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RESEARCH

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Uniqueness and Ulam–Hyers–Rassias stability results for sequential fractional pantograph q-differential equations

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Abstract

We study sequential fractional pantograph q-differential equations. We establish the uniqueness of solutions via Banach's contraction mapping principle. Further, we define and study the Ulam–Hyers stability and Ulam–Hyers–Rassias stability of solutions. We also discuss an illustrative example.

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Keywords: Pantograph equations; Fractional pantograph q-differential equation; Uniqueness; Ulam–Hyers stability

1 Introduction

Differential equations involving q-difference calculus have become a strong tool in modeling many problems in engineering, physics, and mathematics [1–3]. Differential equations with fractional q-difference calculus have been studied by different researchers [4–8]. Many interesting topics concerning fractional q-differential equations (FqDEs) are devoted to the existence and stability of the solutions. In recent years, several scholars have studied the existence, uniqueness, and different types of Ulam stability (US) of solutions of FqDEs; see, for example, [9–12]. Recently, sequential fractional differential equations has been studied by many scholars [13–15].

In the current paper, we discuss the uniqueness and different types of US of solutions for pantograph equations. This equation appears in different fields of pure and applied mathematics such as probability, number theory, quantum mechanics, dynamical systems, etc. [16–18]. The classical form of the pantograph differential equations (PDEs) is given by

$$\begin{cases} \frac{\mathrm{d}w(\mathfrak{s})}{\mathrm{d}\mathfrak{s}} = Aw(\mathfrak{s}) + Bw(\theta\mathfrak{s}), & \mathfrak{s} \in \overline{\Omega} = [0,T], \theta \in \mathrm{J} := (0,1), \\ w(0) = w_0. \end{cases}$$

Several authors have studied the existence, uniqueness, and US of solutions for the above PDEs involving different fractional derivatives. In [19] the authors discussed the existence and uniqueness of PDEs of the form

$$\begin{cases} {}^{C}\mathbb{D}^{\nu} w(\mathfrak{s}) = \varphi(\mathfrak{s}, w(\mathfrak{s}), w(\theta\mathfrak{s})), & \mathfrak{s} \in \overline{\Omega}, \theta \in \mathrm{J}, \\ w(0) = w_0, \end{cases}$$

where ${}^{C}\mathbb{D}^{\nu}$ is the Caputo fractional derivative of order $\nu \in \mathrm{J}$. In [20] the authors studied the existence, uniqueness, and stability of the following fractional pantograph q-differential equation (FPqDE):

$$\begin{cases} {}^{C}\mathbb{D}_q^{\nu} w(\mathfrak{s}) = \varphi(\mathfrak{s}, w(\mathfrak{s}), w(\theta\mathfrak{s})), & \mathfrak{s} \in \overline{\Omega}, \theta \in \mathrm{J}, q \in \mathrm{J}, \\ w(0) + \phi(w) = w_0, \end{cases}$$

where ${}^{C}\mathbb{D}_q^{\nu}$ is the Caputo fractional q-derivative of order $\nu \in \mathrm{J}$. Recently, in [21] the authors discussed the existence and uniqueness of sequential ψ-Hilfer FPDEs of the form

$$\left({}^{H}\mathbb{D}_{0^+}^{\nu,\sigma,\psi} + r\,{}^{H}\mathbb{D}_{0^+}^{\nu-1,\sigma,\psi}\right) w(\mathfrak{s}) = \varphi\big(\mathfrak{s}, w(\mathfrak{s}), w(\theta\mathfrak{s})\big), \quad \mathfrak{s} \in \overline{\Omega}, \theta \in \mathrm{J}, r \in \mathbb{R},$$

via conditions $w(0) = 0$,

$$\sum_{\jmath=1}^{\mathrm{p}} {}_1\mathring{\eta}_\jmath w({}_1\mathring{a}_\jmath) + \sum_{\jmath=1}^{\mathrm{n}} {}_2\mathring{\eta}_\jmath \mathbb{I}_{0^+}^{\sigma_\jmath,\psi} w({}_2\mathring{a}_\jmath) + \sum_{\jmath=1}^{\mathrm{m}} {}_3\mathring{\eta}_\jmath\, {}^{H}\mathbb{D}_{0^+}^{\gamma_\jmath,\psi} w({}_3\mathring{a}_\jmath) = \Lambda,$$

where $\sigma_\jmath > 0$, $\jmath = 1, \dots, \mathrm{n}$, ${}_k\mathring{\eta}_\jmath$ $(k = 1, 2, 3)$, $\Lambda \in \mathbb{R}$, and ${}^{H}\mathbb{D}_{0^+}^{\gamma,\sigma,\psi}$ are the ψ-Hilfer derivatives of order $\gamma \in \{\gamma_\jmath, \nu\}$, $1 < \gamma_\jmath < \nu \le 2$, $0 < \sigma \le 1$, $\mathbb{I}_{0^+}^{\sigma_\jmath,\psi}$ are the ψ-Riemann Liouville fractional integrals, and $\varphi : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

In this work, we discuss the uniqueness and Ulam–Hyers–Rassias stability (UHRS) of solutions for the following sequential FPqDE:

$$\begin{cases} [{}^{C}\mathbb{D}_q^{\nu} + r\,{}^{C}\mathbb{D}_q^{\sigma}] w(\mathfrak{s}) = \varphi(\mathfrak{s}, w(\mathfrak{s}), w(\theta\mathfrak{s}), {}^{C}\mathbb{D}_q^{\sigma} w(\theta\mathfrak{s})), & \mathfrak{s} \in \Omega, \\ w(0) = 0, \\ \lambda_1 w(T) = \lambda_2 w(\eta) + \Lambda, \\ \lambda_1 T^{\nu-\sigma} \neq \lambda_2 \eta^{\nu-\sigma}, \end{cases} \tag{1}$$

where $r \in \mathbb{R}^+$, $1 < \nu \le 2, \sigma, q, \theta \in \mathrm{J}$, $\eta \in \Omega$, $\Lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, ${}^{C}\mathbb{D}_q^{\nu}$ and ${}^{C}\mathbb{D}_q^{\sigma}$ are the Caputo-type q-fractional derivatives, and $\varphi : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function.

The outline of the paper is the following. In Sect. 2, we discuss the main definitions and lemmas by providing a necessary background of q-calculus, including the q-derivative and q-integral. In Sect. 3, we investigate the uniqueness for the FPqDE (1). In Sect. 5, we present an example to apply our outcomes.

2 Preliminaries on fractional q-calculus

In this section, we present essential q-derivative and q-integral notions. For more background information, we refer to [12, 22–24]. For a function w, the q–derivative is defined by

$$\mathbb{D}_q[w](t) = \left(\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\right)_q w(\mathfrak{s}) = \frac{w(\mathfrak{s}) - w(q\mathfrak{s})}{(1-q)\mathfrak{s}} \tag{2}$$

for $\mathfrak{s} \in \mathbb{T} \setminus \{0\}$, where $\mathbb{T} = \mathbb{T}_{s_0} = \{0\} \cup \{s : s = s_0 q^{\aleph}\}$ for $\aleph \in \mathbb{N}$ and $s_0 \in \mathbb{R}$, and [25]

$$\mathbb{D}_q[w](0) = \lim_{\mathfrak{s}\to 0} \mathbb{D}_q[w](t).$$

Also, the higher-order q-derivatives of the function u are defined by

$$\mathbb{D}_q^n[w](\mathfrak{s}) = \mathbb{D}_q\big[\mathbb{D}_q^{n-1}[w]\big](\mathfrak{s})$$

for $n \geq 1$, where $\mathbb{D}_q^0[u](\mathfrak{s}) = w(\mathfrak{s})$ [25]. In fact,

$$\mathbb{D}_q^n[w](\mathfrak{s}) = \frac{1}{\mathfrak{s}^n(1-q)^n} \sum_{k=0}^{n} \frac{(1-q^{-n})_q^{(k)}}{(1-q)_q^{(k)}} q^k w\big(\mathfrak{s}q^k\big) \tag{3}$$

for $\mathfrak{s} \in \mathbb{T} \setminus \{0\}$ [2]. The operator ${}^C\mathbb{D}_q^{\nu}$, the fractional q-derivative in the sense of Caputo [2, 26], of the function w is defined by

$$\begin{cases} {}^C\mathbb{D}_q^{\nu} w(\mathfrak{s}) = \mathbb{I}_q^{n-\nu}\mathbb{D}_q^n w(\mathfrak{s}), & \nu > 0, \\ {}^C\mathbb{D}_q^{0} w(\mathfrak{s}) = w(\mathfrak{s}), \end{cases}$$

where $n = [\nu]$. The fractional q-integral of the Riemann–Liouville type [2, 26] is given by

$$\begin{cases} \mathbb{I}_q^{\nu} w(\mathfrak{s}) = \frac{1}{\Gamma_q(\nu)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-1)} w(\grave{\imath})\, \mathrm{d}_q\grave{\imath}, & \nu > 0, \\ \mathbb{I}_q^{0} w(\mathfrak{s}) = w(\mathfrak{s}), \end{cases}$$

where $\Gamma_q(\nu) = \frac{(1-q)^{(\nu-1)}}{(1-q)^{\nu-1}}$, $\nu \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$, is called the q-gamma function and satisfies

$$\Gamma_q(\nu+1) = [\nu]_q \Gamma_q(\nu), \qquad [\sigma]_q = \frac{1-q^{\grave{a}}}{1-q}, \quad \grave{a} \in \mathbb{R}.$$

We need the following lemmas [2, 26].

Lemma 2.1 *Let* $\nu, \sigma \geq 0$, *and let* φ *be a function defined in* $\bar{\mathrm{J}} := [0,1]$. *Then we have the following formulas*:

$$\mathbb{I}_q^{\nu}\mathbb{I}_q^{\sigma}\varphi(\mathfrak{s}) = \mathbb{I}_q^{\nu+\sigma}\varphi(\mathfrak{s}), \qquad {}^C\mathbb{D}_q^{\mathbb{D}}\mathbb{I}_q^{\nu}\varphi(\mathfrak{s}) = \varphi(\mathfrak{s}).$$

Lemma 2.2 *Let* $\nu > 0$. *Then*

$$\mathbb{I}_q^{\nu}\, {}^C\mathbb{D}_q^{\nu}\varphi(\mathfrak{s}) = \varphi(\mathfrak{s}) - \sum_{j=0}^{[\nu]-1} \frac{\mathfrak{s}^j}{\Gamma_q(j+1)} \mathbb{D}_q^j\varphi(0).$$

Lemma 2.3 *For* $\sigma \in \mathbb{R}_+$ *and* $\epsilon > -1$, *we have*

$$\mathbb{I}_q^{\nu}(\mathfrak{s} - \grave{\imath})^{(\epsilon)} = \frac{\Gamma_q(\epsilon+1)}{\Gamma_q(\nu+\epsilon+1)} (\mathfrak{s} - \grave{\imath})^{(\nu+\epsilon)}.$$

Let us now define the space

$$\mathcal{W} = \{w : w, {}^{C}\mathbb{D}_q^{\sigma} w \in C(\Omega, \mathbb{R})\}$$

equipped with the norm

$$\|w\|_{\mathcal{W}} = \|w\| + \|{}^{C}\mathbb{D}_q^{\sigma} w\| = \sup_{\mathfrak{s}\in J}|w(\mathfrak{s})| + \sup_{\mathfrak{s}\in J}|{}^{C}\mathbb{D}_q^{\sigma} w(\mathfrak{s})|.$$

It is clear that $(\mathcal{W}, \|w\|_{\mathcal{W}})$ *is a Banach space.*

3 Uniqueness results

We prove the following auxiliary lemma, which is pivotal to define the solution for Problem (1).

Lemma 3.1 *Let* $\lambda_1 T^{\nu-\sigma} \neq \lambda_2 \eta^{\nu-\sigma}$. *For* $\psi \in C(\Omega, \mathbb{R})$, *the unique solution of the problem*

$$\begin{cases} [\mathbb{D}_q^{\nu} + r\mathbb{D}_q^{\sigma}]w(\mathfrak{s}) = \psi(\mathfrak{s}), & \mathfrak{s} \in J, \\ w(0) = 0, \\ \lambda_1 w(T) - \lambda_2 w(\eta) = \Lambda, & \Lambda \in \mathbb{R}, \end{cases} \tag{4}$$

where $r > 0$, $1 < \nu \leq 2$, $0 < \sigma \leq 1$ *and* $\eta \in \Omega$, *is given by*

$$\begin{aligned} w(\mathfrak{s}) = {} & \frac{1}{\Gamma_q(\nu)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-1)} \psi(\grave{\imath})\, d_q\grave{\imath} \\ & - \frac{r}{\Gamma_q(\nu-\sigma)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-\sigma-1)} w(\mathfrak{s})\, d_q\grave{\imath} \\ & + \frac{\mathfrak{s}^{\nu-\sigma}}{\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}} \left[\frac{\lambda_2}{\Gamma_q(\nu)} \int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-1)} \psi(\grave{\imath})\, d_q\grave{\imath} \right. \\ & \left. - \frac{r\lambda_2}{\Gamma_q(\nu-\sigma)} \int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-\sigma 1)} w(\grave{\imath})\, d_q\grave{\imath} \right] \\ & - \frac{\mathfrak{s}^{\nu-\sigma}}{\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}} \left[\frac{\lambda_1}{\Gamma_q(\nu)} \int_0^{T} (T - q\grave{\imath})^{(\nu-1)} \psi(\grave{\imath})\, d_q\grave{\imath} \right. \\ & \left. + \frac{r\lambda_1}{\Gamma_q(\nu-\sigma)} \int_0^{T} (T - q\grave{\imath})^{(\nu-\sigma-1)} w(\grave{\imath})\, d_q\grave{\imath} \right] \\ & + \frac{\mathfrak{s}^{\nu-\sigma}}{\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}} \Lambda. \end{aligned} \tag{5}$$

Proof We have

$$\left[{}^{C}\mathbb{D}_q^{\nu} + r\,{}^{C}\mathbb{D}_q^{\sigma}\right] w(\mathfrak{s}) = \psi(\mathfrak{s}). \tag{6}$$

Now we write the linear sequential FDE (6) as

$${}^{C}\mathbb{D}_q^{\sigma}\left[{}^{C}\mathbb{D}_q^{\nu-\sigma} + r\right] w(\mathfrak{s}) = \psi(\mathfrak{s}). \tag{7}$$

By taking the fractional q-integral of order σ for (7) we get

$$w(\mathfrak{s}) = \mathbb{I}_q^{\nu}\psi(\mathfrak{s}) - r\mathbb{I}_q^{\nu-\sigma}w(\mathfrak{s}) + a_0\frac{\mathfrak{s}^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} + b_0, \tag{8}$$

where a_0 and b_0 are arbitrary constants. By the boundary condition $w(0) = 0$ we conclude that $b_0 = 0$. Using the boundary condition $\lambda_1 w(T) - \lambda_2 w(\eta) = \Lambda$, we obtain that

$$\begin{aligned} a_0 = {}& \frac{\Gamma_q(\nu-\sigma+1)}{\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}}\big[\Lambda + \lambda_2\mathbb{I}_q^{\nu}\psi(\eta) - r\lambda_2\mathbb{I}_q^{\nu-\sigma}w(\eta) \\ & - \lambda_1\mathbb{I}_q^{\nu}\psi(T) + r\lambda_1\mathbb{I}_q^{\nu-\sigma}w(T)\big]. \end{aligned}$$

Substituting the values of a_0 and b_0 into (8), we obtain solution (5). This completes the proof. □

In view of Lemma 3.1, we can define the operator: $\mathfrak{G}: \mathcal{W} \to \mathcal{W}$ by

$$\begin{aligned} \mathfrak{G}w(\mathfrak{s}) = {}& \frac{1}{\Gamma_q(\nu)}\int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-1)}\varphi\big(\grave{\imath}, w(\grave{\imath}), w(\theta\grave{\imath}), {}^{C}\mathbb{D}_q^{\sigma}w(\theta\grave{\imath})\big)\,\mathrm{d}_q\grave{\imath} \\ & - \frac{r}{\Gamma_q(\nu-\sigma)}\int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-\sigma-1)}w(\grave{\imath})\,\mathrm{d}_q\grave{\imath} \\ & + \frac{\mathfrak{s}^{\nu-\sigma}}{\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}}\bigg[\frac{\lambda_2}{\Gamma_q(\nu)} \\ & \times \int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-1)}\varphi\big(\grave{\imath}, w(\grave{\imath}), w(\theta\grave{\imath}), {}^{C}\mathbb{D}_q^{\sigma}w(\theta\grave{\imath})\big)\,\mathrm{d}_q\grave{\imath} \\ & - \frac{r\lambda_2}{\Gamma_q(\nu-\sigma)}\int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-\sigma-1)}w(\grave{\imath})\grave{\imath}\bigg] \\ & - \frac{\mathfrak{s}^{\nu-\sigma}}{\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}}\bigg[\frac{\lambda_1}{\Gamma_q(\nu)} \\ & \times \int_0^{T} (T - q\grave{\imath})^{(\nu-1)}\varphi\big(\grave{\imath}, w(\grave{\imath}), w(\theta\grave{\imath}), {}^{C}\mathbb{D}_q^{\sigma}w(\theta\grave{\imath})\big)\,\mathrm{d}_q\grave{\imath} \\ & + \frac{r\lambda_1}{\Gamma_q(\nu-\sigma)}\int_0^{T} (T - q\grave{\imath})^{(\nu-\sigma-1)}w(\grave{\imath})\,\mathrm{d}_q\grave{\imath}\bigg] \\ & + \frac{\mathfrak{s}^{\nu-\sigma}}{\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}}\Lambda. \end{aligned} \tag{9}$$

For convenience, we denote

$$\nabla_1 := \frac{1}{\Gamma_q(\nu+1)}\bigg[T^{\nu} + \frac{T^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}|}\big(|\lambda_2|\eta^{\nu} + |\lambda_1|T^{\nu}\big)\bigg], \tag{10}$$

$$\begin{aligned}
\nabla_2 &:= \frac{r}{\Gamma_q(\nu-\sigma+1)}\bigg[T^{\nu-\sigma} \\
&\quad + \frac{T^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma}-\lambda_2\eta^{\nu-\sigma}|}\big(|\lambda_2|\eta^{\nu-\sigma}+|\lambda_1|T^{\nu-\sigma}\big)\bigg], \\
\Pi_1 &:= \frac{T^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} + \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma}-\lambda_2\eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\quad \times \left(\frac{|\lambda_2|\eta^{\nu}}{\Gamma_q(\nu+1)} + \frac{|\lambda_1|T^{\nu}}{\Gamma_q(\nu+1)}\right), \\
\Pi_2 &:= \frac{rT^{\nu-2\sigma}}{\Gamma_q(\nu-2\sigma+1)} + \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma}-\lambda_2\eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\quad \times \left(\frac{|\lambda_2|\eta^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} + \frac{|\lambda_1|T^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)}\right),
\end{aligned}$$

Our first result is based on Banach's fixed point theorem.

Theorem 3.2 *Let $\varphi:\Omega\times\mathbb{R}^3\to\mathbb{R}$ be continuous function satisfying the condition*

(C1) *there exist nonnegative constants $\breve{\mu}$ such that for all $\mathfrak{s}\in\Omega$ and $w_i,\acute{w}_i\in\mathbb{R}$ $(i=1,2,3)$,*

$$\big|\varphi(\mathfrak{s},w_1,w_2,w_3)-\varphi(\mathfrak{s},\acute{w}_1,\acute{w}_2,\acute{w}_3)\big| \le \breve{\mu}\sum_{i=1}^{3}|w_i-\acute{w}_i|.$$

If

$$\breve{\mu}(2\nabla_1+\Pi_1)+\nabla_2+\Pi_2<1, \tag{11}$$

where ∇_i, Π_i, $i=1,2$, are given by (10), then problem (1) has a unique solution on Ω.

Proof Let us fix $\Delta=\sup_{\mathfrak{s}\in\bar{\mathrm{J}}}\varphi(\mathfrak{s},0,0,0)$, choose

$$\frac{2\Delta\nabla_1+2\nabla_3+\Delta\Pi_1+\Pi_3}{1-[2(\breve{\mu}\nabla_1+\nabla_2)+(\breve{\mu}\Pi_1+\Pi_2)]}\le\ell,$$

where $B_\ell=\{w\in\mathcal{W}:\|w\|_{\mathcal{W}}\le\ell\}$ and

$$\begin{aligned}
\nabla_3 &:= \frac{|\Lambda|}{|\lambda_1 T^{\nu-\sigma}-\lambda_2\eta^{\nu-\sigma}|}, \\
\Pi_3 &:= \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}|\Lambda|}{|\lambda_1 T^{\nu-\sigma}-\lambda_2\eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)}.
\end{aligned}$$

Let $\varphi_w^*(\mathfrak{s})=\varphi(\mathfrak{s},w(\mathfrak{s}),w(\theta\mathfrak{s}),{}^C\mathbb{D}_q^\sigma w(\theta\mathfrak{s}))$. Then we show that $\mathfrak{G}B_\ell\subset B_\ell$. For $w\in B_\ell$, we have

$$\begin{aligned}
\big|\varphi_w^*(\mathfrak{s})\big| &= \big|\varphi\big(\mathfrak{s},w(\mathfrak{s}),w(\theta\mathfrak{s}),{}^C\mathbb{D}_q^\sigma w(\theta\mathfrak{s})\big)\big| \\
&\le \big|\varphi\big(\mathfrak{s},w(\mathfrak{s}),w(\theta\mathfrak{s}),{}^C\mathbb{D}_q^\sigma w(\theta\mathfrak{s})\big)-\varphi(\mathfrak{s},0,0,0)\big| + \big|\varphi(\mathfrak{s},0,0,0)\big| \\
&\le \breve{\mu}\big(\big|w(\mathfrak{s})\big|+\big|w(\eta\mathfrak{s})\big|+\big|{}^C\mathbb{D}_q^\sigma w(\mathfrak{s})\big|\big)+\Delta
\end{aligned}$$

$$\begin{aligned}
&\leq \breve{\mu}\big(2\|w\| + \big\|{}^{C}\mathbb{D}_q^{\sigma} w\big\|\big) + \Delta \\
&= 2\breve{\mu}\|w\|_{\mathcal{W}} + \Delta \leq 2\breve{\mu}\ell + \Delta.
\end{aligned}$$

Using this estimate, we get

$$\begin{aligned}
\big|\mathfrak{G}w(\mathfrak{s})\big| \leq{}& \frac{1}{\Gamma_q(\nu)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-1)} \big|\varphi_w^*(\grave{\imath})\big| \,\mathrm{d}_q\grave{\imath} \\
&+ \frac{k}{\Gamma_q(\nu-\sigma)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-\sigma-1)} \big|w(\grave{\imath})\big| \,\mathrm{d}_q s \\
&+ \frac{\mathfrak{s}^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|} \bigg[\frac{|\lambda_2|}{\Gamma_q(\nu)} \int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-1)} \big|\varphi_w^*(\grave{\imath})\big| \,\mathrm{d}_q\grave{\imath} \\
&+ \frac{r|\lambda_2|}{\Gamma_q(\nu-\sigma)} \int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-\sigma-1)} \big|w(\grave{\imath})\big| \,\mathrm{d}_q\grave{\imath} \bigg] \\
&+ \frac{\mathfrak{s}^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|} \bigg[\frac{|\lambda_1|}{\Gamma_q(\nu)} \int_0^{T} (T - q\grave{\imath})^{(\nu-1)} \big|\varphi_w^*(\grave{\imath})\big| \,\mathrm{d}_q\grave{\imath} \\
&+ \frac{r|\lambda_1|}{\Gamma_q(\nu-\sigma)} \int_0^{T} (T - q\grave{\imath})^{(\nu-\sigma-1)} \big|w(\mathfrak{s})\big| \,\mathrm{d}_q\grave{\imath} \bigg] \\
&+ \frac{\mathfrak{s}^{\nu-\sigma}|\Lambda|}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|},
\end{aligned}$$

which implies that

$$\begin{aligned}
\big\|\mathfrak{G}(w)\big\| \leq{}& \frac{(\breve{\mu}\ell + \Delta)}{\Gamma_q(\nu+1)} \bigg[T^{\nu} \\
&+ \frac{T^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|} \big(|\lambda_2|\eta^{\nu} + |\lambda_1|T^{\nu}\big) \bigg] \\
&+ \frac{r}{\Gamma_q(\nu-\sigma+1)} \bigg[T^{\nu-\sigma} \\
&+ \frac{T^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|} \big(|\lambda_2|\eta^{\nu-\sigma} + |\lambda_1|T^{\nu-\sigma}\big) \bigg] \ell \\
&+ \frac{|\Lambda|}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|} \\
={}& (\breve{\mu}\nabla_1 + \nabla_2)\ell + \Delta\nabla_1 + \nabla_3.
\end{aligned}$$

We also have

$$\begin{aligned}
\big|{}^{C}\mathbb{D}_q^{\nu}\mathfrak{G}w(\mathfrak{s})\big| \leq{}& \frac{1}{\Gamma_q(\nu-\sigma)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-\sigma-1)} \big|\varphi_w^*(\grave{\imath})\big| \,\mathrm{d}_q\grave{\imath} \\
&+ \frac{r}{\Gamma_q(\nu-2\sigma)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-2\sigma-1)} \big|w(\grave{\imath})\big| \,\mathrm{d}_q\grave{\imath} \\
&+ \frac{\Gamma_q(\nu-\sigma+1)\mathfrak{s}^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\times \bigg[\frac{|\lambda_2|}{\Gamma_q(\nu)} \int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-1)} \big|\varphi_w^*(\grave{\imath})\big| \,\mathrm{d}_q\grave{\imath}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{r|\lambda_2|}{\Gamma_q(\nu-\sigma)} \int_0^{\eta} (\eta - q\grave{\imath})^{(\nu-\sigma-1)} \big|w(\mathfrak{s})\big| \, \mathrm{d}_q\grave{\imath} \Bigg] \\
&+ \frac{\Gamma_q(\nu-\sigma+1)\mathfrak{s}^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\times \Bigg[\frac{|\lambda_1|}{\Gamma_q(\nu)} \int_0^T (T - q\mathfrak{s})^{(\nu-1)} \big|\varphi_w^*(\mathfrak{s})\big| \, \mathrm{d}_q\grave{\imath} \\
&+ \frac{r|\lambda_1|}{\Gamma_q(\nu-\sigma)} \int_0^T (T - q\grave{\imath})^{(\nu-\sigma-1)} \big|w(\grave{\imath})\big| \, \mathrm{d}_q\grave{\imath} \Bigg] \\
&+ \frac{\Gamma_q(\nu-\sigma+1)\mathfrak{s}^{\nu-2\sigma}|\Lambda|}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\big\|{}^C\mathbb{D}_q^{\sigma}\mathfrak{G}(w)\big\| \le{}& (\breve{\mu}\ell + \Delta) \Bigg[\frac{T^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} \\
&+ \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\times \left(\frac{|\lambda_2|\eta^{\nu}}{\Gamma_q(\nu+1)} + \frac{|\lambda_1|T^{\nu}}{\Gamma_q(\nu+1)} \right) \Bigg] \\
&+ r \Bigg[\frac{T^{\nu-2\sigma}}{\Gamma_q(\nu-2\sigma+1)} \\
&+ \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\times \left(\frac{|\lambda_2|\eta^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} + \frac{|\lambda_1|T^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} \right) \Bigg] \ell \\
&+ \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}|\Lambda|}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
={}& (\breve{\mu}\Pi_1 + \Pi_2)\ell + \Delta\Pi_1 + \Pi_3.
\end{aligned}
$$

From the definition of $\|\cdot\|_{\mathcal{W}}$ we have

$$
\begin{aligned}
\big\|\mathfrak{G}(w)\big\|_{\mathcal{W}} ={}& 2\big\|\mathfrak{G}(w)\big\| + \big\|{}^C\mathbb{D}_q^{\sigma}\mathfrak{G}(w)\big\| \\
\le{}& \big[2(\breve{\mu}\nabla_1 + \nabla_2) + (\breve{\mu}\Pi_1 + \Pi_2)\big]\ell \\
&+ 2\Delta\nabla_1 + 2\nabla_3 + \Delta\Pi_1 + \Pi_3 \\
\le{}& \ell,
\end{aligned}
$$

which implies that $\mathfrak{G}B_\ell \subset B_\ell$. For $w, \acute{w} \in B_\ell$ and for all $\mathfrak{s} \in \Omega$, we have

$$
\begin{aligned}
&\big|\mathfrak{G}w(\mathfrak{s}) - \mathfrak{G}\acute{w}(\mathfrak{s})\big| \\
&\quad\le \frac{1}{\Gamma_q(\nu)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-1)} \big|\varphi_w^*(\grave{\imath}) - \varphi_{\acute{w}}^*(\grave{\imath})\big| \, \mathrm{d}_q\grave{\imath} \\
&\qquad+ \frac{r}{\Gamma_q(\nu-\sigma)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\imath})^{(\nu-\sigma-1)} \big|w(\grave{\imath}) - \acute{w}(\grave{\imath})\big| \, \mathrm{d}_q\grave{\imath}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\mathfrak{s}^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|} \left[\frac{|\lambda_2|}{\Gamma_q(\nu)} \right. \\
&\times \int_0^{\eta} (\eta - q\grave{\iota})^{(\nu-1)} \left| \varphi_w^*(\grave{\iota}) - \varphi_{\acute{w}}^*(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \\
&\left. + \frac{r|\lambda_2|}{\Gamma_q(\nu - \sigma)} \int_0^{\eta} (\eta - q\grave{\iota})^{(\nu-\sigma-1)} \left| w(\grave{\iota}) - \acute{w}(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \right] \\
&+ \frac{\mathfrak{s}^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}|} \left[\frac{|\lambda_1|}{\Gamma_q(\nu)} \right. \\
&\times \int_0^{T} (T - q\grave{\iota})^{(\nu-1)} \left| \varphi_w^*(\grave{\iota}) - \varphi_{\acute{w}}^*(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \\
&\left. + \frac{r|\lambda_1|}{\Gamma_q(\nu - \sigma)} \int_0^{T} (T - q\grave{\iota})^{(\nu-\sigma-1)} \left| w(\grave{\iota}) - \acute{w}(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \right].
\end{aligned}
$$

Using (C1), we get

$$
\left\| \mathfrak{G}(w) - \mathfrak{G}(\acute{w}) \right\| \leq (\breve{\mu} \nabla_1 + \nabla_2) \| w - \acute{w} \|_{\mathcal{W}}.
$$

We also have

$$
\begin{aligned}
&\left| {}^C\mathbb{D}_q^{\sigma} \mathfrak{G} w(\mathfrak{s}) - {}^C\mathbb{D}_q^{\sigma} \mathfrak{G} \acute{w}(\mathfrak{s}) \right| \\
&\quad \leq \frac{1}{\Gamma_q(\nu - \sigma)} \int_0^{\mathfrak{s}} (\mathfrak{s} - q\grave{\iota})^{(\nu-\sigma-1)} \left| \varphi_w^*(\mathfrak{s}) - \varphi_{\acute{w}}^*(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \\
&\qquad + \frac{r}{\Gamma_q(\nu - 2\sigma)} \int_0^{t} (\mathfrak{s} - q\grave{\iota})^{(\nu-2\sigma-1)} \left| w(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \\
&\qquad + \frac{\Gamma_q(\nu - \sigma + 1)\mathfrak{s}^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}| \Gamma_q(\nu - 2\sigma + 1)} \\
&\qquad \times \left[\frac{|\lambda_2|}{\Gamma_q(\nu)} \int_0^{\eta} (\eta - q\grave{\iota})_q^{(\nu-1)} \left| \varphi_w^*(\grave{\iota}) - \varphi_{\acute{w}}^*(\grave{\iota}) \right| \grave{\iota} \right. \\
&\qquad \left. + \frac{r|\lambda_2|}{\Gamma_q(\nu - \sigma)} \int_0^{\eta} (\eta - q\grave{\iota})^{(\nu-\sigma-1)} \left| w(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \right] \\
&\qquad + \frac{\Gamma_q(\nu - \sigma + 1)\mathfrak{s}^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2 \eta^{\nu-\sigma}| \Gamma_q(\nu - 2\sigma + 1)} \\
&\qquad \times \left[\frac{|\lambda_1|}{\Gamma_q(\nu)} \int_0^{T} (T - q\grave{\iota})^{(\nu-1)} \left| \varphi_w^*(\grave{\iota}) - \varphi_{\acute{w}}^*(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \right. \\
&\qquad \left. + \frac{r|\lambda_1|}{\Gamma_q(\nu - \sigma)} \int_0^{T} (T - q\grave{\iota})^{(\nu-\sigma-1)} \left| w(\grave{\iota}) \right| \mathrm{d}_q \grave{\iota} \right].
\end{aligned}
$$

By (C1) we can write

$$
\left\| {}^C\mathbb{D}_q^{\sigma} \mathfrak{G}(w) - {}^C\mathbb{D}_q^{\sigma} \mathfrak{G}(\acute{w}) \right\| \leq (\breve{\mu} \Pi_1 + \Pi_2) \| w - \acute{w} \|_{\mathcal{W}}.
$$

Consequently, we obtain

$$\begin{aligned}\big\|\mathfrak{G}(w)-\mathfrak{G}(\acute{w})\big\|_{\mathcal{W}} &= 2\big\|\mathfrak{G}(w)-\mathfrak{G}(\acute{w})\big\| + \big\|{}^{C}\mathbb{D}_q^{\sigma}\mathfrak{G}(w)-{}^{C}\mathbb{D}_q^{\sigma}\mathfrak{G}(\acute{w})\big\| \\ &\le \big[(2\nabla_1+\Pi_1)\breve{\mu}+\nabla_2+\Pi_2\big]\|w-\acute{w}\|_{\mathcal{W}}.\end{aligned}$$

By (11) we see that $\mathfrak{G}$ is a contractive operator. Consequently, by the Banach fixed point theorem, $\mathfrak{G}$ has a fixed point, which is a solution of problem (1). This completes the proof. □

4 Ulam–Hyers–Rassias stability results

We discuss the Ulam-type stability of the q-fractional problem (1). For $\mathfrak{s}\in\Omega$, we have the following q-fractional inequalities:

$$\begin{aligned}&\big|\big[{}^{C}\mathbb{D}_q^{\nu}+r\,{}^{C}\mathbb{D}_q^{\sigma}\big]\acute{w}(\mathfrak{s})-\varphi_{\acute{w}}^{*}(\mathfrak{s})\big|\le\mathring{\eta},\\ &\big|\big[{}^{C}\mathbb{D}_q^{\nu}+r\,{}^{C}\mathbb{D}_q^{\sigma}\big]\acute{w}(\mathfrak{s})-\varphi_{\acute{w}}^{*}(\mathfrak{s})\big|\le\phi(\mathfrak{s}),\end{aligned}\tag{12}$$

and

$$\big|\big[{}^{C}\mathbb{D}_q^{\nu}+r\,{}^{C}\mathbb{D}_q^{\sigma}\big]\acute{w}(\mathfrak{s})-\varphi_{\acute{w}}^{*}(\mathfrak{s})\big|\le\mathring{\eta}\phi(\mathfrak{s}),\tag{13}$$

where $\mathring{\eta}\in\mathbb{R}^{+}$, and $\phi:\Omega\to\mathbb{R}_{+}$ is a continuous function. We further define the UHS, GUHS, UHRS, and GUHS.

We say that problem (1) is

S1) UHS if there is $\omega_{\varphi}\in\mathbb{R}_{+}$ such that for each $\mathring{\eta}>0$ and each solution $\acute{w}\in\mathcal{W}$ of inequality (12), there exists a solution $w\in\mathcal{W}$ of problem (1) such that $\|\acute{w}-w\|_{\mathcal{W}}\le\omega_{\varphi}\mathring{\eta}$;

S2) GUHS if there is $\chi_{\varphi}\in C(\mathbb{R}_{+},\mathbb{R}_{+})$, $\chi_{\varphi}(0)=0$, such that for each solution $\acute{w}\in\mathcal{W}$ of inequality (12), there exists a solution $w\in\mathcal{W}$ of problem (1) such that $\|\acute{w}-w\|_{\mathcal{W}}\le\chi_{\varphi}(\mathring{\eta})$;

S3) UHRS with respect to $\phi\in C(\Omega,\mathbb{R}_{+})$ if there is $\omega_{\varphi,\phi}>0$ such that for each $\mathring{\eta}>0$ and for each solution $\acute{w}\in\mathcal{W}$ of inequality (13), there exists a solution $w\in\mathcal{W}$ of problem (1) such that

$$\|\acute{w}-w\|_{\mathcal{W}}\le\omega_{\varphi,\phi}\mathring{\eta}\phi(\mathfrak{s}),\quad \mathfrak{s}\in\Omega;$$

S4) GUHRS with respect to $\phi\in C(\Omega,\mathbb{R}_{+})$ if there is $\omega_{\varphi,\phi}>0$ such that for each solution $\acute{w}\in\mathcal{W}$ of inequality (12), there exists a solution $w\in\mathcal{W}$ of problem (1) such that

$$\|\acute{w}-w\|_{\mathcal{W}}\le\omega_{\varphi,\phi}\phi(\mathfrak{s}),\quad \mathfrak{s}\in\Omega.$$

Remark 4.1 A function $\acute{w}\in W$ is a solution of inequality (12) iff there is $\hbar:\Omega\to\mathbb{R}$ (which depends on $\acute{w}$) such that $|\hbar(\mathfrak{s})|\le\lambda$ for all $\mathfrak{s}\in\Omega$ and

$$\big[{}^{C}\mathbb{D}_q^{\nu}+r\,{}^{C}\mathbb{D}_q^{\sigma}\big]\acute{w}(\mathfrak{s})=\varphi_{\acute{w}}^{*}(\mathfrak{s})+\hbar(\mathfrak{s}),\quad \mathfrak{s}\in\Omega.$$

Theorem 4.1 *Let $\varphi : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying condition (C1). If*

$$\frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)} < 1, \tag{14}$$

then problem (1) is UHS.

Proof Let $\acute{w} \in \mathcal{W}$ be a solution of inequality (12). Let us denote by $w \in \mathcal{W}$ the unique solution of the problem

$$\begin{cases} [{}^{C}\mathbb{D}_q^{\nu} + r\,{}^{C}\mathbb{D}_q^{\sigma}]w(\mathfrak{s}) = \varphi_{\acute{w}}^{*}(\mathfrak{s}), \quad \mathfrak{s} \in \Omega, q \in \mathrm{J}, \\ w(0) = \acute{w}(0), \\ w(T) = \acute{w}(T), \\ w(\eta) = \acute{w}(\eta), \quad \eta \in \Omega, \\ r > 0, \quad 1 < \nu \leq 2, 0 < \sigma \leq 1. \end{cases}$$

According to Lemma 3.1, we have

$$w(\mathfrak{s}) = \mathbb{I}_q^{\nu}\psi_w(\mathfrak{s}) - r\mathbb{I}_q^{\nu-\sigma}w(\mathfrak{s}) + a_0\frac{\mathfrak{s}^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} + b_0, \quad a_0, b_0 \in \mathbb{R},$$

where $\psi_w(\mathfrak{s}) = \varphi_w^{*}(\mathfrak{s})$ for $\mathfrak{s} \in \Omega$. By integration of (12) we obtain

$$\begin{aligned} &\left|\acute{w}(\mathfrak{s}) - \mathbb{I}_q^{\nu}\psi_{\acute{w}}(\mathfrak{s}) + r\mathbb{I}_q^{\nu-\sigma}\acute{w}(\mathfrak{s}) - a_1\frac{\mathfrak{s}^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} - b_1\right| \\ &\quad \leq \frac{\mathring{\eta}\mathfrak{s}^{\nu}}{\Gamma_q(\nu+1)} \leq \frac{\mathring{\eta}T^{\nu}}{\Gamma_q(\nu+1)}. \end{aligned} \tag{15}$$

Then, for any $\mathfrak{s} \in \bar{\mathrm{J}}$, we have

$$\begin{aligned} \acute{w}(\mathfrak{s}) - w(\mathfrak{s}) &= \acute{w}(\mathfrak{s}) - \mathbb{I}_q^{\nu}\big[\psi_w(\mathfrak{s})\big] + r\mathbb{I}_q^{\nu-\sigma}w(\mathfrak{s}) - a_1\frac{\mathfrak{s}^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} - b_1 \\ &\quad + \mathbb{I}_q^{\nu}\big[\psi_{\acute{w}}(\mathfrak{s}) - \psi_w(\mathfrak{s})\big] - r\mathbb{I}_q^{\nu-\sigma}\big(\acute{w}(\mathfrak{s}) - w(\mathfrak{s})\big). \end{aligned}$$

By (C1) and (15) we can write

$$\begin{aligned} \|\acute{w} - w\|_{\mathcal{W}} &\leq \left|\acute{w}(\mathfrak{s}) - \mathbb{I}_q^{\nu}\big[\psi_w(\mathfrak{s})\big] + r\mathbb{I}_q^{\nu-\sigma}w(\mathfrak{s}) - a_1\frac{\mathfrak{s}^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} - b_1\right| \\ &\quad + \frac{1}{\Gamma_q(\nu)}\int_0^{\mathfrak{s}}(\mathfrak{s} - q\grave{\imath})^{(\nu-1)}\big|\psi_{\acute{w}}(\grave{\imath}) - \psi_w(\grave{\imath})\big|\,\mathrm{d}_q\grave{\imath} \\ &\quad + \frac{r}{\Gamma_q(\nu-\sigma)}\int_0^{\mathfrak{s}}(\mathfrak{s} - q\grave{\imath})^{(\nu-\sigma)}\big|\acute{w}(\grave{\imath}) - w(\grave{\imath})\big|\,\mathrm{d}_q\grave{\imath} \\ &\leq \frac{\mathring{\eta}T^{\nu}}{\Gamma_q(\nu+1)} + \frac{\breve{\mu}}{\Gamma_q(\nu+1)}\|\acute{w} - w\|_{\mathcal{W}} \\ &\quad + \frac{r}{\Gamma_q(\nu-\sigma+1)}\|\acute{w} - w\|_{\mathcal{W}}. \end{aligned}$$

This implies that

$$\|\acute{w}-w\|_{\mathcal{W}} \leq \frac{\mathring{\eta} T^{\nu}}{\Gamma_q(\nu+1)} + \left(\frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)}\right)\|\acute{w}-w\|_{\mathcal{W}},$$

from which it follows that

$$\|\acute{w}-w\|_{\mathcal{W}}\left[1-\left(\frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)}\right)\right] \leq \frac{\mathring{\eta} T^{\nu}}{\Gamma_q(\nu+1)}.$$

Then

$$\|\acute{w}-w\|_{\mathcal{W}} \leq \frac{T^{\nu}}{\Gamma_q(\nu+1)[1-(\frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)})]}\mathring{\eta} := \omega_{\varphi}\mathring{\eta}.$$

Thus problem (1) is UHS. □

If we put $\chi_{\varphi} = \omega_{\varphi}\mathring{\eta}$, $\chi_{\varphi}(0) = 0$, then problem (1) is GUHS.

Theorem 4.2 *Let $\varphi : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying condition* $(C1)$, *and let* (14) *hold. Suppose that there is $\rho_{\phi} > 0$ such that*

$$\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-q\grave{\imath})^{(\nu-1)}}{\Gamma_q(\nu)}\phi(\grave{\imath})\,\mathrm{d}_q\grave{\imath} \leq \rho_{\phi}\phi(\mathfrak{s}), \quad \mathfrak{s}\in\Omega, \tag{16}$$

where $\phi \in C(\Omega,\mathbb{R}_+)$ is nondecreasing. Then problem (1) *is UHRS.*

Proof Let $\acute{w}\in\mathcal{W}$ is a solution of inequality (13). By Remark 4.1 we have

$$\begin{aligned}&\left|\acute{w}(\mathfrak{s}) - \mathbb{I}_q^{\nu}\Psi_{\acute{w}}(\mathfrak{s}) + r\mathbb{I}_q^{\nu-\sigma}\acute{w}(\mathfrak{s}) - \frac{\mathfrak{s}^{\nu-\sigma}a_1}{\Gamma_q(\nu-\sigma+1)} - b_1\right| \\ &\quad\leq \mathring{\eta}\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-q\grave{\imath})^{(\nu-1)}}{\Gamma_q(\nu)}\phi(\grave{\imath})\,\mathrm{d}_q\grave{\imath}.\end{aligned}$$

Let $w\in\mathcal{W}$ be the unique solution of the problem

$$\begin{cases}[{}^{C}\mathbb{D}_q^{\nu} + r\,{}^{C}\mathbb{D}_q^{\sigma}]w(\mathfrak{s}) = \varphi_{\acute{w}}^{*}(\mathfrak{s}), \quad \mathfrak{s}\in\Omega, q\in\mathrm{J},\\ w(0) = \acute{w}(0),\\ w(T) = \acute{w}(T),\\ w(\eta) = \acute{w}(\eta), \quad \eta\in\Omega,\\ r>0, \quad 1<\nu\leq 2, 0<\sigma\leq 1.\end{cases}$$

So by Lemma 3.1 we have

$$w(\mathfrak{s}) = \mathbb{I}_q^{\nu}\Psi_w(\mathfrak{s}) + r\mathbb{I}_q^{\nu-\sigma}w(\mathfrak{s}) - a_0\frac{\mathfrak{s}^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} - b_0.$$

Then we get

$$\begin{aligned}
\|\acute{w}-w\|_{\mathcal{W}} &\le \left|\acute{w}(\mathfrak{s}) - \mathbb{I}_q^{\nu}\Psi_{\acute{w}}(\mathfrak{s}) + r\mathbb{I}_q^{\nu-\sigma}\acute{w}(\mathfrak{s}) - a_1\frac{\mathfrak{s}^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} - b_1\right| \\
&\quad + \mathbb{I}_q^{\nu}\big[\left|\Psi_z(\mathfrak{s}) - \Psi_w(\mathfrak{s})\right|\big] + r\mathbb{I}^{\nu-\sigma}\left|\acute{w}(\mathfrak{s}) - w(\mathfrak{s})\right|, \\
&\le \mathring{\eta}\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-q\grave{\imath})^{(\nu-1)}}{\Gamma_q(\nu)}\phi(\grave{\imath})\,\mathrm{d}_q\grave{\imath} \\
&\quad + \frac{1}{\Gamma_q(\nu)}\int_0^{\mathfrak{s}} (\mathfrak{s}-q\grave{\imath})^{(\nu-1)}\left|\Psi_{\acute{w}}(\grave{\imath}) - \Psi_w(\grave{\imath})\right|\mathrm{d}_q\grave{\imath} \\
&\quad + \frac{r}{\Gamma_q(\nu-\sigma)}\int_0^{\mathfrak{s}} (\mathfrak{s}-q\grave{\imath})^{(\nu-\sigma-1)}\left|\big(\acute{w}(\grave{\imath}) - w(\grave{\imath})\big)\right|\mathrm{d}_q\grave{\imath}.
\end{aligned}$$

From (C1) and (16) we can write

$$\|\acute{w}-w\|_{\mathcal{W}} \le \mathring{\eta}\rho_{\phi}\phi(\mathfrak{s}) + \left(\frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)}\right)\|\acute{w}-w\|_{\mathcal{W}}.$$

Indeed,

$$\|\acute{w}-w\|_{\mathcal{W}}\left[1 - \left(\frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)}\right)\right] \le \mathring{\eta}\rho_{\phi}\phi(\mathfrak{s}).$$

Then

$$\begin{aligned}
\|\acute{w}-w\|_{\mathcal{W}} &\le \left[\frac{\rho_{\phi}}{1-(\frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)})}\right]\mathring{\eta}\phi(\mathfrak{s}) \\
&= \omega_{\varphi,\phi}\mathring{\eta}\phi(\mathfrak{s}), \quad \mathfrak{s}\in\Omega.
\end{aligned}$$

Hence problem (1) is stable in the UHR sense. □

5 An illustrative example

Example 5.1 Based on problem (1), we consider the following FqDE:

$$\begin{cases}
[{}^{C}\mathbb{D}_q^{\frac{7}{4}} + \frac{1}{50}\,{}^{C}\mathbb{D}_q^{\frac{4}{5}}]w(\mathfrak{s}) \\
\quad = \frac{2}{13} + \frac{20^2}{63^2\pi^2}\arctan(3\pi w(\mathfrak{s})) + \frac{1}{15^2\pi}\sin(\mathfrak{s})w(\frac{5}{6}\mathfrak{s}) \\
\qquad + \frac{1}{15^2\pi}\,{}^{C}\mathbb{D}^{\frac{3}{4}}w(\frac{5}{6}\mathfrak{s}), \quad \mathfrak{s}\in\overline{\Omega} = [0,1], \\
w(0) = 0, \\
\frac{1}{15}w(1) - \frac{6}{17}w(\frac{3}{4}) = \frac{\sqrt{7}}{8}.
\end{cases} \tag{17}$$

and the q-fractional inequalities

$$\begin{aligned}
&\left|\left[{}^{C}\mathbb{D}_q^{\frac{7}{4}} + \frac{1}{11}\,{}^{C}\mathbb{D}_q^{\frac{4}{5}}\right]\acute{w}(\mathfrak{s}) - \varphi\left(\mathfrak{s}, \acute{w}(\mathfrak{s}), \acute{w}\left(\frac{5}{6}\mathfrak{s}\right), {}^{C}\mathbb{D}_q^{\frac{4}{5}}\acute{w}\left(\frac{5}{6}\mathfrak{s}\right)\right)\right| \le \mathring{\eta}, \\
&\left|\left[{}^{C}\mathbb{D}_q^{\frac{7}{4}} + \frac{1}{11}\,{}^{C}\mathbb{D}_q^{\frac{4}{5}}\right]\acute{w}(\mathfrak{s}) - \varphi\left(\mathfrak{s}, \acute{w}(\mathfrak{s}), \acute{w}\left(\frac{5}{6}\mathfrak{s}\right), {}^{C}\mathbb{D}_q^{\frac{4}{5}}\acute{w}\left(\frac{5}{6}\mathfrak{s}\right)\right)\right| \le \mathring{\eta}\phi(\mathfrak{s})
\end{aligned}$$

for $q \in \bar{\mathrm{J}} = [0,1]$. It is clear that $\nu = \frac{7}{4} \in (1,2]$, $r = \frac{1}{50} \in \mathbb{R}^{+}$, $\sigma = \frac{4}{5} \in (0,1]$, $\theta = \frac{5}{6} \in \bar{\mathrm{J}}$, $T = 1$, and

$$\begin{aligned}
&\varphi\left(\mathfrak{s}, w(\mathfrak{s}), w\left(\frac{5}{6}\mathfrak{s}\right), {}^{C}\mathbb{D}_{q}^{\frac{4}{5}} w\left(\frac{5}{6}\mathfrak{s}\right)\right) \\
&\quad = \frac{2}{13} + \frac{20^2}{63^2\pi^2}\arctan\bigl(3\pi w(\mathfrak{s})\bigr) \\
&\qquad + \frac{1}{15^2\pi}\sin(\mathfrak{s}) w\left(\frac{5}{6}\mathfrak{s}\right) + \frac{1}{15^2\pi}\, {}^{C}\mathbb{D}_{q}^{\frac{3}{4}} w\left(\frac{5}{6}\mathfrak{s}\right).
\end{aligned}$$

For any $w_i, \acute{w}_i \in \mathbb{R}^3$, $i = 1,2,3$, and $\mathfrak{s} \in \overline{\Omega}$, we can write

$$\begin{aligned}
&\bigl|\varphi(\mathfrak{s}, w_1, w_2, w_3) - \varphi(\mathfrak{s}, \acute{w}_1, \acute{w}_2, \acute{w}_3)\bigr| \\
&\quad = \left| \frac{2}{13} + \frac{20^2}{63^2\pi^2}\arctan\bigl(3\pi w(\mathfrak{s})\bigr) \right. \\
&\qquad + \frac{1}{15^2\pi}\sin(\mathfrak{s}) w\left(\frac{5}{6}\mathfrak{s}\right) + \frac{1}{15^2\pi}\, {}^{C}\mathbb{D}_{q}^{\frac{3}{4}} w\left(\frac{5}{6}\mathfrak{s}\right) \\
&\qquad - \left(\frac{2}{13} + \frac{20^2}{63^2\pi^2}\arctan\bigl(3\pi \acute{w}(\mathfrak{s})\bigr) \right. \\
&\qquad \left.\left. + \frac{1}{15^2\pi}\sin(\mathfrak{s}) \acute{w}\left(\frac{5}{6}\mathfrak{s}\right) + \frac{1}{15^2\pi}\, {}^{C}\mathbb{D}_{q}^{\frac{3}{4}} \acute{w}\left(\frac{5}{6}\mathfrak{s}\right)\right)\right| \\
&\quad = \frac{20^2}{63^2\pi^2}\bigl|\arctan\bigl(3\pi w(\mathfrak{s})\bigr) - \arctan\bigl(3\pi \acute{w}(\mathfrak{s})\bigr)\bigr| \\
&\qquad + \frac{1}{15^2\pi}\left|\sin(\mathfrak{s}) w\left(\frac{5}{6}\mathfrak{s}\right) - \sin(\mathfrak{s}) \acute{w}\left(\frac{5}{6}\mathfrak{s}\right)\right| \\
&\qquad + \frac{1}{15^2\pi}\left|{}^{C}\mathbb{D}_{q}^{\frac{3}{4}} w\left(\frac{5}{6}\mathfrak{s}\right) - {}^{C}\mathbb{D}_{q}^{\frac{3}{4}} \acute{w}\left(\frac{5}{6}\mathfrak{s}\right)\right| \\
&\quad \leq \frac{1}{15^2\pi}\sum_{i=1}^{3} |w_i - \acute{w}_i|.
\end{aligned}$$

Hence condition (C1) holds with $\breve{\mu} = \frac{1}{15^2\pi}$. Now we discuss problem (17) for

$$q = \left\{\frac{1}{7}, \frac{1}{2}, \frac{8}{9}\right\}.$$

By using equations (10), assuming that

$$\begin{aligned}
&r = \frac{1}{50} \in \mathbb{R}, \qquad \lambda_1 = \frac{1}{15} \in \mathbb{R}, \qquad \lambda_2 = \frac{6}{17} \in \mathbb{R}, \\
&\Lambda = \frac{\sqrt{7}}{8} \in \mathbb{R}, \qquad \eta = \frac{3}{4} \in \mathbb{R}
\end{aligned}$$

in (17), and applying the MATLAB program (Algorithm 1), we have

$$\begin{aligned}
\nabla_1 &= \frac{1}{\Gamma_q(\nu+1)}\left[T^{\nu} + \frac{T^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}|}\left(|\lambda_2|\eta^{\nu} + |\lambda_1|T^{\nu}\right)\right] \\
&= \frac{1}{\Gamma_q(\frac{7}{4}+1)}\left[1 + \frac{1}{|\frac{1}{15} - \frac{6}{17}(\frac{3}{4})^{\frac{7}{4}-\frac{4}{5}}|}\left(\left|\frac{6}{17}\right|\left(\frac{3}{4}\right)^{\frac{7}{4}} + \left|\frac{1}{15}\right|\right)\right] \\
&\simeq \begin{cases} 2.09415, & q = \frac{1}{7}, \\ 1.26217, & q = \frac{1}{2}, \\ 0.28702, & q = \frac{8}{9}, \end{cases} \\
\nabla_2 &= \frac{r}{\Gamma_q(\nu-\sigma+1)}\left[T^{\nu-\sigma} + \frac{T^{\nu-\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}|}\left(|\lambda_2|\eta^{\nu-\sigma} + |\lambda_1|T^{\nu-\sigma}\right)\right] \\
&= \frac{\frac{1}{50}}{\Gamma_q(\frac{7}{4}-\frac{4}{5}+1)}\left[1 + \frac{1}{|\frac{1}{15} - \frac{6}{17}\times(\frac{3}{4})^{\frac{7}{4}-\frac{4}{5}}|}\left(\left|\frac{6}{17}\right|\left(\frac{3}{4}\right)^{\frac{7}{4}-\frac{4}{5}} + \left|\frac{1}{15}\right|\right)\right] \\
&\simeq \begin{cases} 0.04507, & q = \frac{1}{7}, \\ 0.02601, & q = \frac{1}{2}, \\ 0.00574, & q = \frac{8}{9}, \end{cases} \\
\Pi_1 &= \frac{T^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} + \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\quad\times\left(\frac{|\lambda_2|\eta^{\nu}}{\Gamma_q(\nu+1)} + \frac{|\lambda_1|T^{\nu}}{\Gamma_q(\nu+1)}\right) \\
&= \frac{1}{\Gamma_q(\frac{7}{4}-\frac{4}{5}+1)} + \frac{\Gamma_q(\frac{7}{4}-\frac{4}{5}+1)}{|\frac{1}{15} - \frac{6}{17}\times(\frac{3}{4})^{\frac{7}{4}-\frac{4}{5}}|\Gamma_q(\frac{7}{4}-2\times\frac{4}{5}+1)} \\
&\quad\times\left(\frac{|\frac{6}{17}|(\frac{3}{4})^{\frac{7}{4}}}{\Gamma_q(\frac{7}{4}+1)} + \frac{|\frac{1}{15}|}{\Gamma_q(\frac{7}{4}+1)}\right) \\
&\simeq \begin{cases} 1.21962, & q = \frac{1}{7}, \\ 0.64445, & q = \frac{1}{2}, \\ 0.13680, & q = \frac{8}{9}, \end{cases} \\
\Pi_2 &= \frac{rT^{\nu-2\sigma}}{\Gamma_q(\nu-2\sigma+1)} + \frac{\Gamma_q(\nu-\sigma+1)T^{\nu-2\sigma}}{|\lambda_1 T^{\nu-\sigma} - \lambda_2\eta^{\nu-\sigma}|\Gamma_q(\nu-2\sigma+1)} \\
&\quad\times\left(\frac{|\lambda_2|\eta^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)} + \frac{|\lambda_1|T^{\nu-\sigma}}{\Gamma_q(\nu-\sigma+1)}\right) \\
&= \frac{\frac{1}{50}}{\Gamma_q(\frac{7}{4}-\frac{8}{5}+1)} + \frac{\Gamma_q(\frac{7}{4}-\frac{4}{5}+1)}{|\frac{1}{15} - \frac{6}{17}\times(\frac{3}{4})^{\frac{7}{4}-\frac{4}{5}}|\Gamma_q(\frac{7}{4}-\frac{8}{5}+1)} \\
&\quad\times\left(\frac{|\frac{6}{17}|(\frac{3}{4})^{\frac{7}{4}-\frac{4}{5}}}{\Gamma_q(\frac{7}{4}-\frac{4}{5}+1)} + \frac{|\frac{1}{15}|}{\Gamma_q(\frac{7}{4}-\frac{4}{5}+1)}\right) \\
&\simeq \begin{cases} 0.43588, & q = \frac{1}{7}, \\ 0.17436, & q = \frac{1}{2}, \\ 0.03143, & q = \frac{8}{9}. \end{cases}
\end{aligned}$$

Table 1 Numerical results of ∇_1, ∇_2, Π_1, and Π_2 for $q = \frac{1}{7}$ in Example 5.1

n	$q = \frac{1}{7}$							
	$\Gamma_q(\nu+1)$	$\Gamma_q(\nu-\sigma+1)$	$\Gamma_q(\nu-2\sigma+1)$	∇_1	∇_2	Π_1	Π_2	Σ
1	1.14283	1.18028	3.79986	2.08868	0.04508	1.22423	0.44225	0.49497
2	1.14027	1.18063	3.84742	2.09337	0.04507	1.22027	0.43678	0.48950
3	1.13990	1.18068	3.85421	2.09404	0.04507	1.21971	0.43601	0.48873
4	1.13985	1.18069	3.85518	2.09413	0.04507	1.21963	0.43590	0.48862
5	1.13984	1.18069	3.85532	2.09415	0.04507	1.21962	0.43589	0.48860
6	1.13984	1.18069	3.85534	2.09415	0.04507	1.21962	0.43588	0.48860
7	1.13984	1.18069	3.85534	2.09415	0.04507	1.21962	0.43588	0.48860
8	1.13984	1.18069	3.85534	2.09415	0.04507	1.21962	0.43588	0.48860
9	1.13984	1.18069	3.85534	2.09415	0.04507	1.21962	0.43588	0.48860

Table 2 Numerical results of ∇_1, ∇_2, Π_1, and Π_2 for $q = \frac{1}{2}$ in Example 5.1

n	$q = \frac{1}{7}$							
	$\Gamma_q(\nu+1)$	$\Gamma_q(\nu-\sigma+1)$	$\Gamma_q(\nu-2\sigma+1)$	∇_1	∇_2	Π_1	Π_2	Σ
1	2.10842	2.02631	7.66976	1.13213	0.02626	0.66731	0.21910	0.24951
2	1.99300	2.03657	8.66291	1.19769	0.02613	0.65463	0.19399	0.22443
3	1.94055	2.04137	9.15257	1.23006	0.02607	0.64928	0.18361	0.21407
4	1.91550	2.04369	9.39580	1.24615	0.02604	0.64681	0.17885	0.20933
5	1.90326	2.04484	9.51703	1.25417	0.02602	0.64562	0.17658	0.20706
6	1.89720	2.04541	9.57755	1.25817	0.02601	0.64503	0.17546	0.20595
7	1.89419	2.04569	9.60778	1.26017	0.02601	0.64474	0.17491	0.20540
8	1.89269	2.04583	9.62290	1.26117	0.02601	0.64460	0.17463	0.20512
9	1.89194	2.04590	9.63045	1.26167	0.02601	0.64453	0.17450	0.20499
10	1.89156	2.04594	9.63423	1.26192	0.02601	0.64449	0.17443	0.20492
11	1.89137	2.04595	9.63612	1.26205	0.02601	0.64447	0.17439	0.20488
12	1.89128	2.04596	9.63706	1.26211	0.02601	0.64446	0.17438	0.20487
13	1.89123	2.04597	9.63753	1.26214	0.02601	0.64446	0.17437	0.20486
14	1.89121	2.04597	9.63777	1.26216	0.02601	0.64446	0.17436	0.20485
15	1.89120	2.04597	9.63789	1.26216	0.02601	0.64445	0.17436	0.20485
16	1.89119	2.04597	9.63794	1.26217	0.02601	0.64445	0.17436	0.20485
17	1.89119	2.04597	9.63797	1.26217	0.02601	0.64445	0.17436	0.20485
18	1.89119	2.04597	9.63799	1.26217	0.02601	0.64445	0.17436	0.20485
19	1.89119	2.04597	9.63800	1.26217	0.02601	0.64445	0.17436	0.20485
20	1.89119	2.04597	9.63800	1.26217	0.02601	0.64445	0.17436	0.20485
21	1.89119	2.04597	9.63800	1.26217	0.02601	0.64445	0.17436	0.20485

Tables 1, 2, and 3 show these results. Also, we can see a graphical representation of ∇_i, Π_i $(i = 1, 2)$ and Σ in Figs. 1, 2, and 3. Using the given data, we find that

$$\Sigma = (2\nabla_1 + \Pi_1)\breve{\mu} + \nabla_2 + \Pi_2 \simeq \begin{Bmatrix} 0.48860, q = \frac{1}{7}, \\ 0.20485, q = \frac{1}{2}, \\ 0.03817, q = \frac{8}{9}, \end{Bmatrix} < 1,$$

Hence by Theorem 3.2 problem (17) has a unique solution. Also, from (14) we have

$$\breve{\Sigma} = \frac{\breve{\mu}}{\Gamma_q(\nu+1)} + \frac{r}{\Gamma_q(\nu-\sigma+1)} \simeq \begin{Bmatrix} 0.01818, q = \frac{1}{7}, \\ 0.01052, q = \frac{1}{2}, \\ 0.00233, q = \frac{8}{9}, \end{Bmatrix} < 1.$$

Table 3 Numerical results of ∇_1, ∇_2, Π_1, and Π_2 for $q = \frac{8}{9}$ in Example 5.1

n	$q=\frac{8}{9}$							
	$\Gamma_q(\nu+1)$	$\Gamma_q(\nu-\sigma+1)$	$\Gamma_q(\nu-2\sigma+1)$	∇_1	∇_2	Π_1	Π_2	Σ
1	21.16106	8.65603	14.61554	0.11280	0.00615	0.15435	0.11498	0.12166
2	17.64177	8.77836	19.44710	0.13530	0.00606	0.14941	0.08641	0.09307
3	15.46858	8.86537	23.57141	0.15431	0.00600	0.14652	0.07129	0.07794
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
34	8.40872	9.26182	52.74519	0.28387	0.00575	0.13693	0.03186	0.03860
35	8.39834	9.26262	52.82649	0.28422	0.00574	0.13692	0.03181	0.03855
36	8.38914	9.26333	52.89874	0.28453	0.00574	0.13690	0.03177	0.03851
37	8.38098	9.26396	52.96295	0.28481	0.00574	0.13689	0.03173	0.03847
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
56	8.32322	9.26844	53.42139	0.28679	0.00574	0.13681	0.03146	0.03820
57	8.32246	9.26850	53.42747	0.28681	0.00574	0.13680	0.03145	0.03820
58	8.32178	9.26855	53.43288	0.28684	0.00574	0.13680	0.03145	0.03820
59	8.32118	9.26860	53.43768	0.28686	0.00574	0.13680	0.03145	0.03819
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
65	8.31875	9.26879	53.45715	0.28694	0.00574	0.13680	0.03144	0.03818
66	8.31848	9.26881	53.45925	0.28695	0.00574	0.13680	0.03143	0.03818
67	8.31825	9.26882	53.46112	0.28696	0.00574	0.13680	0.03143	0.03818
68	8.31804	9.26884	53.46279	0.28697	0.00574	0.13680	0.03143	0.03818
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
73	8.31730	9.26890	53.46871	0.28699	0.00574	0.13680	0.03143	0.03818
74	8.31720	9.26891	53.46953	0.28700	0.00574	0.13680	0.03143	0.03817
75	8.31711	9.26891	53.47026	0.28700	0.00574	0.13680	0.03143	0.03817
76	8.31703	9.26892	53.47091	0.28700	0.00574	0.13680	0.03143	0.03817
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
83	8.31666	9.26895	53.47382	0.28701	0.00574	0.13680	0.03143	0.03817
84	8.31663	9.26895	53.47408	0.28702	0.00574	0.13680	0.03143	0.03817
85	8.31661	9.26895	53.47430	0.28702	0.00574	0.13679	0.03143	0.03817
86	8.31658	9.26895	53.47450	0.28702	0.00574	0.13679	0.03143	0.03817

Table 4 and Fig. 4 show these results and graphical representation of $\breve{\Sigma}$ respectively. So by Theorem 4.1 problem (17) is UHS such that

$$\begin{aligned} \|\acute{w}-w\|_{\mathcal{W}} &\leq \frac{T^{\nu}}{\Gamma_q(\nu+1)[1-(\frac{\breve{\mu}}{\Gamma_q(\nu+1)}+\frac{r}{\Gamma_q(\nu-\sigma+1)})]}\mathring{\eta} \\ &= \frac{1}{\Gamma_q(\frac{7}{4}+1)[1-(\frac{\breve{\mu}}{\Gamma_q(\frac{7}{4}+1)}+\frac{\frac{1}{50}}{\Gamma_q(\frac{7}{4}-\frac{4}{5}+1)})]}\mathring{\eta} \\ &= \omega_{\varphi}\mathring{\eta} \simeq \begin{cases} 0.89356\mathring{\eta}, & q=\frac{1}{7}, \\ 0.53439\mathring{\eta}, & q=\frac{1}{2}, \\ 0.12046\mathring{\eta}, & q=\frac{8}{9}. \end{cases} \end{aligned}$$

Let $\phi(\mathfrak{s}) = \mathfrak{s}^2$. Then

$$\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-q\grave{\imath})^{(\nu-1)}}{\Gamma_q(\nu)}\phi(\grave{\imath})\,d_q\grave{\imath} = \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-q\grave{\imath})^{(\frac{7}{4}-1)}}{\Gamma_q(\frac{7}{4})}\grave{\imath}^2\,d_q\grave{\imath}$$

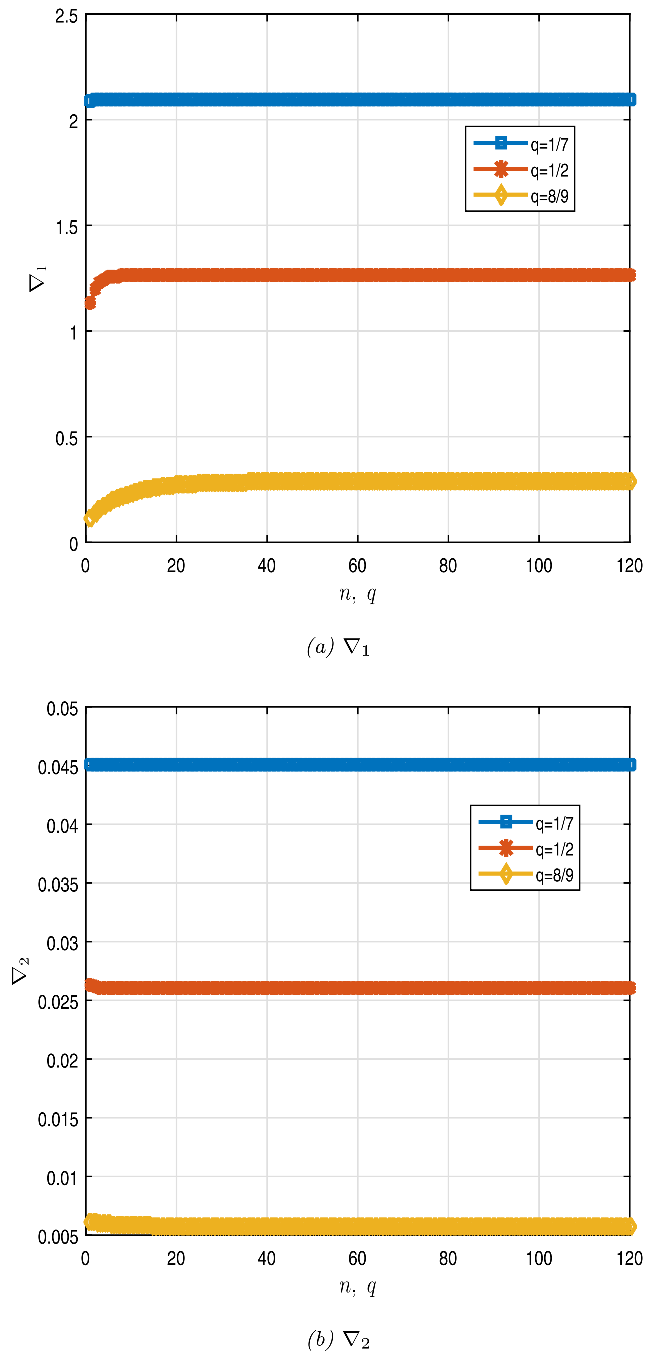

Figure 1 Graphical representation of ∇_i for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$ in Example 5.1

$$\leq \begin{cases} 0.88636, & q = \frac{1}{7}, \\ 0.64374, & q = \frac{1}{2}, \\ 0.46437, & q = \frac{8}{9} \end{cases}$$

$$\leq \rho_\phi \times \mathfrak{s}^2 = \rho_\phi \phi(\mathfrak{s}).$$

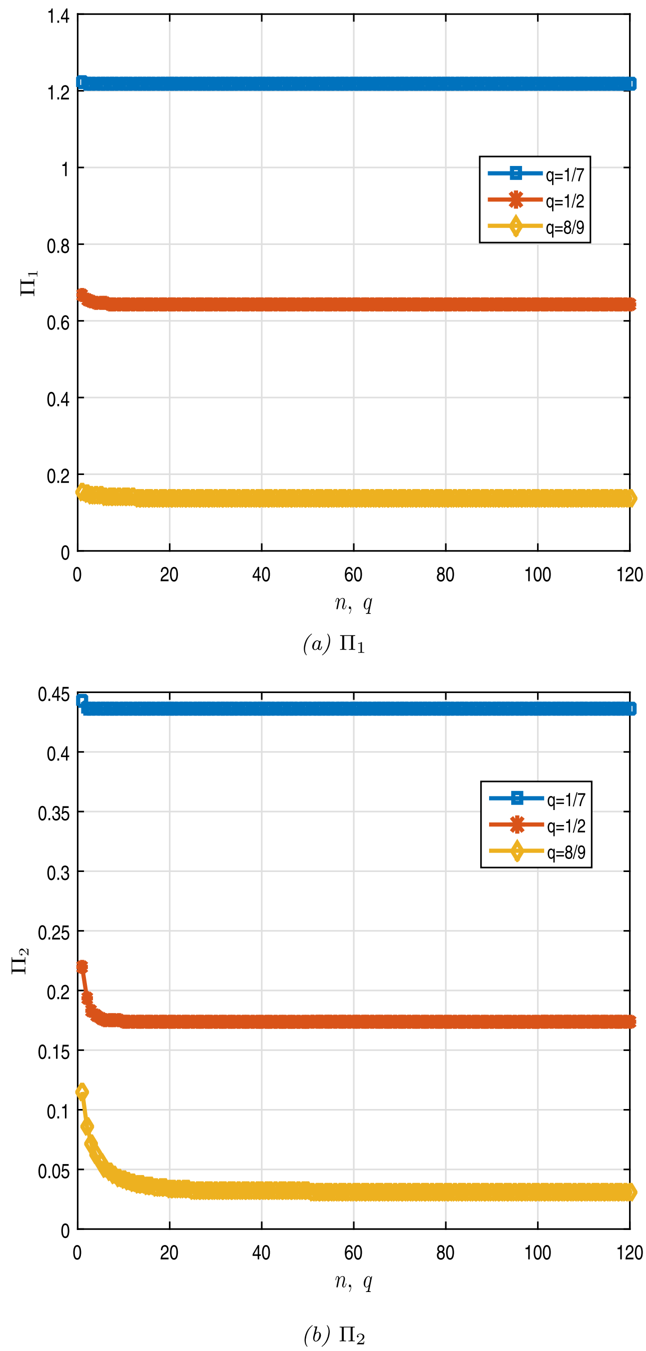

Figure 2 Graphical representation of Π_i for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$ in Example 5.1

Thus condition (16) is satisfied with $\phi(\mathfrak{s}) = \mathfrak{s}^2$ and

$$\rho_\phi = 0.88636, 0.64374, 0.46437$$

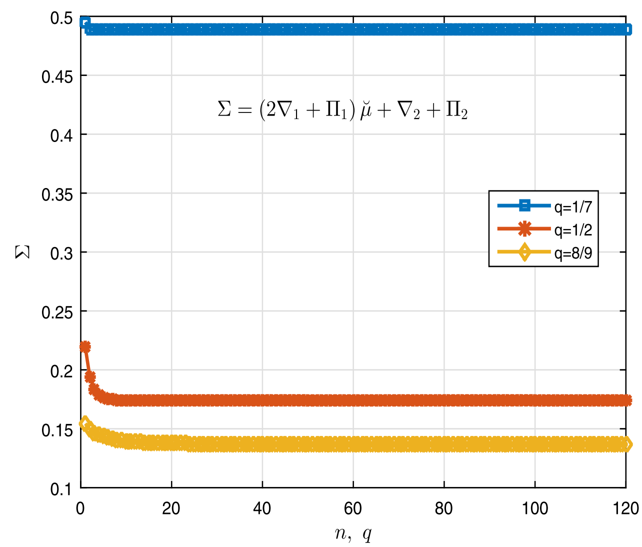

Figure 3 Graphical representation of $\Sigma = (2\nabla_1 + \Pi_1)\breve{\mu} + \nabla_2 + \Pi_2$ in Example 5.1

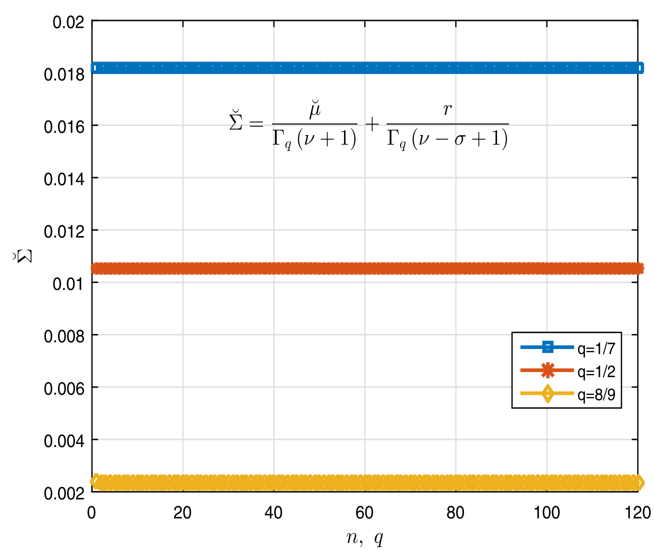

Figure 4 Graphical representation of $\breve{\Sigma}$ in Example 5.1

for $q \in \{\frac{1}{7}, \frac{1}{2}, \frac{8}{9}\}$, respectively. Table 5 shows these results. Also, we can see a graphical representation of

$$\int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - q\grave{\imath})^{(\nu-1)}}{\Gamma_q(\nu)} \phi(\grave{\imath})\, \mathrm{d}_q\grave{\imath}$$

for $\mathfrak{s} \in \Omega$ with step 0.1 in Fig. 5. From Theorem 4.2 it follows that problem (17) is UHRS such that

$$\|\acute{w} - w\|_{\mathcal{W}} \leq \omega_{\varphi,\phi}\mathring{\eta}\phi(\mathfrak{s}), \quad \mathfrak{s} \in \Omega.$$

Table 4 Numerical results of $\check{\Sigma}$ for $q = \frac{1}{7}$ in Example 5.1

n	$\Gamma_q(\nu+1)$	$\Gamma_q(\nu-\sigma+1)$	$\check{\Sigma}$	ω_φ
	$q=\frac{1}{7}$			
1	1.14283	1.18028	0.01818	0.89123
2	1.14027	1.18063	0.01818	0.89323
3	1.13990	1.18068	0.01818	0.89351
4	1.13985	1.18069	0.01818	0.89355
5	1.13984	1.18069	0.01818	0.89356
6	1.13984	1.18069	0.01818	0.89356
	$q=\frac{1}{2}$			
1	2.10842	2.02631	0.01054	0.47934
2	1.99300	2.03657	0.01053	0.50710
3	1.94055	2.04137	0.01053	0.52080
4	1.91550	2.04369	0.01052	0.52761
⋮	⋮	⋮	⋮	⋮
13	1.89123	2.04597	0.01052	0.53438
14	1.89121	2.04597	0.01052	0.53439
15	1.89120	2.04597	0.01052	0.53439
	$q=\frac{8}{9}$			
1	21.16106	8.65603	0.00238	0.04737
2	17.64177	8.77836	0.00236	0.05682
3	15.46858	8.86537	0.00235	0.06480
4	13.99343	8.93115	0.00234	0.07163
5	12.92932	8.98279	0.00234	0.07752
6	12.12863	9.02441	0.00233	0.08264
⋮	⋮	⋮	⋮	⋮
53	8.32612	9.26821	0.00233	0.12038
54	8.32504	9.26830	0.00233	0.12040
55	8.32407	9.26837	0.00233	0.12041
56	8.32322	9.26844	0.00233	0.12043
57	8.32246	9.26850	0.00233	0.12044
58	8.32178	9.26855	0.00233	0.12045
59	8.32118	9.26860	0.00233	0.12046
60	8.32065	9.26864	0.00233	0.12046

Table 5 Numerical results of $\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-q\grave{\imath})^{(\nu-1)}}{\Gamma_q(\nu)} \phi(\grave{\imath})\, \mathrm{d}_q\grave{\imath}$ for $q \in \{\frac{1}{7}, \frac{1}{2}, \frac{8}{9}\}$ in Example 5.1

$\mathfrak{s}$	$q=\frac{1}{7}$	$q=\frac{1}{2}$	$q=\frac{8}{9}$
0.00	0.00000	0.00000	0.00000
0.10	0.00141	0.00102	0.00074
0.20	0.00978	0.00711	0.00513
0.30	0.03045	0.02211	0.01595
0.40	0.06814	0.04949	0.03570
0.50	0.12727	0.09243	0.06668
0.60	0.21205	0.15401	0.11109
0.70	0.32650	0.23713	0.17106
0.80	0.47453	0.34464	0.24861
0.90	0.65992	0.47928	0.34574
1.00	0.88636	0.64374	0.46437

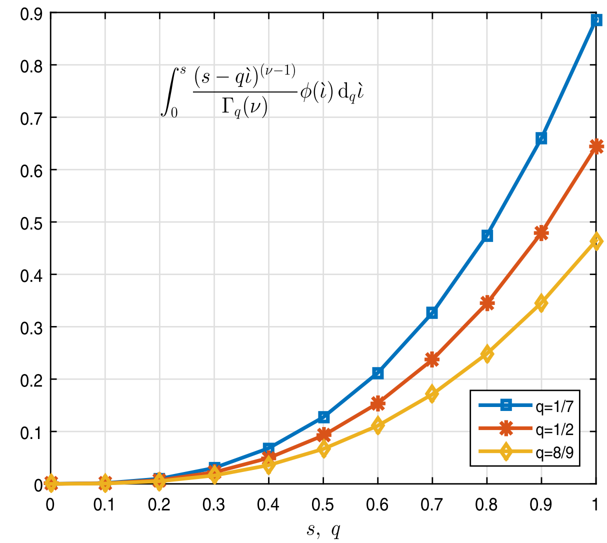

Figure 5 Graphical representation of $\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-q\grave{\imath})^{(\nu-1)}}{\Gamma_q(\nu)}\phi(\grave{\imath})\,\mathrm{d}_q\grave{\imath}$ for $\mathfrak{s} \in [0, 1]$ in Example 5.1

6 Conclusion

In this research work, we have discussed the uniqueness and Ulam-type stability of solutions of sequential FPqDEs. We have established the uniqueness by applying Banach's contraction mapping principle. Furthermore, studied the stability in the sense of UHS and UHRS. We have also provided an example to illustrate our results.

Appendix

Algorithm 1 (MATLAB lines for calculation ∇_i, Π_i, and Σ, $\breve{\Sigma}$ in Example 5.1)

```
clear;
format long;
syms t;
q=[1/7 1/2 8/9];
[xq yq]=size(q);
nu=7/4; sigma =4/5; r=1/50; uptheta=5/6; T = 1;
lambda_1=1/15; lambda_2=6/17; Lambda =sqrt(7)/8;
eta =3/4;
mu=1/(15^2*pi);
k=120;
t0 = 0;
column=1;
for s=1:yq
    for n=1:k
        paramsmatrix(n, column)=n;
        Gammanu=qGamma(q(s),nu+1,n);
        paramsmatrix(n, column+1)=Gammanu;
        nabla_1=1/Gammanu*(T^nu+T^(nu-sigma)...
        /abs(lambda_1*T^(nu-sigma)-lambda_2*eta^(nu-sigma))...
        *(abs(lambda_2)*eta^nu+ abs(lambda_1)*T^nu));
        paramsmatrix(n, column+2)=nabla_1;
        Gammanu_sigma=qGamma(q(s),nu-sigma+1,n);
        paramsmatrix(n, column+3)=Gammanu_sigma;
        nabla_2=r/Gammanu_sigma*(T^(nu-sigma)...
        +T^(nu-sigma)/abs(lambda_1*T^(nu-sigma)...
        -lambda_2*eta^(nu-sigma))...
        *(abs(lambda_2)*eta^(nu-sigma)+abs(lambda_1)*T^(nu-sigma)));
        paramsmatrix(n, column+4)=nabla_2;
        Gammanu_2sigma=qGamma(q(s),nu-2*sigma+1,n);
        paramsmatrix(n, column+5)=Gammanu_2sigma;
        Pi_1=T^(nu-sigma)/Gammanu_sigma+Gammanu_sigma*T^(nu-2*sigma)...
        /(abs(lambda_1*T^(nu-sigma)-lambda_2*eta^(nu-sigma))...
```

```
                        *Gammanu_2sigma)*(abs(lambda_2)*eta^nu/Gammanu...
                        +abs(lambda_1)*T^nu/Gammanu);
                        paramsmatrix(n, column+6)=Pi_1;
                        Pi_2=r*T^(nu-2*sigma)/Gammanu_2sigma+Gammanu_sigma...
                        *T^(nu-2*sigma)/(abs(lambda_1*T^(nu-sigma)...
                        -lambda_2*eta^(nu-sigma))*Gammanu_2sigma)...
                        *(abs(lambda_2)*eta^(nu-sigma)/Gammanu_sigma...
                        +abs(lambda_1)*T^(nu-sigma)/Gammanu_sigma);
                        paramsmatrix(n, column+7)=Pi_2;
                        paramsmatrix(n, column+8)=(2*nabla_1+Pi_1)*mu+nabla_2+Pi_2;
                        paramsmatrix(n, column+9)=mu/Gammanu+r/Gammanu_sigma;
            end;
            column=column +10;
    end;
    t0 = 0;
    column=1;
    for s=1:yq
            row=1;
            t=t0;
            while t<T
                    MR(row,column)=t;
                    MR(row,column+1)=qintegral(q(s),sigma,t,k,power(t,2));
                    t=t+0.1;
                    row=row+1;
            end;
            column=column+2;
    end;
```

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Declarations

Ethics approval and consent to participate
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Competing interests
The authors declare that they have no competing interests.

Author contributions
MH: Actualization, methodology, formal analysis, validation, investigation, and initial draft. FM: Actualization, validation, methodology, formal analysis, investigation, and initial draft. MES: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft; he was the major contributor in writing the manuscript. MKAK: Actualization, methodology, formal analysis, validation, investigation, initial draft, supervision of the original draft, and editing. All authors read and approved the final manuscript.

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