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Blow-up time estimate for porous-medium problems with gradient terms under Robin boundary conditions

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Abstract

This paper deals with the blow-up phenomena connected to the following porous-medium problem with gradient terms under Robin boundary conditions:

$$\begin{cases} u_t = \Delta u^m + k_1 u^p - k_2 |\nabla u|^q & \text{in } \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} + \gamma u = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded and convex domain with smooth boundary $\partial\Omega$. The constants p, q, m are positive, and $p > q > m > 1$, $q > 2$. By making use of the Sobolev inequality and the differential inequality technique, we obtain a lower bound for the blow-up time of the solution. In addition, an example is given as an application of the abstract results obtained in this paper. Our results can be regarded as an answer to the open question raised by Li et al. in (Z. Angew. Math. Phys. 70:1–18, 2019).

MSC: 35K59; 35K20; 35B44

Keywords: Porous-medium problem; Blow-up; Gradient terms; Lower bound

1 Introduction

In recent years, blow-up phenomena for parabolic problems have been actively studied by many researchers. Such an issue is quite important, since it is often used to describe many physical and biological phenomena; we refer, for instance, to [2–9]. Most of these papers deal with the existence, blow-up, and other qualitative properties of solutions to related problems. In many situations, the techniques used in the study of blow-up phenomena often lead to the upper bound for the blow-up time when blow-up occurs. However, in applications, the lower bound seems to be more important, due to the explosive nature of the solution (see [10–14]). In this paper, we investigate the blow-up phenomena connected

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to the following porous-medium problem under Robin boundary conditions:

$$\begin{cases} u_t = \Delta u^m + k_1 u^p - k_2 |\nabla u|^q & \text{in } \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} + \gamma u = 0 & \text{on } \partial \Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded and convex domain with smooth boundary $\partial \Omega$, $\frac{\partial u}{\partial \nu}$ is the normal derivative of u, t^* is the maximal existence time of u, and $\overline{\Omega}$ is the closure of Ω. Additionally, we set $\mathbb{R}_+ = (0, +\infty)$ and assume that k_1, k_2 are nonnegative constants, u_0 is a nonnegative $C^1(\overline{\Omega})$ function satisfying compatibility conditions.

There is a vast literature on the work of the blow-up phenomena for parabolic equations with the gradient terms. We mainly concentrate on the following papers [1, 15–17]. Marras, Vernier Piro and Viglialoro in [16] considered the following problem

$$\begin{cases} u_t = \Delta u + k_1(t)u^p - k_2(t)|\nabla u|^q & \text{in } \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} + \gamma u = 0 & \text{on } \partial \Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) whose boundary is sufficiently smooth. When $\Omega \subset \mathbb{R}^3$, the authors derived a lower bound for the blow-up time under suitable assumptions. Li, Pintus and Viglialoro [1] investigated the following problem

$$\begin{cases} u_t = \Delta u^m + k_1 u^p - k_2 |\nabla u|^q & \text{in } \Omega \times (0, t^*), \\ u = 0 & \text{on } \partial \Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded and smooth domain. When $\Omega \subset \mathbb{R}^3$, the authors obtained a lower bound for the blow-up time by using the differential inequality technique.

Furthermore, we note that an open problem in [1] is how to give the lower bound of the blow-up time when the Dirichlet boundary conditions in (3) are replaced by Robin ones. Under this situation and inspired by the aforementioned works, we studied the blow-up phenomena of problem (1). The innovation of this paper is to establish suitable auxiliary functions. Since auxiliary functions defined in problems (2) and (3) are not suitable for our study, we need to construct new auxiliary functions. Moreover, instead of considering the lower bound for the blow-up time of the problem in $\Omega \subset \mathbb{R}^3$, we study the more general case when $\Omega \subset \mathbb{R}^n$ ($n \geq 3$). Combining the Sobolev inequality and a differential inequality technique, we give a lower bound for the blow-up time.

The rest of this paper is organized as follows. In Sect. 2, we derive a lower bound for the blow-up time when blow-up occurs. In Sect. 3, an example is given to illustrate the application of the results obtained in this paper.

2 Lower bound for blow-up time

In this section, we aim to obtain a lower bound for the blow-up time t^* that gives a safe interval of the existence of the solution. To achieve this, we define $L_0 = \min_{\partial \Omega}(x \cdot \nu)$, $d = \max_\Omega |x|$, where x is a vector relative to the origin 0 and ν is a unit normal vector directed

outward on $\partial\Omega$. Suppose that k_1, k_2 are nonnegative constants and m, p, q are positive constants with

$$q > p > m > 1, \quad q > 2. \tag{4}$$

We now define the following auxiliary function

$$\Psi(t) = \int_\Omega u^\alpha \, dx$$

with

$$\alpha > \max\{1, 3 - m\}. \tag{5}$$

Let λ_1 be the first eigenvalue of the plastic-membrane problem

$$\begin{cases} \Delta\omega + \lambda\omega = 0, & x \in \Omega, \\ \frac{\partial\omega}{\partial\nu} + \gamma\omega = 0, & x \in \partial\Omega \end{cases}$$

with

$$\lambda_1 > \frac{\gamma(n+d)(\alpha+q-1)}{2L_0}. \tag{6}$$

It follows from [18] that

$$W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega), \quad n \geq 3.$$

Hence, we have

$$\int_\Omega \left(u^{\frac{\alpha+m-1}{2}}\right)^{\frac{2n}{n-2}} dx \leq C_s^{\frac{2n}{n-2}} \left(\int_\Omega u^{\alpha+m-1} \, dx + \int_\Omega \left|\nabla u^{\frac{\alpha+m-1}{2}}\right|^2 dx\right)^{\frac{n}{n-2}}, \tag{7}$$

where $Cs = Cs(n, \Omega)$ is a Sobolev embedding constant depending on n and Ω. In this section, we need to use the Sobolev inequality (7). Now, we state our main result as follows.

Theorem 2.1 *Let u be a nonnegative classical solution of* (1). *Assume conditions* (4)–(6) *hold. If the solution u becomes unbounded in the measure* $\Psi(t)$ *at some finite time* t^*, *then*

$$t^* \geq \int_{\Psi(0)}^{+\infty} \frac{d\tau}{C_1 \tau^{\frac{\alpha+m-1}{\alpha}}},$$

where

$$C_1 = \frac{4\alpha^2 k_1(q-p)}{(q-m)(2\alpha+n(m-1))} \varepsilon_1^{-\frac{p-m}{q-p}} C_s^{\frac{n(m-1)}{\alpha}}$$

$$\times \left[\left(\frac{2\alpha+n(m-1)}{2n(m-1)}\right)^{\frac{-n(m-1)}{2\alpha}} + \varepsilon_2^{\frac{n(m-1)}{2\alpha}}\right], \tag{8}$$

$$\varepsilon_1 = \frac{k_2(q-m)(2q)^q}{k_1(p-m)M^2(\alpha+q-1)^q}, \tag{9}$$

$$\varepsilon_2 = \frac{2m(\alpha-1)(q-m)[2\alpha+n(m-1)]\varepsilon_1^{\frac{p-m}{q-p}}}{k_1 n(m-1)(q-p)(\alpha+m-1)^2}, \tag{10}$$

$$M = \frac{2L_0\lambda_1 - \gamma(n+d)(\alpha+q-1)}{2L_0 + \gamma d(\alpha+q-1)}. \tag{11}$$

Proof Using the divergence theorem and assumption (5), we derive

$$\Psi'(t)$$

$$= \alpha \int_\Omega u^{\alpha-1} u_t \, dx = \alpha \int_\Omega u^{\alpha-1} \left(\Delta u^m + k_1 u^p - k_2 |\nabla u|^q \right) dx$$

$$= \alpha \int_\Omega u^{\alpha-1} \Delta u^m \, dx + \alpha k_1 \int_\Omega u^{\alpha+p-1} \, dx - \alpha k_2 \int_\Omega u^{\alpha-1} |\nabla u|^q \, dx$$

$$= -\alpha \int_\Omega \left[\nabla(u^{\alpha-1}) \cdot \nabla(u^m) \right] dx + m\alpha \int_{\partial\Omega} u^{\alpha+m-2} \frac{\partial u}{\partial \nu} \, dS + \alpha k_1 \int_\Omega u^{\alpha+p-1} \, dx$$

$$\quad - \alpha k_2 \int_\Omega u^{\alpha-1} |\nabla u|^q \, dx$$

$$= -m\alpha(\alpha-1) \int_\Omega u^{\alpha+m-3} |\nabla u|^2 \, dx - m\alpha\gamma \int_{\partial\Omega} u^{\alpha+m-1} \, dS + \alpha k_1 \int_\Omega u^{\alpha+p-1} \, dx$$

$$\quad - \alpha k_2 \int_\Omega u^{\alpha-1} |\nabla u|^q \, dx$$

$$\leq -m\alpha(\alpha-1) \int_\Omega u^{\alpha+m-3} |\nabla u|^2 \, dx - \alpha k_2 \int_\Omega u^{\alpha-1} |\nabla u|^q \, dx$$

$$\quad + \alpha k_1 \int_\Omega u^{\alpha+p-1} \, dx. \tag{12}$$

First, we deal with the second term of (12). For this purpose, we consider the following fact that

$$\left(\frac{q}{2} \right)^2 u^{\frac{(q-2)(\alpha+q-1)}{q}} \left| \nabla u^{\frac{\alpha+q-1}{q}} \right|^2 = \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2.$$

Using the Hölder inequality, we have

$$\int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2 dx = \left(\frac{q}{2} \right)^2 \int_\Omega u^{\frac{(q-2)(\alpha+q-1)}{q}} \left| \nabla u^{\frac{\alpha+q-1}{q}} \right|^2 dx$$

$$= \left(\frac{q}{2} \right)^2 \left(\int_\Omega u^{\alpha+q-1} \, dx \right)^{\frac{q-2}{q}} \left(\int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{q}} \right|^q dx \right)^{\frac{2}{q}}. \tag{13}$$

Applying the general Poincaré inequality [19] to the term $\int_\Omega u^{\alpha+q-1} \, dx$ in (13), we derive

$$\lambda_1 \int_\Omega u^{\alpha+q-1} \, dx = \lambda_1 \int_\Omega \left(u^{\frac{\alpha+q-1}{2}} \right)^2 dx$$

$$\leq \int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2 dx - \int_{\partial\Omega} u^{\frac{\alpha+q-1}{2}} \frac{\partial(u^{\frac{\alpha+q-1}{2}})}{\partial\nu} \, dS$$

$$= \int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2 dx + \frac{\gamma(\alpha+q-1)}{2} \int_{\partial\Omega} u^{\alpha+q-1} \, dS. \tag{14}$$

Combining the Lemma in [20], the Hölder inequality and the Young inequality, we obtain

$$\int_{\partial\Omega} u^{\alpha+q-1} \, dS$$

$$\leq \frac{n}{L_0} \int_\Omega u^{\alpha+q-1} \, dx + \frac{(\alpha+q-1)d}{L_0} \int_\Omega u^{\alpha+q-2} |\nabla u| \, dx$$

$$\leq \frac{n}{L_0} \int_\Omega u^{\alpha+q-1} \, dx + \frac{2d}{L_0} \left(\int_\Omega u^{\alpha+q-1} \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \frac{n+d}{L_0} \int_\Omega u^{\alpha+q-1} \, dx + \frac{d}{L_0} \int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2 dx. \tag{15}$$

We insert (15) into (14) to deduce

$$M \int_\Omega u^{\alpha+q-1} \, dx \leq \int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2 dx, \tag{16}$$

where $M > 0$ is given in (11) and assumption (6) guarantees $M > 0$. From (16) and (13), we derive

$$M \int_\Omega u^{\alpha+q-1} \, dx \leq \int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{2}} \right|^2 dx$$

$$= \left(\frac{q}{2} \right)^2 \left(\int_\Omega u^{\alpha+q-1} \, dx \right)^{\frac{q-2}{q}} \left(\int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{q}} \right|^q dx \right)^{\frac{2}{q}},$$

which, is equivalent to

$$M^{\frac{q}{2}} \left(\frac{2}{q} \right)^q \int_\Omega u^{\alpha+q-1} \, dx \leq \int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{q}} \right|^q dx. \tag{17}$$

Hence, multiplying both sides of the inequality (17) by $(\frac{q}{\alpha+q-1})^q$, we obtain the relationship of the second term in (12)

$$M^{\frac{q}{2}} \left(\frac{2}{q} \right)^q \left(\frac{q}{\alpha+q-1} \right)^q \int_\Omega u^{\alpha+q-1} \, dx$$

$$\leq \left(\frac{q}{\alpha+q-1} \right)^q \int_\Omega \left| \nabla u^{\frac{\alpha+q-1}{q}} \right|^q dx$$

$$= \int_\Omega u^{\alpha-1} |\nabla u|^q \, dx. \tag{18}$$

We substitute (18) into (12) to obtain

$$\Psi'(t) \leq -m\alpha(\alpha-1) \int_\Omega u^{\alpha+m-3} |\nabla u|^2 \, dx$$

$$- \alpha k_2 M^{\frac{q}{2}} \left(\frac{2}{\alpha+q-1} \right)^q \int_\Omega u^{\alpha+q-1} \, dx + \alpha k_1 \int_\Omega u^{\alpha+p-1} \, dx$$

$$= -\frac{4m\alpha(\alpha-1)}{(\alpha+m-1)^2} \int_\Omega \left| \nabla u^{\frac{\alpha+m-1}{2}} \right|^2 dx$$

$$-\alpha k_2 M^{\frac{q}{2}}\left(\frac{2}{\alpha+q-1}\right)^q \int_\Omega u^{\alpha+q-1}\,dx + \alpha k_1 \int_\Omega u^{\alpha+p-1}\,dx. \tag{19}$$

Secondly, we pay attention to the third term of (19). It follows from the Hölder inequality and the Young inequality that

$$
\begin{aligned}
\int_\Omega u^{\alpha+p-1}\,dx &\le \left(\int_\Omega u^{\alpha+q-1}\,dx\right)^{\frac{p-m}{q-m}}\left(\int_\Omega u^{\alpha+m-1}\,dx\right)^{\frac{q-p}{q-m}}\\
&\le \left(\varepsilon_1\int_\Omega u^{\alpha+q-1}\,dx\right)^{\frac{p-m}{q-m}}\left(\varepsilon_1^{-\frac{p-m}{q-p}}\int_\Omega u^{\alpha+m-1}dx\right)^{\frac{q-p}{q-m}}\\
&= \frac{p-m}{q-m}\varepsilon_1\int_\Omega u^{\alpha+q-1}\,dx + \frac{q-p}{q-m}\varepsilon_1^{-\frac{p-m}{q-p}}\int_\Omega u^{\alpha+m-1}\,dx,
\end{aligned} \tag{20}
$$

where $0 < \frac{p-m}{q-m} < 1$ due to (4). The substitution of (20) into (19) leads to

$$
\begin{aligned}
\Psi'(t) &\\
\le &-\frac{4m\alpha(\alpha-1)}{(\alpha+m-1)^2}\int_\Omega\left|\nabla u^{\frac{\alpha+m-1}{2}}\right|^2\,dx - \alpha k_2 M^{\frac{q}{2}}\left(\frac{2}{\alpha+q-1}\right)^q\int_\Omega u^{\alpha+q-1}\,dx\\
&+ \alpha k_1\left(\frac{p-m}{q-m}\varepsilon_1\int_\Omega u^{\alpha+q-1}\,dx + \frac{q-p}{q-m}\varepsilon_1^{-\frac{p-m}{q-p}}\int_\Omega u^{\alpha+m-1}\,dx\right)\\
= &-\frac{4m\alpha(\alpha-1)}{(\alpha+m-1)^2}\int_\Omega\left|\nabla u^{\frac{\alpha+m-1}{2}}\right|^2\,dx + \alpha k_1\frac{q-p}{q-m}\varepsilon_1^{-\frac{p-m}{q-p}}\int_\Omega u^{\alpha+m-1}\,dx,
\end{aligned} \tag{21}
$$

where ε_1 is defined in (9). For the term $\int_\Omega u^{\alpha+m-1}\,dx$ of (21), using the Hölder inequality and the Sobolev inequality (7), we have

$$
\begin{aligned}
&\int_\Omega u^{\alpha+m-1}\,dx\\
&\le \left(\int_\Omega u^\alpha\,dx\right)^{\frac{2\alpha+2(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega\left(u^{\frac{\alpha+m-1}{2}}\right)^{\frac{2n}{n-2}}\,dx\right)^{\frac{(n-2)(m-1)}{2\alpha+n(m-1)}}\\
&\le \left(\int_\Omega u^\alpha\,dx\right)^{\frac{2\alpha+2(m-1)}{2\alpha+n(m-1)}}\\
&\quad\times\left[C_s^{\frac{2n}{n-2}}\left(\int_\Omega u^{\alpha+m-1}\,dx + \int_\Omega\left|\nabla u^{\frac{\alpha+m-1}{2}}\right|^2\,dx\right)^{\frac{n}{n-2}}\right]^{\frac{(n-2)(m-1)}{2\alpha+n(m-1)}}\\
&= C_s^{\frac{2n(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega u^\alpha\,dx\right)^{\frac{2\alpha+2(m-1)}{2\alpha+n(m-1)}}\\
&\quad\times\left(\int_\Omega u^{\alpha+m-1}\,dx + \int_\Omega\left|\nabla u^{\frac{\alpha+m-1}{2}}\right|^2\,dx\right)^{\frac{n(m-1)}{2\alpha+n(m-1)}}.
\end{aligned} \tag{22}
$$

It is easy to see that $0 < \frac{n(m-1)}{2\alpha+n(m-1)} < 1$. In view of the inequality

$$(a+b)^j \le a^j + b^j, \quad a,b>0, 0\le j<1,$$

we can rewrite (22) as

$$
\int_\Omega u^{\alpha+m-1}\,\mathrm{d}x
$$

$$
\leq C_s^{\frac{2n(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{2\alpha+2(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega u^{\alpha+m-1}\,\mathrm{d}x\right)^{\frac{n(m-1)}{2\alpha+n(m-1)}}
$$

$$
+ C_s^{\frac{2n(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{2\alpha+2(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega \big|\nabla u^{\frac{\alpha+m-1}{2}}\big|^2\,\mathrm{d}x\right)^{\frac{n(m-1)}{2\alpha+n(m-1)}}. \tag{23}
$$

To the first term of the right-hand side of (23), we make use of the Hölder inequality and the Young inequality to derive

$$
C_s^{\frac{2n(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{2\alpha+2(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega u^{\alpha+m-1}\,\mathrm{d}x\right)^{\frac{n(m-1)}{2\alpha+n(m-1)}}
$$

$$
\leq \left[C_s^{\frac{n(m-1)}{\alpha}}\left(\frac{2\alpha+n(m-1)}{2n(m-1)}\right)^{-\frac{n(m-1)}{2\alpha}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{\alpha+m-1}{\alpha}}\right]^{\frac{2\alpha}{2\alpha+n(m-1)}}
$$

$$
\times\left(\frac{2\alpha+n(m-1)}{2n(m-1)}\int_\Omega u^{\alpha+m-1}\,\mathrm{d}x\right)^{\frac{n(m-1)}{2\alpha+n(m-1)}}
$$

$$
\leq \frac{2\alpha}{2\alpha+n(m-1)}C_s^{\frac{n(m-1)}{\alpha}}\left(\frac{2\alpha+n(m-1)}{2n(m-1)}\right)^{-\frac{n(m-1)}{2\alpha}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{\alpha+m-1}{\alpha}}
$$

$$
+\frac{1}{2}\int_\Omega u^{\alpha+m-1}\,\mathrm{d}x. \tag{24}
$$

To the second term of the right-hand side of (23), it follows from the Hölder inequality and the Young inequality that

$$
C_s^{\frac{2n(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{2\alpha+2(m-1)}{2\alpha+n(m-1)}}\left(\int_\Omega \big|\nabla u^{\frac{\alpha+m-1}{2}}\big|^2\,\mathrm{d}x\right)^{\frac{n(m-1)}{2\alpha+n(m-1)}}
$$

$$
\leq \left[C_s^{\frac{n(m-1)}{\alpha}}\varepsilon_2^{-\frac{n(m-1)}{2\alpha}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{\alpha+m-1}{\alpha}}\right]^{\frac{2\alpha}{2\alpha+n(m-1)}}
$$

$$
\times\left(\varepsilon_2\int_\Omega \big|\nabla u^{\frac{\alpha+m-1}{2}}\big|^2\right)^{\frac{n(m-1)}{2\alpha+n(m-1)}}
$$

$$
\leq \frac{2\alpha}{2\alpha+n(m-1)}C_s^{\frac{n(m-1)}{\alpha}}\varepsilon_2^{-\frac{n(m-1)}{2\alpha}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{\alpha+m-1}{\alpha}}
$$

$$
+\frac{n(m-1)}{2\alpha+n(m-1)}\varepsilon_2\int_\Omega \big|\nabla u^{\frac{\alpha+m-1}{2}}\big|^2\,\mathrm{d}x, \tag{25}
$$

where ε_2 is defined in (10). It follows from (23)–(25) that

$$
\int_\Omega u^{\alpha+m-1}\,\mathrm{d}x
$$

$$
\leq \frac{4\alpha}{2\alpha+n(m-1)}C_s^{\frac{n(m-1)}{\alpha}}\left[\left(\frac{2\alpha+n(m-1)}{2n(m-1)}\right)^{-\frac{n(m-1)}{2\alpha}}+\varepsilon_2^{-\frac{n(m-1)}{2\alpha}}\right]
$$

$$
\times\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{\alpha+m-1}{\alpha}}+\frac{2n(m-1)}{2\alpha+n(m-1)}\varepsilon_2\int_\Omega \big|\nabla u^{\frac{\alpha+m-1}{2}}\big|^2\,\mathrm{d}x. \tag{26}
$$

Thirdly, we insert (26) into (22) to obtain

$$
\begin{aligned}
\Psi'(t) \\
\leq -&\frac{4m\alpha(\alpha-1)}{(\alpha+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\alpha+m-1}{2}}\right|^2 dx + \alpha k_1 \frac{q-p}{q-m}\varepsilon_1^{-\frac{p-m}{q-p}}\\
&\times\left\{\frac{4\alpha}{2\alpha+n(m-1)}C_s^{\frac{n(m-1)}{\alpha}}\left[\left(\frac{2\alpha+n(m-1)}{2n(m-1)}\right)^{-\frac{n(m-1)}{2\alpha}}+\varepsilon_2^{\frac{n(m-1)}{2\alpha}}\right]\right.\\
&\times\left.\left(\int_\Omega u^\alpha\, dx\right)^{\frac{\alpha+m-1}{\alpha}}+\frac{2n(m-1)}{2\alpha+n(m-1)}\varepsilon_2\int_\Omega\left|\nabla u^{\frac{\alpha+m-1}{2}}\right|^2 dx\right\}\\
=&\, C_1 \Psi^{\frac{\alpha+m-1}{\alpha}}(t),
\end{aligned}
\tag{27}
$$

where C_1 is defined in (8). Integrating (27) from 0 to t, we deduce

$$
t \geq \int_{\Psi(0)}^{\Psi(t)} \frac{d\tau}{C_1 \tau^{\frac{\alpha+m-1}{\alpha}}}.
$$

Since u blows up at finite time t^* in measure $\Psi(t)$, passing the limits as $t \to t^*$, we obtain

$$
t^* \geq \int_{\Psi(0)}^{+\infty} \frac{d\tau}{C_1 \tau^{\frac{\alpha+m-1}{\alpha}}},
$$

where $\frac{\alpha+m-1}{\alpha} > 1$ in view of (4) and (5).   □

3 Application
In what follows, an example is given to illustrate the application of the abstract results.

Example 3.1 Let u be a nonnegative classical solution of the problem (1)

$$
\begin{cases}
u_t = \Delta u^2 + u^3 - |\nabla u|^4 & \text{in } \Omega \times (0, t^*),\\
\frac{\partial u}{\partial \nu} + \frac{1}{3}u = 0 & \text{on } \partial\Omega \times (0, t^*),\\
u(x,0) = \frac{1}{2} - \frac{1}{36}|x|^2 & \text{in } \overline{\Omega},
\end{cases}
$$

where $\Omega = \{x = (x_1, x_2, x_3) \,|\, |x|^2 = x_1^2 + x_2^2 + x_3^2 < 9\}$ is a ball of $\mathbb{R}^3$. Here, we choose $k_1 = k_2 = 1$, $m = 2$, $p = 3$, $q = 4$, $\alpha = 2$, $\gamma = \frac{1}{3}$, and $u_0(x) = \frac{1}{2} - \frac{1}{36}|x|^2$. By direct computation, we have $\lambda_1 \approx 2.4674$. It is easy to check that (6) holds. Moreover, we compute (9)–(11) to obtain $M \approx 0.4368$, $\varepsilon_1 \approx 68.6981$, $\varepsilon_2 \approx 142.4850$. Using the Theorems 2.1 and 3.2 in [21], we have the embedding constant $C_s \approx 7.5931$. Inserting the above parameters into (8), we have $C_1 \approx 14.6651$.

Since we assume that u becomes unbounded at t^* in measure $\Psi(t)$, it follows from the Theorem 2.1 that u blows up in finite time and

$$
\begin{aligned}
t^* &\geq \int_{\Psi(0)}^{+\infty} \frac{d\tau}{C_1 \tau^{\frac{\alpha+m-1}{\alpha}}}\\
&\approx \int_{14.3391}^{+\infty} \frac{d\tau}{14.6651\tau^{\frac{3}{2}}} \approx 3.5879 \times 10^{-2},
\end{aligned}
$$

where

$$\Psi(0) = \int_\Omega u_0^2 \, dx = \int_\Omega \left(\frac{1}{2} - \frac{1}{36} |x|^2 \right)^2 dx$$

$$= \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^3 \left(\frac{1}{2} - \frac{1}{36} |r|^2 \right)^2 r^2 \sin\varphi \, dr \approx 14.3391.$$

4 Conclusion

In this paper, we make use of the Sobolev inequality (7) to overcome the difficulty mentioned in [1] when finding the lower bound for the blow-up time under Robin boundary conditions. The results of this paper provide an answer to the open question raised in [1].

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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