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On the class of uncertainty inequalities for the coupled fractional Fourier transform

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Abstract
The coupled fractional Fourier transform $\mathcal{F}_{\alpha,\beta}$ is a two-dimensional fractional Fourier transform depending on two angles α and β, which are coupled in such a way that the transform parameters are $\gamma = (\alpha + \beta)/2$ and $\delta = (\alpha - \beta)/2$. It generalizes the two-dimensional Fourier transform and serves as a prominent tool in some applications of signal and image processing. In this article, we formulate a new class of uncertainty inequalities for the coupled fractional Fourier transform (CFrFT). Firstly, we establish a sharp Heisenberg-type uncertainty inequality for the CFrFT and then formulate some logarithmic and local-type uncertainty inequalities. In the sequel, we establish several concentration-based uncertainty inequalities, including Nazarov, Amrein–Berthier–Benedicks, and Donoho–Stark's inequalities. Towards the end, we formulate Hardy's and Beurling's inequalities for the CFrFT.

Keywords: Coupled fractional Fourier transform; Uncertainty principle; Heisenberg's inequality; Logarithmic and local inequalities; Concentration-based uncertainty principle; Hardy's and Beurling's inequalities

1 Introduction

While solving some deep problems in quantum mechanics arising from classical quadratic Hamiltonians, Victor Namias introduced the fractional Fourier transform (FrFT) by using the fact that the Hermite functions $h_n(x)$ are eigenfunctions of the Fourier transform with eigenvalues $e^{in\pi/2}$ [1]. The fractional Fourier transform (FrFT) has proved to be an important tool in harmonic analysis, and it has received significant attention due to its wide applicability in optics, quantum mechanics, neural networks, differential equations, optics, pattern recognition, radar, sonar, and other communication systems [2, 3]. The extension of the FrFT to higher dimension has been studied by several authors, and the most commonly used approach is based on the kernel which is the tensor product of n-copies of the usual fractional kernel, each of which relies on an angle $\alpha_i, i = 1, 2, \dots, n$ [4]. Recently, Zayed [5] introduced a new variant of fractional Fourier transform $\mathcal{F}_{\alpha,\beta}$ in a two-dimensional space, wherein the kernel is not a tensor product of two copies of the usual fractional kernel, but relies on two angles α and β that are coupled, yielding a new pair of transform parameters: $\gamma = (\alpha + \beta)/2$ and $\delta = (\alpha - \beta)/2$. Mathematically, the coupled fractional Fourier transform of any function $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is defined as

$$\mathcal{F}_{\alpha,\beta}[f](\mathbf{u}) = \tilde{d}(\gamma) \int_{\mathbb{R}^2} f(\mathbf{t}) e^{-i(\tilde{a}(\gamma)(|\mathbf{t}|^2+|\mathbf{u}|^2)-\mathbf{t}\cdot M\mathbf{u})}\, d\mathbf{t}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$ are such that $\alpha + \beta \notin 2\pi\mathbb{Z}$ and

$$\left.\begin{aligned} &\gamma = \frac{\alpha+\beta}{2}, \qquad \delta = \frac{\alpha-\beta}{2}, \qquad \tilde{a}(\gamma) = \frac{\cos\gamma}{2}, \qquad \tilde{b}(\gamma,\delta) = \frac{\cos\delta}{\sin\gamma}, \\ &\tilde{c}(\gamma,\delta) = \frac{\sin\delta}{\sin\gamma}, \qquad \tilde{d}(\gamma) = \frac{ie^{-i\gamma}}{2\pi\sin\gamma}, \qquad M = \begin{pmatrix} \tilde{b}(\gamma,\delta) & \tilde{c}(\gamma,\delta) \\ -\tilde{c}(\gamma,\delta) & \tilde{b}(\gamma,\delta) \end{pmatrix}. \end{aligned}\right\} \tag{1.2}$$

Since its inception, the coupled fractional Fourier transform (1.1) has received instant recognition not only from a theoretical perspective but has also been of utmost significance in diverse aspects of science and engineering. For instance, Kamalakkannan et al. derived several new properties of the transform, including additive property, and then extended some of them to $L^2(\mathbb{R}^2)$ [6]. Kamalakkannan and Roopkumar extended the two-dimensional coupled fractional Fourier transform to the n-dimensional fractional Fourier transform and studied the corresponding convolution structures [7]. Recently, Shah and Teali intertwined the merits of coupled fractional Fourier transform and the Wigner distribution and showed its applicability to LFM signals [8]. Moreover, Parseval's formula corresponding to (1.1) is given by

$$\left\langle \mathcal{F}_{\alpha,\beta}[f], \mathcal{F}_{\alpha,\beta}[g] \right\rangle_2 = \langle f, g\rangle_2. \tag{1.3}$$

In addition, it is imperative to mention that the coupled fractional Fourier transform shares a nice bond with the classical Fourier transform and obeys the following relationship:

$$\mathcal{F}_{\alpha,\beta}[f](\mathbf{u}) = 2\pi \tilde{d}(\gamma) e^{-i\tilde{a}(\gamma)|\mathbf{u}|^2} \mathcal{F}\big[e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})\big]\big(-M^{-1}\mathbf{u}\big). \tag{1.4}$$

Heisenberg's uncertainty principle plays a central role in quantum mechanics, which asserts that the position and momentum of a particle cannot be simultaneously measured with absolute precision [9]. That is, the more precisely momentum is known, the more uncertain the position is, and vice versa. The harmonic analysis version of this principle says that a nonzero function f and its Fourier transform $\mathcal{F}[f]$ cannot be simultaneously localized with absolute precision [10]. This standard inequality has received much attention over time and therefore has been developed in diverse fields of harmonic analysis ranging from the fractional Fourier to the much recent quadratic-phase Fourier transforms [11–13]. Since its first appearance in the literature, the uncertainty principle has been extensively studied in different settings, and numerous ramifications of this standard inequality have appeared in the open literature over time, for instance, the logarithmic uncertainty principles, entropy-based uncertainty principles, Pitt's uncertainty principles, local-type uncertainty principles, Hardy's and Beurling's uncertainty inequalities, Sobolev-type inequalities, and so on [14–19]. However, to the best of our knowledge, an exclusive study of uncertainty principles for the coupled fractional Fourier transform has not been carried out yet. Taking this opportunity, we shall establish some prominent uncertainty inequalities in the context of the coupled fractional Fourier transform. The highlights of the article are given below:

- To establish the Heisenberg-type uncertainty inequality for the coupled fractional Fourier transform.
- To study Beckner's as well as local-type uncertainty inequalities.
- To derive the concentration-based uncertainty inequalities, such as Nazarov's, Amrein–Berthier–Benedicks's, and Donoho–Stark's inequalities.
- To develop another pair of inequalities, namely Hardy's and Beurling's inequalities for the coupled fractional Fourier transform.

The remainder of the article is structured as follows. In Sect. 2, we study Heisenberg-type uncertainty inequality for the coupled fractional Fourier transform. Section 3 is dedicated to formulating the logarithmic and local-type uncertainty inequalities. Section 4 is concerned with the study of the concentration-based uncertainty inequalities. Section 5 is devoted to establishing Hardy's and Beurling's uncertainty inequalities for the coupled fractional Fourier transform. Finally, a conclusion is extracted in Sect. 6.

2 Heisenberg-type uncertainty inequality

In this section, our major aim is to establish a sharper variant of Heisenberg's inequality associated with the coupled fractional Fourier transform. To meet our intention, we first recall Pitt's inequality for the Fourier transform. Given $f \in \mathcal{S}(\mathbb{R}^2) \subseteq L^2(\mathbb{R}^2)$, the Schwartz class in $L^2(\mathbb{R}^2)$, Pitt's inequality is given by [14]

$$\int_{\mathbb{R}^2} |\mathbf{u}|^{-\kappa} \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \le C_\kappa \int_{\mathbb{R}^2} |\mathbf{t}|^{\kappa} \left|f(\mathbf{t})\right|^2 d\mathbf{t}, \quad 0 \le \kappa < 1, \tag{2.1}$$

where

$$C_\kappa = \pi^\kappa \left[\Gamma\left(\frac{2-\kappa}{4}\right) \Big/ \Gamma\left(\frac{2+\kappa}{4}\right)\right]^2 \tag{2.2}$$

and Γ is the conventional Euler gamma function.

Lemma 2.1 *Let $f(\mathbf{t}) \in L^2(\mathbb{R}^2)$ and $2 \le \kappa < 3$. Then*

$$\int_{\mathbb{R}^2} |\mathbf{t}|^{\kappa} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^{\kappa} \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \ge \frac{1}{2\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \|f\|^2. \tag{2.3}$$

Proof Let $J(\mathbf{t}) = \mathbf{t}f(\mathbf{t})$, then for $2 \le \kappa < 3$, inequality (2.1) yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mathbf{t}|^{\kappa} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^{\kappa} \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \\
&= \int_{\mathbb{R}^2} |\mathbf{t}|^{\kappa-2} \left|J(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^{\kappa} \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \\
&\ge \frac{1}{\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \int_{\mathbb{R}^2} |\mathbf{u}|^{2-\kappa} \left|\mathcal{F}[J](\mathbf{u})\right|^2 d\mathbf{u} \int_{\mathbb{R}^2} |\mathbf{u}|^{\kappa} \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \\
&= \frac{1}{\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \int_{\mathbb{R}^2} \left|\mathbf{u}^{1-\kappa/2}\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \int_{\mathbb{R}^2} \left|\mathbf{u}^{\kappa/2}\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u}.
\end{aligned} \tag{2.4}$$

Implementing Cauchy's inequality in (2.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mathbf{t}|^\kappa \left|f(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^\kappa \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \\
&\quad \geq \frac{1}{\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \left| \int_{\mathbb{R}^2} \mathbf{u}^{1-\kappa/2} \mathcal{F}[J](\mathbf{u}) \overline{\mathbf{u}^{\kappa/2} \mathcal{F}[f](\mathbf{u})}\, d\mathbf{u} \right| \\
&\quad = \frac{1}{\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \left| \int_{\mathbb{R}^2} \mathcal{F}[J](\mathbf{u}) \overline{\mathbf{u} \mathcal{F}[f](\mathbf{u})}\, d\mathbf{u} \right|.
\end{aligned}$$

Using the differential property of the Fourier transform, that is, $\mathcal{F}[\frac{nf}{d\mathbf{t}}](\mathbf{u}) = i\mathbf{u}\mathcal{F}[f](\mathbf{u})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mathbf{t}|^\kappa \left|f(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^\kappa \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \\
&\quad = \frac{1}{\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \left| \int_{\mathbb{R}^2} \mathcal{F}[J](\mathbf{u}) \overline{\mathcal{F}\left[\frac{df}{d\mathbf{t}}\right](\mathbf{u})}\, d\mathbf{u} \right|.
\end{aligned}$$

Using Parseval's formula of the Fourier transform, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\mathbf{t}|^\kappa \left|f(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^\kappa \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} &= \frac{1}{\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \left| \int_{\mathbb{R}^2} J(\mathbf{t}) \overline{\frac{df}{d\mathbf{t}}}\, d\mathbf{t} \right| \\
&= \frac{1}{\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \left| \int_{\mathbb{R}^2} \mathbf{t} f(\mathbf{t}) \overline{\frac{df}{d\mathbf{t}}}\, d\mathbf{t} \right|.
\end{aligned}$$

Finally, using $\int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\frac{df}{d\mathbf{t}}}\, d\mathbf{t} = \frac{1}{2}|f(\mathbf{t})|^2$ while doing integration by parts, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} |\mathbf{t}|^\kappa \left|f(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^\kappa \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} &= \frac{1}{2\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \left| \int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \right| \\
&= \frac{1}{2\pi^{\kappa-2}} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \|f\|^2.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Next, we obtain a sharper Heisenberg-type uncertainty inequality for the coupled fractional Fourier transform.

Theorem 2.2 *Let* $f(\mathbf{t})$ *be any square integrable function and* $2 \leq \kappa \leq 3$, *then*

$$\int_{\mathbb{R}^2} |\mathbf{t}|^\kappa \left|f(\mathbf{t})\right|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\mathbf{u}|^\kappa \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} \geq \frac{\sin^{2k}\gamma}{8\pi^\kappa} \left[\Gamma\left(\frac{\kappa}{4}\right) \Big/ \Gamma\left(\frac{4-\kappa}{4}\right)\right]^2 \|f\|^2. \tag{2.5}$$

Proof Define a function

$$I(\mathbf{u}) = \int_{\mathbb{R}^2} f(\mathbf{t}) e^{-i(\tilde{a}(\gamma)|\mathbf{t}|^2 - \mathbf{t}\cdot M\mathbf{u})}\, d\mathbf{t}, \tag{2.6}$$

which can also be expressed as

$$I(\mathbf{u}) = \frac{1}{\tilde{d}(\gamma)} e^{i\tilde{a}(\gamma)|\mathbf{u}|^2} \mathcal{F}_{\alpha,\beta}[f](\mathbf{u}). \tag{2.7}$$

From (2.6), we observe that $I(-M^{-1}\mathbf{u})$ is the Fourier transform of $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2}f(\mathbf{t})$. Using inequality (2.3) for the function $J(\mathbf{t})$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi^{\kappa-2}}\left[\Gamma\left(\frac{\kappa}{4}\right)\Big/\Gamma\left(\frac{4-\kappa}{4}\right)\right]^2\|f\|^2 \\
&\leq \int_{\mathbb{R}^2}|\mathbf{t}|^\kappa\left|J(\mathbf{t})\right|^2 d\mathbf{t}\int_{\mathbb{R}^2}|\mathbf{u}|^\kappa\left|\mathcal{F}[J](\mathbf{u})\right|^2 d\mathbf{u} \\
&= \int_{\mathbb{R}^2}|\mathbf{t}|^\kappa\left|J(\mathbf{t})\right|^2 d\mathbf{t}\int_{\mathbb{R}^2}|\mathbf{u}|^\kappa\left|I(-M^{-1}\mathbf{u})\right|^2 d\mathbf{u} \\
&= \frac{1}{\sin^2\gamma}\int_{\mathbb{R}^2}|\mathbf{t}|^\kappa\left|J(\mathbf{t})\right|^2 d\mathbf{t}\int_{\mathbb{R}^2}\left|\det(-M)\mathbf{u}\right|^\kappa\left|I(\mathbf{u})\right|^2 d\mathbf{u}.
\end{aligned} \tag{2.8}$$

Note that $|I(\mathbf{u})| = |2\pi\sin\gamma\,\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|$ and $|J(\mathbf{t})| = |f(\mathbf{t})|$, inequality (2.8) yields

$$\int_{\mathbb{R}^2}|\mathbf{t}|^\kappa\left|f(\mathbf{t})\right|^2 d\mathbf{t}\int_{\mathbb{R}^2}|\mathbf{u}|^\kappa\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} \geq \frac{\sin^{2k}\gamma}{8\pi^\kappa}\left[\Gamma\left(\frac{\kappa}{4}\right)\Big/\Gamma\left(\frac{4-\kappa}{4}\right)\right]^2\|f\|^2,$$

which is the desired result. □

Remark 2.3

(i) For $\kappa = 2$, Theorem 2.2 reduces to the classical Heisenberg uncertainty inequality for CFrFT:

$$\left\{\int_{\mathbb{R}^2}|\mathbf{t}|^2\left|f(\mathbf{t})\right|^2 d\mathbf{t}\right\}^{1/2}\left\{\int_{\mathbb{R}^2}|\mathbf{u}|^2\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}\right\}^{1/2} \geq \frac{\sin^4\gamma}{8\pi^2}\|f\|^2.$$

(ii) For $\kappa = 2$ and $\alpha = \beta$, Theorem 2.2 yields the Heisenberg uncertainty principle for the ordinary fractional Fourier transform:

$$\left\{\int_{\mathbb{R}^2}|\mathbf{t}|^2\left|f(\mathbf{t})\right|^2 d\mathbf{t}\right\}^{1/2}\left\{\int_{\mathbb{R}^2}|\mathbf{u}|^2\left|\mathcal{F}_{\alpha}[f](\mathbf{u})\right|^2 d\mathbf{u}\right\}^{1/2} \geq \frac{\sin^4\alpha}{8\pi^2}\|f\|^2.$$

(iii) For $\kappa = 2$ and $\alpha = \beta = \pi/2$, Theorem 2.2 reduces to the Heisenberg uncertainty principle for the classical Fourier transform:

$$\left\{\int_{\mathbb{R}^2}|\mathbf{t}|^2\left|f(\mathbf{t})\right|^2 d\mathbf{t}\right\}^{1/2}\left\{\int_{\mathbb{R}^2}|\mathbf{u}|^2\left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u}\right\}^{1/2} \geq \frac{1}{8\pi^2}\|f\|^2.$$

3 Logarithmic, local, and entropy-based inequalities

This section is entirely devoted to the logarithmic, local, and entropy-based uncertainty inequalities for the coupled fractional Fourier transform.

3.1 Logarithmic uncertainty inequalities

Beckner first introduced the logarithmic uncertainty inequality to the class of quantitative uncertainty principles which investigates the localization of a function in its time and Fourier transform domains via the logarithmic approximations derived from the classic Pitt inequality (2.1). For any $f \in \mathcal{S}(\mathbb{R})$, logarithmic inequality reads as follows [14]:

$$\int_{\mathbb{R}^2}\log|\mathbf{t}|\left|f(\mathbf{t})\right|^2 d\mathbf{t} + \int_{\mathbb{R}^2}\log|\mathbf{u}|\left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \geq \left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \log\pi\right)\int_{\mathbb{R}^2}|f(\mathbf{t})|^2 d\mathbf{t}. \tag{3.1}$$

This inequality has gained significant attention over the past few decades leading to its various modifications and refinements [9, 16]. As such, we are deeply motivated to formulate logarithmic-type uncertainty inequality for the CFrFT given by (1.1).

Theorem 3.1 *Let f be any function belonging to $\mathcal{S}(\mathbb{R})$. Then we have*

$$4\pi^2\int_{\mathbb{R}^2}\log|\mathbf{u}|\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}+\int_{\mathbb{R}^2}\log|\mathbf{t}|\left|f(\mathbf{t})\right|^2 d\mathbf{t}$$
$$\geq\left(\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\log\pi+8\pi^2\log|\sin\gamma|\right)\|f\|_2^2. \tag{3.2}$$

Proof Corresponding to $J(\mathbf{t})=e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2}f(\mathbf{t})$, we have $\mathcal{F}[J](\mathbf{u})=I(-M^{-1}\mathbf{u})$, where I is defined by (2.6). Implementing Pitt's inequality (2.1) on $J(\mathbf{t})$, we obtain

$$\int_{\mathbb{R}^2}|\mathbf{u}|^{-\kappa}\left|I\left(-M^{-1}\mathbf{u}\right)\right|^2 d\mathbf{u}\leq C_\kappa\int_{\mathbb{R}^2}|\mathbf{t}|^{\kappa}\left|J(\mathbf{t})\right|^2 d\mathbf{t}. \tag{3.3}$$

Or

$$\frac{1}{\sin^2\gamma}\int_{\mathbb{R}^2}\left|\det(-M)\mathbf{u}\right|^{-\kappa}\left|I(\mathbf{u})\right|^2 d\mathbf{u}\leq C_\kappa\int_{\mathbb{R}^2}|\mathbf{t}|^{\kappa}\left|J(\mathbf{t})\right|^2 d\mathbf{t}. \tag{3.4}$$

Equivalently,

$$\frac{1}{\sin^{2(1-\kappa)}\gamma}\int_{\mathbb{R}^2}|\mathbf{u}|^{-\kappa}\left|I(\mathbf{u})\right|^2 d\mathbf{u}\leq C_\kappa\int_{\mathbb{R}^2}|\mathbf{t}|^{\kappa}\left|J(\mathbf{t})\right|^2 d\mathbf{t}. \tag{3.5}$$

Since $|I(\mathbf{u})|=|2\pi\sin\gamma\,\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|$ and $|J(\mathbf{t})|=|f(\mathbf{t})|$, inequality (3.5) yields

$$4\pi^2\sin^{2\kappa}\gamma\int_{\mathbb{R}^2}|\mathbf{u}|^{-\kappa}\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}\leq C_\kappa\int_{\mathbb{R}^2}|\mathbf{t}|^{\kappa}\left|f(\mathbf{t})\right|^2 d\mathbf{t}, \tag{3.6}$$

which is Pitt's inequality for the coupled fractional Fourier transform. Also, for every $0\leq\kappa<1$, we define

$$\Gamma(\kappa)=4\pi^2\sin^{2\kappa}\gamma\int_{\mathbb{R}^2}|\mathbf{u}|^{-\kappa}\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}-C_\kappa\int_{\mathbb{R}^2}|\mathbf{t}|^{\kappa}\left|f(\mathbf{t})\right|^2 d\mathbf{t}. \tag{3.7}$$

Differentiating (3.7) yields

$$\Gamma'(\kappa)=8\pi^2\sin^{2\kappa}\gamma\log|\sin\gamma|\int_{\mathbb{R}^2}|\mathbf{u}|^{-\kappa}\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}-4\pi^2\sin^{2\kappa}\gamma$$
$$-C_\kappa\int_{\mathbb{R}^2}|\mathbf{t}|^{\kappa}\log|\mathbf{t}|\left|f(\mathbf{t})\right|^2 d\mathbf{t}-C'_\kappa\int_{\mathbb{R}^2}|\mathbf{t}|^{\kappa}\left|f(\mathbf{t})\right|^2 d\mathbf{t}, \tag{3.8}$$

where

$$C'_\kappa=-\frac{\pi^\kappa}{2}\left\{\frac{\Gamma^2(\frac{2+\kappa}{4})\Gamma(\frac{2-\kappa}{4})\Gamma'(\frac{2-\kappa}{4})+\Gamma^2(\frac{2-\kappa}{4})\Gamma(\frac{2+\kappa}{4})\Gamma'(\frac{2+\kappa}{4})}{\Gamma^2(\frac{2+\kappa}{4})}\right\}$$
$$+\pi^\kappa\log\pi\left\{\Gamma^2\left(\frac{2-\kappa}{4}\right)\Big/\Gamma^2\left(\frac{2+\kappa}{4}\right)\right\}. \tag{3.9}$$

Substituting $\kappa = 0$ in (3.9) gives

$$C_0' = \left(\log\pi - \frac{\Gamma'(1/2)}{\Gamma(1/2)}\right). \tag{3.10}$$

By virtue of (2.1), we have $\Gamma(\kappa) \le 0$ for all $0 \le \kappa < 1$ and $\Gamma(0) = 0$. Thus, for any $n > 0$, we must have $\Gamma'(0+n) \le 0$ provided $n \to 0$. Therefore, we have

$$\begin{aligned}&8\pi^2 \log|\sin\gamma| \int_{\mathbb{R}^2} \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} - 4\pi^2 \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} \\ &\quad - C_0 \int_{\mathbb{R}^2} \log|\mathbf{t}| \left|f(\mathbf{t})\right|^2 d\mathbf{t} - C_0' \int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \le 0.\end{aligned}$$

Invoking Parseval's formula (1.3), we obtain

$$\begin{aligned}&4\pi^2 \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} + \int_{\mathbb{R}^2} \log|\mathbf{t}| \left|f(\mathbf{t})\right|^2 d\mathbf{t} \\ &\quad \ge \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \log\pi + 8\pi^2 \log|\sin\gamma|\right) \|f\|_2^2.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Remark 3.2
(i) For $\alpha = \beta$, Theorem 3.1 yields the logarithmic uncertainty inequality for the ordinary two-dimensional fractional Fourier transform:

$$\begin{aligned}&4\pi^2 \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|\mathcal{F}_{\alpha}[f](\mathbf{u})\right|^2 d\mathbf{u} + \int_{\mathbb{R}^2} \log|\mathbf{t}| \left|f(\mathbf{t})\right|^2 d\mathbf{t} \\ &\quad \ge \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \log\pi + 8\pi^2 \log|\sin\alpha|\right) \|f\|_2^2.\end{aligned}$$

(ii) For $\alpha = \beta = \pi/2$, Theorem 2.2 yields the logarithmic uncertainty inequality for the classical Fourier transform.

In the following, we formulate some Sobolev-type inequalities for CFrFT (1.1). To carry our endeavor, we shall recall some basic definitions and results.

Definition 3.3 Given the operator $\mathscr{D} = (\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2})$, the Sobolev space $\mathbb{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$ is defined as

$$\mathbb{S}(\mathbb{R}^2) = \left\{f \in L^2(\mathbb{R}^2) : \mathscr{D}f \in L^2(\mathbb{R}^2)\right\}. \tag{3.11}$$

Definition 3.4 For $1 \le p < \infty$ and $j > 0$, the weighted Lebesgue space $\mathcal{W}_j^p(\mathbb{R}^2)$ on $\mathbb{R}^2$ is defined as

$$\mathcal{W}_j^p(\mathbb{R}^2) = \left\{f \in \mathbf{L}_{\mathrm{loc}}^p(\mathbb{R}^2) : \langle\mathbf{t}\rangle^j f \in L^p(\mathbb{R}^2)\right\}, \tag{3.12}$$

where $\langle\mathbf{t}\rangle = (1 + |\mathbf{t}|^2)^{1/2}$ is the weight function.

The logarithmic Sobolev-type inequality for any $f \in \mathbb{S}(\mathbb{R}^2)$ reads as follows [16]:

$$\int_{\mathbb{R}^2} |f(\mathbf{t})|^2 \log\left(\frac{|f(\mathbf{t})|^2}{\|f\|_2^2}\right) d\mathbf{t} \leq \log\left(\frac{1}{\pi e\|f\|_2^2} \int_{\mathbb{R}^2} |\mathscr{D} f(\mathbf{t})|^2 \, d\mathbf{t}\right). \tag{3.13}$$

Beckner derived a new class of Sobolev-type inequality, which yields a better estimate than Gross's inequality (3.13) and is given by [14]

$$\begin{aligned} &\int_{\mathbb{R}^2} |f(\mathbf{t})|^2 \log\left(\frac{|f(\mathbf{t})|^2}{\|f\|_2^2}\right) d\mathbf{t} \\ &\quad \leq \int_{\mathbb{R}^2} |\mathcal{F}[f](\mathbf{u})|^2 \log\left(\frac{1}{4\pi}\left(\frac{\Gamma(2)}{\Gamma(1)}\right)|\mathbf{u}|^2\right) d\mathbf{u} - 2\|f\|_2^2\left(\frac{\Gamma'(1)}{\Gamma(1)}\right). \end{aligned} \tag{3.14}$$

In continuation, Kubo et al. [16] uses Beckner's inequality (3.1) to formulate another type of logarithmic Sobolev-type inequality. For any nonzero function $f \in \mathcal{W}_j^p(\mathbb{R}^2)$, the inequality reads

$$-\int_{\mathbb{R}^2} |f(\mathbf{t})| \log\left(\frac{|f(\mathbf{t})|}{\|f\|_1}\right) d\mathbf{t} \leq 2\int_{\mathbb{R}^2} |f(\mathbf{t})| \log\left(C_{2,j}\left(1 + |\mathbf{t}|^j\right)\right) d\mathbf{t}, \tag{3.15}$$

where

$$C_{2,j} = \left\{\frac{2\pi\,\Gamma(2/j)\Gamma(2/j')}{j\Gamma(2)\Gamma(1)}\right\}^{1/2}, \qquad \frac{1}{j} + \frac{1}{j'} = 1. \tag{3.16}$$

Moreover, the duality relation says that, for any $f \in \mathbb{S}(\mathbb{R}^2) \cap \mathcal{W}_1^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} |f(\mathbf{t})|^2 \log\left(\frac{1+|\mathbf{t}|^2}{2}\right) d\mathbf{t} + \int_{\mathbb{R}^2} \log|\mathbf{u}|^2 |\mathcal{F}[f](\mathbf{u})|^2 \, d\mathbf{u} \geq \left(\frac{\Gamma'(1)}{\Gamma(1)}\right) \int_{\mathbb{R}^2} |f(\mathbf{t})|^2 \, d\mathbf{t}. \tag{3.17}$$

We are now in a position to develop an analogue of Sobolev's inequality (3.17) for the coupled fractional Fourier transform defined in (1.1).

Theorem 3.5 *Let $\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})$ denote the coupled fractional Fourier transform of any nonzero function $f \in \mathbb{S}(\mathbb{R}^2) \cap \mathcal{W}_1^2(\mathbb{R}^2)$. Then the following inequality holds*:

$$\begin{aligned} &\int_{\mathbb{R}^2} |f(\mathbf{t})|^2 \log\left(\frac{1+|\mathbf{t}|^2}{2}\right) d\mathbf{t} + 4\pi^2 \int_{\mathbb{R}^2} \log|\mathbf{u}| |\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|^2 \, d\mathbf{u} \\ &\quad \geq \left(\frac{\Gamma'(1)}{\Gamma(1)} - 4\pi^2 \log\left(\frac{1}{\sin^2\gamma}\right)\right)\|f\|_2^2. \end{aligned} \tag{3.18}$$

Proof The logarithmic Sobolev-type inequality for any $f \in \mathbb{S}(\mathbb{R}^2)$ for the Fourier transform reads [16]

$$\int_{\mathbb{R}^2} |f(\mathbf{t})|^2 \log\left(\frac{|f(\mathbf{t})|^2}{\|f\|_2^2}\right) d\mathbf{t} \leq \log\left(\frac{1}{\pi e\|f\|_2^2} \int_{\mathbb{R}^2} |\mathscr{D} f(\mathbf{t})|^2 \, d\mathbf{t}\right). \tag{3.19}$$

Applying (3.17) for the function $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})$, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|J(\mathbf{t})\right|^2 \log\left(\frac{1+|\mathbf{t}|^2}{2}\right) d\mathbf{t} + \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|I\left(-M^{-1}\mathbf{u}\right)\right|^2 d\mathbf{u} \\ &\quad \geq \left(\frac{\Gamma'(1)}{\Gamma(1)}\right) \int_{\mathbb{R}^2} \left|J(\mathbf{t})\right|^2 d\mathbf{t}, \end{aligned} \tag{3.20}$$

where I follows from (2.6). Equivalently,

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|J(\mathbf{t})\right|^2 \log\left(\frac{1+|\mathbf{t}|^2}{2}\right) d\mathbf{t} + \frac{1}{\sin^2\gamma} \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|I(\mathbf{u})\right|^2 d\mathbf{u} \\ &\quad + \frac{1}{\sin^2\gamma} \int_{\mathbb{R}^2} \log\left(\frac{1}{\sin^2\gamma}\right) \left|I(\mathbf{u})\right|^2 d\mathbf{u} \\ &\quad \geq \left(\frac{\Gamma'(1)}{\Gamma(1)}\right) \int_{\mathbb{R}^2} \left|J(\mathbf{t})\right|^2 d\mathbf{t}. \end{aligned} \tag{3.21}$$

Using the identities $|J(\mathbf{t})| = |f(\mathbf{t})|$ and $|I(\mathbf{u})| = |2\pi \sin\gamma \mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|$ in (3.21), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 \log\left(\frac{1+|\mathbf{t}|^2}{2}\right) d\mathbf{t} + 4\pi^2 \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} \\ &\quad + 4\pi^2 \int_{\mathbb{R}^2} \log\left(\frac{1}{\sin^2\gamma}\right) \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} \geq \left(\frac{\Gamma'(1)}{\Gamma(1)}\right) \int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 d\mathbf{t}. \end{aligned} \tag{3.22}$$

Equivalently, (3.22) can be recast as

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 \log\left(\frac{1+|\mathbf{t}|^2}{2}\right) d\mathbf{t} + 4\pi^2 \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u} \\ &\quad \geq \left(\frac{\Gamma'(1)}{\Gamma(1)} - 4\pi^2 \log\left(\frac{1}{\sin^2\gamma}\right)\right) \|f\|_2^2, \end{aligned}$$

which is the desired inequality. □

Remark 3.6

(i) For $\alpha = \beta$, Theorem 5.2 yields Sobolev's inequality for the ordinary two-dimensional fractional Fourier transform:

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 \log\left(\frac{1+|\mathbf{t}|^2}{2}\right) d\mathbf{t} + 4\pi^2 \int_{\mathbb{R}^2} \log|\mathbf{u}| \left|\mathcal{F}_{\alpha}[f](\mathbf{u})\right|^2 d\mathbf{u} \\ &\quad \geq \left(\frac{\Gamma'(1)}{\Gamma(1)} - 4\pi^2 \log\left(\frac{1}{\sin^2\alpha}\right)\right) \|f\|_2^2. \end{aligned}$$

(ii) For $\alpha = \beta = \pi/2$, Theorem 5.2 yields Sobolev's inequality for the classical Fourier transform.

3.2 Local-type uncertainty inequalities

The classical Heisenberg uncertainty principle states that it is impossible to localize a signal in the natural and its corresponding spectral domain precisely and simultaneously. However, it does not tell anything about the possibility of I being localized in ε-neighborhood of two or more distinct points. The local uncertainty principle refines and pinpoints this flaw. The purpose of this section is to derive some local uncertainty principles for the coupled fractional Fourier transform.

Theorem 3.7 *Let $E \subset \mathbb{R}^2$ with a finite measure, then the CFrFT $\mathcal{F}_{\alpha,\beta}[f]$ of any $f \in L^2(\mathbb{R}^2)$ satisfies the following inequality*:

$$\int_{\mathbb{R}^2} |\mathbf{t}|^{2\kappa} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \geq \frac{4\pi^2}{C_\kappa |E|^{2\kappa} \sin^2\gamma} \int_E \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}, \quad 0 < \kappa < 1, \tag{3.23}$$

where C_κ is a constant.

Proof For a finite measurable set $E \subset \mathbb{R}^2$, the classical local uncertainty principle is given by [9]

$$\int_E \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u} \leq C_\kappa |E|^{2\kappa} \left\| |\mathbf{t}|^\kappa f(\mathbf{t}) \right\|_2^2. \tag{3.24}$$

Therefore, for the function $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|} f(\mathbf{t})$, we have

$$\int_E \left|\mathcal{F}[J](\mathbf{u})\right|^2 d\mathbf{u} \leq C_\kappa |E|^{2\kappa} \left\| |\mathbf{t}|^\kappa J(\mathbf{t}) \right\|_2^2. \tag{3.25}$$

Or

$$\frac{1}{\sin^2\gamma} \int_E \left|I\left(-M^{-1}\mathbf{u}\right)\right|^2 d\mathbf{u} \leq C_\kappa |E|^{2\kappa} \left\| |\mathbf{t}|^\kappa J(\mathbf{t}) \right\|_2^2.$$

Equivalently,

$$\frac{1}{\sin^4\gamma} \int_E \left|I(\mathbf{u})\right|^2 d\mathbf{u} \leq C_\kappa |E|^{2\kappa} \left\| |\mathbf{t}|^\kappa J(\mathbf{t}) \right\|_2^2. \tag{3.26}$$

Invoking the relations $|J(\mathbf{t})| = |f(\mathbf{t})|$ and $|I(\mathbf{u})| = |2\pi \sin\gamma \mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|$ in (3.26), we obtain

$$\int_{\mathbb{R}^2} |\mathbf{t}|^{2\kappa} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \geq \frac{4\pi^2}{C_\kappa |E|^{2\kappa} \sin^2\gamma} \int_E \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u},$$

which is the desired result. □

Next, we develop one more local-type uncertainty principle by invoking Sobolev's uncertainty principle (3.18).

Theorem 3.8 *Let $\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})$ denote the coupled fractional Fourier transform of any $f \in \mathbb{S}(\mathbb{R}^2) \cap \mathcal{W}_1^2(\mathbb{R}^2)$. Then the following uncertainty inequality holds*:

$$\int_{\mathbb{R}^2} |\mathbf{t}|^2 \left|f(\mathbf{t})\right|^2 d\mathbf{t} \geq \left[2\left(\frac{\sin^2\gamma \|f\|^2}{\|\mathscr{D}f\|}\right)^{4\pi^2} \exp\left\{\left(\frac{\Gamma'(1)}{\Gamma(1)} - 4\pi^2\right)\|f\|^2\right\} - \|f\|^2\right]. \tag{3.27}$$

Proof By using (3.18), we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|f(\mathbf{t})\right|^2\log\left(\frac{1+|\mathbf{t}|^2}{2}\right)d\mathbf{t}+4\pi^2\int_{\mathbb{R}^2}\log|\mathbf{u}|\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}\\&\quad\geq\left(\frac{\Gamma'(1)}{\Gamma(1)}-4\pi^2\right)\|f\|_2^2.\end{aligned}\tag{3.28}$$

By virtue of Jensen's inequality, inequality (3.28) can be redrafted as

$$\begin{aligned}&\log\int_{\mathbb{R}^2}\frac{|f(\mathbf{t})|^2}{\|f\|_2^2}\left(\frac{1+|\mathbf{t}|^2}{2}\right)d\mathbf{t}+2\pi^2\int_{\mathbb{R}^2}\log|\mathbf{u}|^2\frac{|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|^2}{\|f\|_2^2}d\mathbf{u}\\&\quad\geq\left(\frac{\Gamma'(1)}{\Gamma(1)}-4\pi^2\right)\|f\|_2^2.\end{aligned}\tag{3.29}$$

Setting $d\mu=|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|^2\,d\mathbf{u}/\|f\|_2^2$ and invoking Jensen's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\log|\mathbf{u}|^2\left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}&=\|f\|_2^2\int_{\mathbb{R}^2}\log|\mathbf{u}|^2\,d\mu\\&\leq\|f\|_2^2\log\left(\int_{\mathbb{R}^2}|\mathbf{u}|^2\,d\mu\right)\\&=\|f\|_2^2\log\left(\int_{\mathbb{R}^2}|\mathbf{u}|^2\frac{|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})|^2}{\|f\|_2^2}d\mathbf{u}\right)\\&=\|f\|_2^2\log\left(\frac{1}{\sin^2\gamma}\int_{\mathbb{R}^2}|\mathbf{u}|^2\frac{|\mathcal{F}[J](\mathbf{u})|^2}{\|G\|_2^2}d\mathbf{u}\right)\\&=\|f\|_2^2\log\left(\frac{1}{\sin^2\gamma\,\|G\|_2^2}\int_{\mathbb{R}^2}\left|\mathscr{D}f(\mathbf{t})\right|^2 d\mathbf{t}\right)\\&=\|f\|_2^2\log\left(\frac{1}{\sin^2\gamma\,\|f\|_2^2}\int_{\mathbb{R}^2}\left|\mathscr{D}f(\mathbf{t})\right|^2 d\mathbf{t}\right)<\infty.\end{aligned}\tag{3.30}$$

Substituting (3.30) in (3.29), we obtain

$$\begin{aligned}&\log\left\{\left(\int_{\mathbb{R}^2}\frac{|f(\mathbf{t})|^2}{\|f\|^2}\left(\frac{1+|\mathbf{t}|^2}{2}\right)d\mathbf{t}\right)\left(\frac{1}{\sin^2\gamma\,\|f\|_2^2}\int_{\mathbb{R}^2}\left|\mathscr{D}f(\mathbf{t})\right|^2 d\mathbf{t}\right)^{2\pi^2}\right\}\\&\quad\geq\left(\frac{\Gamma'(1)}{\Gamma(1)}-\log\left(4\pi^2\right)\right)\|f\|^2.\end{aligned}$$

Equivalently,

$$\log\left\{\left(\frac{1}{2}\int_{\mathbb{R}^2}\frac{|\mathbf{t}f(\mathbf{t})|^2}{\|f\|^2}d\mathbf{t}+\frac{1}{2}\right)\left(\frac{1}{\sin^2\gamma\,\|f\|_2^2}\int_{\mathbb{R}^2}\left|\mathscr{D}f(\mathbf{t})\right|^2 d\mathbf{t}\right)^{2\pi^2}\right\}\geq\left(\frac{\Gamma'(1)}{\Gamma(1)}-4\pi^2\right)\|f\|^2,$$

which yields

$$\int_{\mathbb{R}^2}|\mathbf{t}|^2\left|f(\mathbf{t})\right|^2 d\mathbf{t}\geq\left[2\left(\frac{\sin^2\gamma\,\|f\|^2}{\|\mathscr{D}f\|}\right)^{4\pi^2}\exp\left\{\left(\frac{\Gamma'(1)}{\Gamma(1)}-4\pi^2\right)\|f\|^2\right\}-\|f\|^2\right].$$

This completes the proof of Theorem 3.8. □

3.3 Entropy-based inequality

The entropy-based inequality for the classical Fourier transform uses the well-known Shanon entropy to localize the signal I in the time and frequency domains as

$$E(f) = -\int_{\mathbb{R}^2} f(\mathbf{t}) \log\bigl(f(\mathbf{t})\bigr)\, d\mathbf{t}. \tag{3.31}$$

From equality (3.31), we infer that the more the spikes in the signal, the more negative the entropy is. Therefore, we can say that $E(f)$ provides the localization of f in terms of entropy. The entropic uncertainty principle of any normalized signal $f \in L^2(\mathbb{R}^2)$ reads as follows [9]:

$$E\bigl(|f|^2\bigr) + E\bigl(\bigl|\mathcal{F}[f]\bigr|^2\bigr) \ge \log(\pi e). \tag{3.32}$$

We now obtain an analogue of the uncertainty inequality (3.32) for the coupled fractional Fourier transform.

Theorem 3.9 *Let $\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})$ be the coupled fractional Fourier transform of any $f \in L^2(\mathbb{R}^2)$ with $\|f\| = 1$, then*

$$E\bigl(|f|^2\bigr) + 4\pi^2 E\bigl(\bigl|\mathcal{F}_{\alpha,\beta}[f]\bigr|^2\bigr) \ge \log(\pi e) + 4\pi^2 \log\bigl(4\pi^2 \sin^2\gamma\bigr). \tag{3.33}$$

Proof Employing inequality (3.32) for the function $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})$, we obtain

$$E\bigl(|J|^2\bigr) + E\bigl(\bigl|\mathcal{F}[J]\bigr|^2\bigr) \ge \log(\pi e). \tag{3.34}$$

Using (3.32), the above inequality can be recast as

$$-\int_{\mathbb{R}^2} \bigl|J(\mathbf{t})\bigr|^2 \log\bigl|J(\mathbf{t})\bigr|^2\, d\mathbf{t} - \int_{\mathbb{R}^2} \bigl|\mathcal{F}[J](\mathbf{u})\bigr|^2 \log\bigl|\mathcal{F}[J](\mathbf{u})\bigr|^2\, d\mathbf{u} \ge \log(\pi e). \tag{3.35}$$

Since $\|J\|_2 = \|f\|_2$ and $\mathcal{F}[J](\mathbf{u}) = I(-M^{-1}\mathbf{u})$, where I is given by (2.6), inequality (3.35) takes the form

$$-\int_{\mathbb{R}^2} \bigl|f(\mathbf{t})\bigr|^2 \log\bigl|f(\mathbf{t})\bigr|^2\, d\mathbf{t} - \int_{\mathbb{R}^2} \bigl|I\bigl(-M^{-1}\mathbf{u}\bigr)\bigr|^2 \log\bigl|I\bigl(-M^{-1}\mathbf{u}\bigr)\bigr|^2\, d\mathbf{u} \ge \log(\pi e).$$

Equivalently,

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \bigl|f(\mathbf{t})\bigr|^2 \log\bigl|f(\mathbf{t})\bigr|^2\, d\mathbf{t} - 4\pi^2 \int_{\mathbb{R}^2} \bigl|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\bigr|^2 \log\bigl|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\bigr|^2\, d\mathbf{u} \\
&\quad \ge \log(\pi e) + 4\pi^2 \log\bigl(4\pi^2 \sin^2\gamma\bigr).
\end{aligned}$$

Employing the definition of Shanon's entropy, we obtain the desired inequality as

$$E\bigl(|f|^2\bigr) + 4\pi^2 E\bigl(\bigl|\mathcal{F}_{\alpha,\beta}[f]\bigr|^2\bigr) \ge \log(\pi e) + 4\pi^2 \log\bigl(4\pi^2 \sin^2\gamma\bigr).$$

This completes the proof of Theorem 3.9. □

4 Concentration-based uncertainty inequalities

The main goal of this section is to develop various concentration-based uncertainty inequalities for the coupled fractional Fourier transform including Nazarov's, Amrein–Benedicks's, and Donoho–Stark's inequalities.

4.1 Nazarov's inequality

Nazarov's inequality considers a support of the function f rather than the dispersion as employed by Heisenberg's uncertainty inequality. For any $f \in L^2(\mathbb{R}^2)$ and $T_1, T_2 \subset \mathbb{R}^2$ with finite measure, the classical Nazarov uncertainty inequality

$$\int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \le Ce^{C|T_1||T_2|}\left(\int_{\mathbb{R}^2/T_1} \left|f(\mathbf{t})\right|^2 d\mathbf{t} + \int_{\mathbb{R}^2/T_2} \left|\mathcal{F}[f](\mathbf{u})\right|^2 d\mathbf{u}\right), \quad C > 0, \tag{4.1}$$

where $|T_1|$ and $|T_2|$ denote the Lebesgue measures of T_1 and T_2, respectively [15].

In the following theorem, we derive an analogue of Nazarov's uncertainty principle for the coupled fractional Fourier transform.

Theorem 4.1 *Let $T_1, T_2 \subset \mathbb{R}^2$ with finite measure and $\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})$ be the coupled fractional Fourier transform of any $f \in L^2(\mathbb{R}^2)$. Then the following uncertainty inequality holds*:

$$\begin{aligned}\int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \\ \le Ce^{C|T_1||T_2|}\left(\int_{\mathbb{R}^2/T_1} \left|f(\mathbf{t})\right|^2 d\mathbf{t} + 4\pi^2 \int_{\mathbb{R}^2/(T_2 \sin^2\gamma)} \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}\right), \quad C > 0,\end{aligned} \tag{4.2}$$

where $|T_1|$ and $|T_2|$ denote the measures of T_1 and T_2.

Proof Implementing Nazarov's inequality (4.1) for $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})$, we have

$$\int_{\mathbb{R}^2} \left|J(\mathbf{t})\right|^2 d\mathbf{t} \le Ce^{C|T_1||T_2|}\left(\int_{\mathbb{R}^2/T_1} \left|J(\mathbf{t})\right|^2 d\mathbf{t} + \int_{\mathbb{R}^2/T_2} \left|\mathcal{F}[J](\mathbf{u})\right|^2 d\mathbf{u}\right). \tag{4.3}$$

Invoking the identities $\|J\|_2 = \|f\|_2$ and $\mathcal{F}[G](\mathbf{u}) = I(-M^{-1}\mathbf{u})$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 d\mathbf{t} &\le Ce^{C|T_1||T_2|}\left(\int_{\mathbb{R}^2/T_1} \left|f(\mathbf{t})\right|^2 d\mathbf{t} + \int_{\mathbb{R}^2/T_2} \left|I\left(-M^{-1}\mathbf{u}\right)\right|^2 d\mathbf{u}\right) \\ &= Ce^{C|T_1||T_2|}\left(\int_{\mathbb{R}^2/T_1} \left|f(\mathbf{t})\right|^2 d\mathbf{t} + \frac{1}{\sin^2\gamma} \int_{\mathbb{R}^2/(T_2 \sin^2\gamma)} \left|I(\mathbf{u})\right|^2 d\mathbf{u}\right).\end{aligned} \tag{4.4}$$

Using (2.7), we can express inequality (4.4) as

$$\int_{\mathbb{R}^2} \left|f(\mathbf{t})\right|^2 d\mathbf{t} \le Ce^{C|T_1||T_2|}\left(\int_{\mathbb{R}^2/T_1} \left|f(\mathbf{t})\right|^2 d\mathbf{t} + 4\pi^2 \int_{\mathbb{R}^2/(T_2 \sin^2\gamma)} \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}\right),$$

which is the desired result. □

4.2 Amrein–Berthier–Benedicks's uncertainty principle

This subsection aims to obtain the Amrein–Berthier–Benedick uncertainty principle for the coupled fractional Fourier transform. To carry our endeavor, we have the following lemma.

Lemma 4.2 ([17]) *Let $f \in L^1(\mathbb{R}^2)$ and $E_1, E_2 \subset \mathbb{R}^2$ with finite measure satisfying* $\operatorname{supp}(f) \subseteq E_1$ *and* $\operatorname{supp}(\mathcal{F}[f]) \subseteq E_2$. *Moreover, if $|E_1||E_2| < \infty$, then $f = 0$.*

Theorem 4.3 *Let $\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})$ be the coupled fractional Fourier transform of any $f \in L^1(\mathbb{R}^2)$ and E_1, E_2 be any two subsets of $\mathbb{R}^2$ satisfying* $\operatorname{supp}(f(\mathbf{t})) \subseteq E_1$ *and* $\operatorname{supp}(\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})) \subseteq E_2$. *If $|E_1||E_2| < \infty$, then $f = 0$.*

Proof Clearly, $\operatorname{supp}(J(\mathbf{t})) = \operatorname{supp}(f) \subseteq E_1$ for the function $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})$. Also, $\operatorname{supp}(\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})) \subseteq E_2$ yields $\operatorname{supp}(I(\mathbf{u})) \subseteq E_2$ by virtue of (2.7). Hence, $\operatorname{supp}(I(-M^{-1}\mathbf{u})) \subseteq E_2/\sin^2\gamma$. Observe that $I(-M^{-1}\mathbf{u})$ is the Fourier transform corresponding to $J(\mathbf{t})$, therefore, as a consequence of Lemma 4.1, it follows that $J(\mathbf{t}) = 0$; i.e., $e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t}) = 0$. Thus, we conclude that $f = 0$. □

4.3 Donoho–Stark's uncertainty principle

In this subsection, we shall establish an analogue of Donoho–Stark's inequality for the CFrFT transform. This uncertainty inequality investigates the case wherein f and the corresponding coupled fractional Fourier transform $\mathcal{F}_{\alpha,\beta}[f]$ are zero almost everywhere outside the sets of finite measure. To begin with, we recall the following prerequisites.

Definition 4.4 For any measurable set $E \subset \mathbb{R}^2$, any function $f \in L^2(\mathbb{R}^2)$ is said to be ε-concentrated ($\varepsilon > 0$) on E if

$$\left(\int_{\mathbb{R}^2/E} \left|f(\mathbf{t})\right|^2 d\mathbf{t}\right)^{1/2} \leq \varepsilon \|f\|_2. \tag{4.5}$$

Lemma 4.5 ([18]) *For any measurable sets $E_1, E_2 \subseteq \mathbb{R}^2$ and $f \in L^2(\mathbb{R}^2)$ such that f is ε_{E_1}-concentrated on E_1 and $\mathcal{F}[f]$ is ε_{E_2}-concentrated on E_2,*

$$|E_1||E_2| \geq (1 - \varepsilon_{E_1} - \varepsilon_{E_2})^2, \tag{4.6}$$

where $|E_1|$ and $|E_2|$ denote the Lebesgue measures E_1 and E_2, respectively.

Theorem 4.6 *Let $E_1, E_2 \subset \mathbb{R}^2$ be any two measurable sets and assume that a nonzero square integrable function f is ε_{E_1}-concentrated on E_1 and the corresponding coupled fractional Fourier transform $\mathcal{F}_{\alpha,\beta}[f]$ is ε_{E_2}-concentrated on E_2. Then we have*

$$|E_1||E_2| \geq \sin^2\gamma\,(1 - \varepsilon_{E_1} - \varepsilon_{E_2})^2, \tag{4.7}$$

where $|E_1|$ and $|E_2|$ denote the Lebesgue measures E_1 and E_2, respectively.

Proof Since $\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})$ is ε_{E_2}-concentrated on E_2, we can write

$$\left(\int_{\mathbb{R}^2/E_2} \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right|^2 d\mathbf{u}\right)^{1/2} \leq \varepsilon_{E_2} \left\|\mathcal{F}_{\alpha,\beta}[f]\right\|_2. \tag{4.8}$$

Implementing (2.7) on (4.8), we obtain

$$\left(\int_{\mathbb{R}^2/E_2} \left|I(\mathbf{u})\right|^2 d\mathbf{u}\right)^{1/2} \leq \varepsilon_{E_2} \|F\|_2. \tag{4.9}$$

From (4.9), we conclude that $I(-M^{-1}\mathbf{u})$ is ε_{E_2}-concentrated on $E_2/\sin^2\gamma \subseteq \mathbb{R}^2$. Since I is ε_{E_1}-concentrated on E_1, so

$$\left(\int_{\mathbb{R}^2/E_1} \left|f(\mathbf{t})\right|^2 d\mathbf{t}\right)^{1/2} \leq \varepsilon_{E_1}\|f\|_2. \tag{4.10}$$

Furthermore, $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})$ implies that $|J(\mathbf{t})| = |f(\mathbf{t})|$ and $\mathcal{F}[J](\mathbf{u}) = I(-M^{-1}\mathbf{u})$. Thus, (4.10) takes the form

$$\left(\int_{\mathbb{R}^2/E_1} \left|J(\mathbf{t})\right|^2 d\mathbf{u}\right)^{1/2} \leq \varepsilon_{E_1}\|G\|_2. \tag{4.11}$$

Therefore, as a consequence of Lemma 4.5, we have

$$|E_1|\left|\frac{E_2}{\sin^2\gamma}\right| \geq (1-\varepsilon_{E_1}-\varepsilon_{E_2})^2,$$

so that

$$|E_1||E_2| \geq \sin^2\gamma(1-\varepsilon_{E_1}-\varepsilon_{E_2})^2.$$

This completes the proof of Theorem 4.6. □

5 Hardy's and Beurling's uncertainty inequalities

Apart from the classical uncertainty inequality, G.H. Hardy introduced a new variant of the uncertainty inequality regarding the decaying of a function f at infinity in its respective time and frequency domains [19]. Mathematically, if $f \in L^2(\mathbb{R}^2)$ such that

$$\left|f(\mathbf{t})\right| = \mathcal{O}\left(e^{-\pi\alpha|\mathbf{t}|^2}\right) \quad \text{and} \quad \left|\mathcal{F}[f](\mathbf{u})\right| = \mathcal{O}\left(e^{-\mathbf{u}^2/4\pi\alpha}\right) \tag{5.1}$$

for some $\alpha > 0$, then f must be of the following type:

$$f(\mathbf{t}) = Ce^{-\pi\alpha|\mathbf{t}|^2}, \quad C \in \mathbb{C}. \tag{5.2}$$

We now develop an analogue of Hardy's inequality for the coupled fractional Fourier transform.

Theorem 5.1 *Let $\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})$ be the coupled fractional Fourier transform of any $f \in L^2(\mathbb{R}^2)$ such that*

$$\left|f(\mathbf{t})\right| = \mathcal{O}\left(e^{-\pi\sigma|\mathbf{t}|^2}\right) \quad \textit{and} \quad \left|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\right| = \mathcal{O}\left(e^{-\mathbf{u}^2/4\pi\sigma\sin^2\gamma}\right) \tag{5.3}$$

for some $\alpha > 0$. Then f must be of the form

$$f(\mathbf{t}) = Ce^{i\tilde{a}(\gamma)|\mathbf{t}|^2-\pi\sigma|\mathbf{t}|^2} f(\mathbf{t}), \quad C \in \mathbb{C}. \tag{5.4}$$

Proof For $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})$, we observe that $|J(\mathbf{t})| = |f(\mathbf{t})|$ and $\mathcal{F}[J](\mathbf{u}) = I(-M^{-1}\mathbf{u})$, where I is given by (2.6). Moreover, we have

$$\big|J(\mathbf{t})\big| = \mathcal{O}\big(e^{-\pi\sigma|\mathbf{t}|^2}\big) \quad \text{and} \quad \big|I(\mathbf{u})\big| = \big|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\big| = \mathcal{O}\big(e^{-\mathbf{u}^2/4\pi\sigma\sin^2\gamma}\big). \tag{5.5}$$

Since $I(-M^{-1}\mathbf{u})$ is the Fourier transform corresponding to the function $J(\mathbf{t})$, implementing Hardy's uncertainty inequality of Fourier transform for the function J, we have

$$J(\mathbf{t}) = Ce^{-\pi\sigma|\mathbf{t}|^2}, \quad C \in \mathbb{C}. \tag{5.6}$$

Equivalently, (5.6) can be written in the form

$$f(\mathbf{t}) = Ce^{i\tilde{a}(\gamma)|\mathbf{t}|^2 - \pi\sigma|\mathbf{t}|^2} f(\mathbf{t}), \quad C \in \mathbb{C},$$

which proves the result. □

Beurling's uncertainty inequality is a new twist on Hardy's uncertainty inequality, demonstrating that a nontrivial function f and its Fourier transform $\mathcal{F}[f]$ cannot accept a simultaneous rapid in their respective domains. Formally, if $f, \mathcal{F}[f] \in L^1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big|f(\mathbf{t})\big|\big|\mathcal{F}[f](\mathbf{u})\big| e^{|\mathbf{t}\cdot\mathbf{u}|}\, d\mathbf{t}\, d\mathbf{u} < \infty, \tag{5.7}$$

then we must have $f = 0$.

Next, we establish an analogue of Beurling's inequality for the CFrFT.

Theorem 5.2 *Let $f \in L^1(\mathbb{R}^2)$ be such that $\mathcal{F}_{\alpha,\beta}[f] \in L^1(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big|f(\mathbf{t})\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\big| e^{|\mathbf{t}\cdot M\mathbf{u}|}\, d\mathbf{t}\, d\mathbf{u} < \infty, \tag{5.8}$$

then $f = 0$.

Proof For the function $J(\mathbf{t}) = e^{-i\tilde{a}(\gamma)|\mathbf{t}|^2} f(\mathbf{t})$, we observe that $|J(\mathbf{t})| = |f(\mathbf{t})|$ and $\mathcal{F}[J](\mathbf{u}) = I(-M^{-1}\mathbf{u})$, where I follows from (2.6). Therefore, we have

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big|J(\mathbf{t})\big|\big|I\big(-M^{-1}\mathbf{u}\big)\big| e^{|\mathbf{t}\cdot\mathbf{u}|}\, d\mathbf{t}\, d\mathbf{u} \\ &\quad = \frac{2\pi}{\sin\gamma}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big|J(\mathbf{t})\big|\big|\mathcal{F}_{\alpha,\beta}[f](\mathbf{u})\big| e^{|\mathbf{t}\cdot M\mathbf{u}|}\, d\mathbf{t}\, d\mathbf{u} < \infty. \end{aligned} \tag{5.9}$$

As a consequence of (5.7), we have $J(\mathbf{t}) = 0$, so that $f = 0$. □

6 Conclusion

In this study, we have achieved our primary goal of establishing various classes of uncertainty principles associated with the coupled fractional Fourier transform. More precisely, we derived an analogue of Pitts, Heisenberg's, logarithmic, Hardy's, and Beurling's uncertainty inequalities for the coupled fractional Fourier transform. Finally, we established some concentration-based inequalities for the underlying transform. In fact, we have shown that all of the uncertainty inequalities are governed by a fractional parameter γ. This study is new to the literature and is expected to contribute to the theory and applications of signal processing.

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References

1. Namias, V.: The fractional order Fourier transform and its application to quantum mechanics. J. Inst. Math. Appl. **25**, 241–265 (1980)
2. Debnath, L., Shah, F.A.: Wavelet Transforms and Their Applications. Birkhäuser, New York (2015)
3. Debnath, L., Shah, F.A.: Lecture Notes on Wavelet Transforms. Birkhäuser, Boston (2017)
4. Ozaktas, H., Zalevsky, Z., Kutay, M.: The Fractional Fourier Transform with Applications in Optics and Signal Processing. Wiley, New York (2001)
5. Zayed, A.: Two-dimensional fractional Fourier transform and some of its properties. Integral Transforms Spec. Funct. **29**(7), 553–570 (2018)
6. Kamalakkannana, R., Roopkumar, R., Zayed, A.: On the extension of the coupled fractional Fourier transform and its properties. Integral Transforms Spec. Funct. **33**(1), 65–80 (2021)
7. Kamalakkannana, R., Roopkumar, R.: Multidimensional fractional Fourier transform and generalized fractional convolution. Integral Transforms Spec. Funct. **31**(2), 152–165 (2020)
8. Teali, A.A., Shah, F.A., Tantary, A.Y.: Coupled fractional Wigner distribution in $L^2(\mathbb{R}^2)$. Fractals (2022, in press)
9. Folland, G.B., Sitaram, A.: The uncertainty principle: a mathematical survey. J. Fourier Anal. Appl. **3**, 207–238 (1997)
10. Havin, V., Jöricke, B.: The Uncertainty Principle in Harmonic Analysis. Springer, Berlin (1994)
11. Shah, F.A., Nisar, K.S., Lone, W.Z., Tantary, A.Y.: Uncertainty principles for the quadratic-phase Fourier transforms. Math. Methods Appl. Sci. **44**, 10416–10431 (2021)
12. Shah, F.A., Lone, W.Z.: Quadratic-phase wavelet transform with applications to generalized differential equations. Math. Methods Appl. Sci. (2021). https://doi.org/10.1002/mma.7842
13. Bahri, M., Shah, F.A., Tantary, A.Y.: Uncertainty principles for the continuous shearlet transforms in arbitrary space dimensions. Integral Transforms Spec. Funct. **31**(7), 538–555 (2020)
14. Beckner, W.: Pitt's inequality and the uncertainty principle. Proc. Am. Math. Soc. **123**, 1897–1905 (1995)
15. Jaming, P.: Nazarov's uncertainty principles in higher dimensions. J. Approx. Theory **149**, 30–41 (2007)
16. Kubo, H., Ogawa, T., Su, T.S.: Beckner type of the logarithmic Sobolev and a new type of Shanon's inequalities and an application to the uncertainty principle. Proc. Am. Math. Soc. **147**, 1511–1518 (2019)
17. Amrein, W.O., Berthier, A.M.: On support properties of L^p-functions and their Fourier transforms. J. Funct. Anal. **24**, 258–267 (1977)
18. Donoho, D.L., Stark, P.B.: Uncertainty principles and signal recovery. SIAM J. Appl. Math. **49**, 906–931 (1989)
19. Hardy, G.H.: A theorem concerning Fourier transforms. J. Lond. Math. Soc. **8**(3), 227–231 (1933)