $\|f - h\|_{L^1_\kappa} < \varepsilon^2$. The continuity of h implies $\Omega h = 0$ everywhere, and hence

$$\Omega f \le \Omega(f - h) + \Omega h = \Omega(f - h),$$

so

$$\mu_{\kappa,m}\big(\{x : \Omega f(x) > \varepsilon\}\big) \le \mu_{\kappa,m}\big(\{x : \Omega(f - h)(x) > \varepsilon\}\big) \lesssim \frac{\|f - h\|_{L^1_\kappa}}{\varepsilon} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and small we conclude $\Omega f = 0$ a.e. Now we turn our attention to the proof of the convergence $F_r \to f$ in $L^1(\mu_{\kappa,m})$. We have

$$\int_{\mathbb{R}} \big|F_r(x) - f(x)\big|\, d\mu_{\kappa,m}(x)$$
$$= \int_{\mathbb{R}} \left| \frac{1}{\mu_{\kappa,m}(]-r,r[)} \int_{-r}^{r} \tau_x f\big((-1)^m y\big) - f(x)\, d\mu_{\kappa,m}(y) \right| d\mu_{\kappa,m}(x)$$

$$\leq \frac{1}{\mu_{\kappa,m}(]-r,r[)} \int_{\mathbb{R}} \left(\int_{-r}^{r} \left| \tau_{(-1)^m y} f(x) - f(x) \right| d\mu_{\kappa,m}(y) \right) d\mu_{\kappa,m}(x)$$

$$\leq \frac{1}{\mu_{\kappa,m}(]-r,r[)} \int_{-r}^{r} \| \tau_{(-1)^m y} f - f \|_{L^1_\kappa} \, d\mu_{\kappa,m}(y). \tag{5.3}$$

By [8, Theorem 3.2], we have $\| \tau_{(-1)^m y} f - f \|_{L^1_\kappa} \to 0$ as $y \to 0$. Thus, the right-hand side of (5.3) converges to 0 as $r \to 0$. □

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