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The proof of a formula concerning the asymptotic behavior of the reciprocal sum of the square of multiple-angle Fibonacci numbers

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Abstract

Let $(F_n)_n$ be the Fibonacci sequence defined by $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$ and $F_1 = 1$. In this paper, we prove that for any integer $m \geq 1$ there exists a positive constant C_m for which

$$\lim_{n\to\infty}\left\{\left(\sum_{k=n}^{\infty}\frac{1}{F_{mk}^2}\right)^{-1} - \left(F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn}C_m\right)\right\} = 0.$$

Furthermore, we show that C_m tends to 2/5 as $m \to \infty$ (indeed, we provide quantitative versions of the previous results as well as an explicit form for C_m). This confirms some questions proposed by Lee and Park [J. Inequal. Appl. 2020(1):91 2020].

MSC: 11B39; 11B05

Keywords: Fibonacci; Series; Upper bounds; Inequalities; Asymptotic; Recurrence sequences

1 Introduction

It is well known that if a series $\sum_{k\geq 1} a_k$ is convergent, then its "tail" $(\sum_{k=n}^{\infty} a_k)_n$ tends to 0 as $n \to \infty$. In particular,

$$\lim_{n\to\infty}\left(\sum_{k=n}^{\infty} a_k\right)^{-1} = \infty.$$

In the past years, many mathematicians have been interested in studying the properties and forms of the reciprocal tails (as above) of the convergent series, where a_n is related to some recurrence sequences. Here, we restrict ourselves only to cases in which a_n is related to the *Fibonacci sequence* $(F_n)_{n\geq 0}$ which is defined by the binary recurrence

$$F_{n+2} = F_{n+1} + F_n,$$

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with initial values, $F_0 = 0$ and $F_1 = 1$. In 2008, Ohtsuka and Nakamura [11] studied the partial infinite sums of reciprocal Fibonacci numbers and showed that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even}, n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd}, n \geq 1; \end{cases}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ is even}, n \geq 2; \\ F_{n-1}F_n, & \text{if } n \text{ is odd}, n \geq 1. \end{cases} \tag{1}$$

In the same year, Choi and Choo [2] provided formulas related to the sums of reciprocals of the products of Fibonacci and Lucas numbers, namely

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}} \right)^{-1} \right\rfloor \quad \text{and} \quad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k F_{k+m}} \right)^{-1} \right\rfloor$$

(recall that the Lucas sequence $(L_n)_n$ satisfies the same recurrence as Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$). For more facts in this topic, we recommend to the reader the papers [1, 3–6, 10, 12, 13].

To study the analytic behavior of these sequences, one introduces another (more qualitative) definition. We then say that $f_n \sim g_n$ if $f_n - g_n$ tends to 0 as $n \to \infty$. Very recently, motivated by this definition, Lee and Park [8, 9] proved, among other things, that

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \sim F_{n-2}, \qquad \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \sim F_n^2 - F_{n-1}^2 + \frac{2}{3}(-1)^n$$

and

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_{3k}^2} \right)^{-1} \sim F_{3n}^2 - F_{3n-3}^2 + \frac{4}{9}(-1)^n.$$

Moreover, as formula (5.1) of [8, Sect. 5], they stated (without proof) the following expected general formula:

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} \sim F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn} C_m, \tag{2}$$

which should hold for any positive integer m, where C_m is a positive constant. Additionally, they remarked that "it looks not easy to find the explicit values of C_m satisfying (2) except for $m = 1$ and 3 (for which $C_1 = 2/3$ and $C_3 = 4/9$). By using computer software programs (MAPLE 17 and wolframalpha.com), they estimated this constant for $m \in [1, 9]$. These computations suggest (as written by them): "We might expect that C_m tends to 2/5 as $m \to \infty$".

The aim of this paper is to confirm the expectation by proving these facts (indeed, we provide quantitative versions for them as well as a completely explicit formula for C_m). More precisely, we have the following.

Theorem 1 *For any integer $m \geq 1$, there exists a positive constant C_m such that*

$$\left| \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} - \left(F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn} C_m \right) \right| < \frac{9.83}{\alpha^{2m(n-1)}} \tag{3}$$

for all $n \geq \max\{\frac{4}{m}, 2\}$ (where, as usual, $\alpha = (1+\sqrt{5})/2$ denotes the golden ratio). In particular,

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} \sim F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn} C_m. \tag{4}$$

Moreover, for all $m \geq 1$,

$$C_m = \frac{2}{5} \left(1 + \frac{r_m + \beta s_m}{2 - (-1)^m L_{4m}} \right), \tag{5}$$

where $\beta = (1-\sqrt{5})/2$,

$$r_m := \left(1 - (-1)^m\right) L_{2m} + 5\left(1 + (-1)^m\right) F_{m-1} F_m F_{2m},$$

and

$$s_m := \left(1 + (-1)^m\right) F_{2m} (L_{2m} - 2).$$

Furthermore, the estimate

$$|C_m - 2/5| < 1.2/\alpha^{2m} \tag{6}$$

holds for all $m \geq 1$, and so C_m tends to $2/5$ as $m \to \infty$.

Now, we shall present two interesting consequences of the previous result. First, observe that it is immediate, after a standard calculation, that C_m in formula (5) agrees with values $C_1 = 2/3$ and $C_3 = 4/9$ (provided in [8]). Moreover, the first 10 values of C_m for m odd are as follows:

$$\frac{2}{3}, \frac{4}{9}, \frac{50}{123}, \frac{338}{843}, \frac{1156}{2889}, \frac{15{,}842}{39{,}603}, \frac{108{,}578}{271{,}443}, \frac{372{,}100}{930{,}249}, \frac{5{,}100{,}818}{12{,}752{,}043}, \frac{34{,}961{,}522}{87{,}403{,}803},$$

which are rational numbers. However,

$$C_2 = \frac{2}{15}(1-2\beta), \qquad C_4 = \frac{6}{35}(1-2\beta), \qquad C_6 = \frac{8}{45}(1-2\beta),$$
$$C_8 = \frac{42}{235}(1-2\beta), \qquad C_{10} = \frac{22}{123}(1-2\beta), \qquad C_{12} = \frac{144}{161}(1-2\beta),$$

which (since $1 - 2\beta = \sqrt{5}$) is the same list as

$$C_2 = \frac{2\sqrt{5}}{15}, \qquad C_4 = \frac{6\sqrt{5}}{35}, \qquad C_6 = \frac{8\sqrt{5}}{45},$$

$$C_8 = \frac{42\sqrt{5}}{235}, \qquad C_{10} = \frac{22\sqrt{5}}{123}, \qquad C_{12} = \frac{144}{805}\sqrt{5}$$

are irrational numbers.

This suggests that $C_{2m-1} \in \mathbb{Q}_{>0}$ and $C_{2m} \in \sqrt{5} \cdot \mathbb{Q}_{>0}$ for all $m \geq 1$. Indeed, the next result confirms this fact by providing a cleaner formula for C_m depending on the parity of m. More precisely,

Corollary 1 *Let m be a positive integer. We have*

(i) *If m is even, then*

$$C_m = \frac{2(L_{2m} - 2)}{25F_{2m}}\sqrt{5}.$$

(ii) *If m is odd, then*

$$C_m = \frac{2(L_{2m} + 2)}{5L_{2m}}.$$

In particular, C_m is a rational number if and only if m is odd.

The previous explicit formulas for C_m provide better bounds to $C_m - 2/5$ which together with Theorem 1 allows to prove the following.

Corollary 2 *We have that*

(i) *Let m be an even positive integer. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} \right\rfloor = F_{mn}^2 - F_{m(n-1)}^2$$

holds for all $n \geq 3$.

(ii) *Let m be an odd positive integer. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{mn}^2 - F_{m(n-1)}^2, & \text{if } n \text{ is even;} \\ F_{mn}^2 - F_{m(n-1)}^2 - 1, & \text{if } n \text{ is odd,} \end{cases}$$

holds for all $n \geq 3$.

The proofs of these results combine several estimates, properties of Fibonacci and Lucas numbers as well as some facts about the convergence of series. The computations in this work were performed with MATHEMATICA software.

2 Auxiliary results

In this section, we present a few auxiliary facts which will be very useful in all proofs.

Lemma 1 *Let $(F_n)_n$ and $(L_n)_n$ be the Fibonacci and Lucas sequences, respectively, and $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. We have*

(i) (*Binet's formula for* F_n) *formula*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

holds for all $n \geq 1$.

(ii) (*Binet's formula for* L_n) *formula*

$$L_n = \alpha^n + \beta^n$$

holds for all $n \geq 1$.

The next two lemmas follow from the previous one.

Lemma 2 *We have that*

(i) $F_{2n} = F_n L_n$.
(ii) $L_n = F_{n+1} + F_{n-1}$.
(iii) $L_{2n} = 5F_n^2 + 2(-1)^n$.
(iv) $L_{2n} = L_n^2 - 2(-1)^n$.
(v) (*D'Ocagne's identity*) $(-1)^n F_{m-n} = F_m F_{n+1} - F_{m+1} F_n$.

We know that $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{5})$ is a quadratic field extension of $\mathbb{Q}$ with $\mathbb{Q}$-basis $\{1, \beta\}$. The next result asserts the exact coefficients of the $\mathbb{Q}$-linear combinations for powers of β, namely,

Lemma 3 *For any* $n \geq 1$, *one has that*

$$\beta^n = \beta F_n + F_{n-1}.$$

The last ingredients are some known lower and upper bounds for F_n, that is,

Lemma 4 *The inequalities*

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$$

hold for all $n \geq 1$.

We refer the reader to [7] for the proofs of these results as well as for the history, properties, and rich applications of the Fibonacci sequence and some of its generalizations.

With these tools in hand, we are now in a position to prove our results.

3 The proof of Theorem 1

By Lemma 1(i), we have that

$$F_{mk} = \frac{\alpha^{mk}}{\sqrt{5}}\left(1 - \left(\frac{\beta}{\alpha}\right)^{mk}\right),$$

hence

$$\frac{1}{F_{mk}^2} = \frac{5}{\alpha^{2mk}(1-(\beta/\alpha)^{mk})^2}.$$

However, $1/(1-x)^2 = 1+2x+3x^2+\cdots$ holds for $|x|<1$. Thus, since $|\beta/\alpha|<1$ and $\alpha\beta=-1$, we have

$$\begin{aligned}\frac{1}{F_{mk}^2} &= \frac{5}{\alpha^{2mk}}\left(1+2\left(\frac{\beta}{\alpha}\right)^{mk}+3\left(\frac{\beta}{\alpha}\right)^{2mk}+\cdots\right)\\ &= \frac{5}{\alpha^{2mk}}+5\sum_{i=2}^{\infty} i(-1)^{mk}\left(\frac{\beta}{\alpha}\right)^{imk}.\end{aligned}$$

By summing up from $k=n$ to infinity and after a straightforward calculation, we arrive at

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{F_{mk}^2} &= \frac{5}{\alpha^{2m(n-1)}(\alpha^{2m}-1)}+5\sum_{i=2}^{\infty} i\sum_{k=n}^{\infty}(-1)^{mk}\left(\frac{\beta}{\alpha}\right)^{imk}\\ &= \frac{5}{\alpha^{2m(n-1)}(\alpha^{2m}-1)}+5\sum_{i=2}^{\infty} i(-1)^{mn}\left(\frac{\beta}{\alpha}\right)^{imn}\frac{1}{1-(-1)^m(\beta/\alpha)^{im}}\\ &= \frac{5}{\alpha^{2m(n-1)}(\alpha^{2m}-1)}\bigl(1+f(m,n)\bigr),\end{aligned} \tag{7}$$

where $f(m,n)$ denotes the following summatory

$$f(m,n) := \left(\frac{\alpha^{2m}-1}{\alpha^{2m}}\right)\sum_{i=2}^{\infty} i\left(\frac{\beta}{\alpha}\right)^{mn(i-1)}\frac{1}{1-(-1)^m(\beta/\alpha)^{im}}.$$

Note that, by the reverse triangle inequality (and $\beta/\alpha = -1/\alpha^2$, which follows from $\beta = -1/\alpha$), one has

$$\bigl|1-(-1)^m(\beta/\alpha)^{im}\bigr| \geq 1-\left|\frac{\beta}{\alpha}\right|^{2m} > 1-\frac{1}{\alpha^4} > 0.85.$$

Thus, we deduce the following upper bound for $|f(m,n)|$:

$$\begin{aligned}\bigl|f(m,n)\bigr| &\leq \frac{1}{0.85}\frac{\alpha^{2m}-1}{\alpha^{2m}}\sum_{i=2}^{\infty} i\left(\frac{1}{\alpha^2}\right)^{mn(i-1)}\\ &\leq \frac{1.18}{\alpha^{2mn}}\sum_{i=2}^{\infty} i\left(\frac{1}{\alpha^{2mn}}\right)^{i-2}\\ &= \frac{1.18}{\alpha^{2mn}}\left(2+\frac{3}{\alpha^{2mn}}+\sum_{i=4}^{\infty}\frac{i}{\alpha^{2mn(i-2)}}\right)\\ &= \frac{1.18}{\alpha^{2mn}}\left(2+\frac{3}{\alpha^{2mn}}+\sum_{i=3}^{\infty}\frac{i+1}{\alpha^{mn(2i-2)}}\right)\\ &= \frac{1.18}{\alpha^{2mn}}\left(2+\frac{3}{\alpha^{2mn}}+\sum_{i=3}^{\infty}\left(\frac{i+1}{\alpha^{imn}}\cdot\frac{1}{\alpha^{(i-2)mn}}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1.18}{\alpha^{2mn}}\left(2.065 + \frac{1}{60}\sum_{i=1}^{\infty}\frac{1}{\alpha^{imn}}\right)\\
&\le \frac{1.18}{\alpha^{2mn}}\left(2.065 + \frac{0.171}{60}\right)\\
&< \frac{2.45}{\alpha^{2mn}},
\end{aligned}
$$

where we used that

$$\alpha^{imn} > 1.6^{imn} \ge 6.55^{i} > 60(i+1)$$

for all $i \ge 3$ (since $mn \ge 4$) together with

$$\frac{1}{\alpha^{mn}} + \frac{1}{\alpha^{2mn}} + \frac{1}{\alpha^{3mn}} + \cdots < \frac{1}{\alpha^{4}} + \frac{1}{\alpha^{8}} + \frac{1}{\alpha^{12}} + \cdots = \frac{1}{\alpha^{4}-1} < 0.171,$$

because $\alpha^{mn} \ge \alpha^4$. Thus $f(m,n)$ tends to 0 as $\min\{m,n\} \to \infty$. Furthermore, $|f(m,n)| < 0.06$ for all integers m and n with $mn \ge 4$.

Turning back to (7), we have that

$$\sum_{k=n}^{\infty}\frac{1}{F_{mk}^2} = \frac{5}{\alpha^{2m(n-1)}(\alpha^{2m}-1)}\cdot\bigl(1+f(m,n)\bigr),$$

and hence

$$\left(\sum_{k=n}^{\infty}\frac{1}{F_{mk}^2}\right)^{-1} = \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}\left(\frac{1}{1+f(m,n)}\right). \tag{8}$$

Since $|f(m,n)| < 1$, for $mn \ge 4$, then

$$\frac{1}{1+f(m,n)} = 1 - f(m,n) + \bigl(f(m,n)\bigr)^2 - \cdots = 1 - f(m,n) + R_{m,n},$$

where $R_{m,n} := (f(m,n))^2 - (f(m,n))^3 + (f(m,n))^4 - \cdots$. For our purposes, we need to find an upper bound for $|R_{m,n}|$. For that, one has

$$
\begin{aligned}
|R_{m,n}| &\le \left|\bigl(f(m,n)\bigr)^2\bigl(1 - f(m,n) + \bigl(f(m,n)\bigr)^2 - \cdots\bigr)\right|\\
&= \left|\frac{(f(m,n))^2}{1+f(m,n)}\right|\\
&\le \frac{(f(m,n))^2}{1-|f(m,n)|}\\
&\le 1.07\bigl(f(m,n)\bigr)^2 < \frac{6.4}{\alpha^{4mn}},
\end{aligned} \tag{9}
$$

where we used that

$$\bigl|1+f(m,n)\bigr| \ge 1 - \bigl|f(m,n)\bigr| > 1 - \frac{2.37}{\alpha^{2mn}} > 1 - 0.06 = 0.94.$$

Turning back to (8), we have

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{F_{mk}^2}\right)^{-1} &= \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}\big(1-f(m,n)+R_{m,n}\big)\\ &= \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5} - \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}f(m,n)\\ &\quad + \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}R_{m,n}\\ &= \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5} - \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}f(m,n)\\ &\quad +T_{m,n},\end{aligned}$$

where

$$T_{m,n} := \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}\cdot R_{m,n}$$

satisfies (by (9))

$$|T_{m,n}| = \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)|R_{m,n}|}{5} < \frac{\alpha^{2mn}}{5}\cdot\frac{6.4}{\alpha^{4mn}} < \frac{1.3}{\alpha^{2mn}}. \tag{10}$$

Hence

$$\left(\sum_{k=n}^{\infty}\frac{1}{F_{mk}^2}\right)^{-1} = \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5} - \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}f(m,n) + T_{m,n}. \tag{11}$$

Now, let us work with the second term of the right-hand side of (11). By the definition of $f(m,n)$, we can write

$$\begin{aligned}&\frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5}f(m,n)\\ &\quad = \frac{2\alpha^{2mn}(\alpha^{2m}-1)^2}{5\alpha^{4m}}\left(\frac{\beta}{\alpha}\right)^{mn}\frac{1}{1-(-1)^m(\beta/\alpha)^{2m}} + D_{m,n},\end{aligned}$$

where

$$D_{m,n} := \frac{\alpha^{2mn}(\alpha^{2m}-1)^2}{5\alpha^{4m}}\sum_{i=3}^{\infty} i\left(\frac{\beta}{\alpha}\right)^{mn(i-1)}\frac{1}{1-(-1)^m(\beta/\alpha)^{im}}.$$

Since $\beta/\alpha = -1/\alpha^2$, we deduce that

$$\begin{aligned}&\frac{2\alpha^{2mn}(\alpha^{2m}-1)^2}{5\alpha^{4m}}\left(\frac{\beta}{\alpha}\right)^{mn}\frac{1}{1-(-1)^m(\beta/\alpha)^{2m}}\\ &\quad = \frac{2\alpha^{2mn}(\alpha^{2m}-1)^2}{5\alpha^{4m}}\left(\frac{-1}{\alpha^2}\right)^{mn}\frac{1}{1-(-1/\alpha^4)^m}\\ &\quad = \frac{2(-1)^{mn}}{5}\cdot\frac{\alpha^{4m}-2\alpha^{2m}+1}{\alpha^{4m}}\left(1+\sum_{i=1}^{\infty}\left(\frac{-1}{\alpha^4}\right)^{mi}\right)\end{aligned}$$

$$= \frac{2(-1)^{mn}}{5}(1 - S_m)(1 + E_m)$$
$$= \frac{2(-1)^{mn}}{5} + (-1)^{mn} G_m,$$

where

$$G_m := \frac{2}{5}(E_m - S_m - E_m S_m) \tag{12}$$

and E_m and S_m are defined as

$$S_m := \frac{2}{\alpha^{2m}} - \frac{1}{\alpha^{4m}} \quad \text{and} \quad E_m := \sum_{i=1}^{\infty}\left(\frac{-1}{\alpha^4}\right)^{mi} = \frac{(-1)^m}{\alpha^{4m} - (-1)^m}. \tag{13}$$

Now, we use that $S_m < 2/\alpha^{2m}$ and

$$|E_m| \leq \sum_{i=1}^{\infty}\left|\frac{-1}{\alpha^4}\right|^{mi} = \frac{1}{\alpha^{4m} - 1} = \frac{1}{\alpha^{4m}}\frac{\alpha^{4m}}{\alpha^{4m} - 1} \leq \frac{1}{\alpha^{4m}}\frac{\alpha^4}{\alpha^4 - 1} < \frac{1.2}{\alpha^{4m}}$$

(since $x \mapsto x/(x-1)$ is a decreasing function for $x > 1$, and so the maximum of $\alpha^{4m}/(\alpha^{4m} - 1)$ is attained at $m = 1$) to infer that

$$|G_m| \leq \frac{2}{5}\big(|E_m| + |S_m| + |E_m S_m|\big) \leq \frac{2}{5}\left(\frac{1.2}{\alpha^{4m}} + \frac{2}{\alpha^{2m}} + \frac{2.4}{\alpha^{6m}}\right) \leq \frac{1.2}{\alpha^{2m}}, \tag{14}$$

which proves (6). Observe that, in particular, G_m tends to 0 as $m \to \infty$.

For the remaining terms, i.e., $D_{m,n}$, first we can realize that

$$\frac{\alpha^{2mn}(\alpha^{2m} - 1)^2}{5\alpha^{4m}} < \frac{\alpha^{2mn}}{5}, \tag{15}$$

since $(\alpha^{2m} - 1)^2 < \alpha^{4m}$. Therefore, by using again $\beta/\alpha = -1/\alpha^2$, we get

$$\begin{aligned}
\left|\sum_{i=3}^{\infty} i\frac{(\beta/\alpha)^{mn(i-1)}}{1 - (-1)^m(\beta/\alpha)^{im}}\right| &= \left|\sum_{i=3}^{\infty} i\left(\frac{-1}{\alpha^2}\right)^{mn(i-1)}\frac{1}{1 - (-1)^m(\beta/\alpha)^{im}}\right| \\
&< 1.32 \cdot \sum_{i=3}^{\infty} i\frac{1}{\alpha^{2mn(i-1)}} \\
&< 1.32 \cdot \sum_{i=3}^{\infty} \frac{1}{\alpha^{2mn(i-1)-2i}},
\end{aligned}$$

where we used that $i < 2^i < \alpha^{2i}$ holds for every integer $i \geq 3$. Furthermore, since $mn \geq 4$, we have for $i \geq 2$

$$a^{2mn(i-1)-2i} \geq \left(\alpha^{2mn(\frac{i-1}{i})-2}\right)^i \geq \left(\alpha^{\frac{2mn}{2}-2}\right)^i \geq \left(\alpha^2\right)^i > 2.6^i.$$

Thus,

$$\left|\sum_{i=3}^{\infty} i\frac{(\beta/\alpha)^{mn(i-1)}}{1 - (-1)^m(\beta/\alpha)^{im}}\right|$$

$$\begin{aligned}
&\le 1.32 \cdot \sum_{i=3}^{\infty} \frac{1}{\alpha^{2mn(i-1)-2i}} = \frac{1.32}{\alpha^{4mn-4}} \sum_{i=3}^{\infty} \frac{1}{\alpha^{2mn(i-3)-2(i-2)}} \\
&= \frac{1.32}{\alpha^{4mn-4}} \sum_{i=1}^{\infty} \frac{1}{\alpha^{2mn(i-1)-2i}} = \frac{1.32}{\alpha^{4mn-4}} \left(\frac{1}{\alpha^{-2}} + \sum_{i=2}^{\infty} \frac{1}{\alpha^{2mn(i-1)-2i}} \right) \\
&< \frac{1.32}{\alpha^{4mn-4}} \left(\frac{1}{\alpha^{-2}} + \sum_{i=2}^{\infty} \frac{1}{2.6^{i}} \right) \\
&< \frac{3.78}{\alpha^{4mn-4}}.
\end{aligned} \tag{16}$$

By combining (15) and (16), we infer that

$$|D_{m,n}| < \frac{0.76}{\alpha^{2mn-4}}. \tag{17}$$

In particular,

$$\lim_{\min\{m,n\}\to\infty} D_{m,n} = 0.$$

Summarizing, we have

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} = \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5} - \frac{2(-1)^{mn}}{5} - (-1)^{mn} G_m + D_{m,n} + T_{m,n}. \tag{18}$$

On the other hand, we can apply Binet's formula again to obtain

$$\begin{aligned}
F_{mn}^2 - F_{m(n-1)}^2 &= \left(\frac{\alpha^{mn} - \beta^{mn}}{\sqrt{5}} \right)^2 + \left(\frac{\alpha^{m(n-1)} - \beta^{m(n-1)}}{\sqrt{5}} \right)^2 \\
&= \frac{\alpha^{2mn} - 2(-1)^{mn} + \beta^{2mn} - \alpha^{2m(n-1)} + 2(-1)^{m(n-1)} - \beta^{2m(n-1)}}{5} \\
&= \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5} - \frac{2(-1)^{mn}}{5}\left(1 + (-1)^{m}\right) + \left(\frac{\beta^{2mn} - \beta^{2m(n-1)}}{5} \right), \\
&= \frac{\alpha^{2m(n-1)}(\alpha^{2m}-1)}{5} - \frac{2(-1)^{mn}}{5} - \frac{2(-1)^{m(n-1)}}{5} - \left(\frac{\alpha^{2m}-1}{5\alpha^{2mn}} \right),
\end{aligned}$$

where we used that $\alpha\beta = -1$. Now, we can combine the previous relation with the formula in (18) to write

$$A_{m,n} = \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} - \left(F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn} C_m \right), \tag{19}$$

where

$$C_m := \frac{2}{5} + (-1)^m G_m \quad \text{and} \quad A_{m,n} := \frac{1-\alpha^{2m}}{5\alpha^{2mn}} + D_{m,n} + T_{m,n}.$$

Now, we use the formula for $A_{m,n}$, estimates (17) and (10) to obtain

$$|A_{m,n}| < \frac{0.2}{\alpha^{2m(n-1)}} + \frac{5.3}{\alpha^{2mn}} + \frac{1.3}{\alpha^{2mn}} = \frac{1}{\alpha^{2m(n-1)}} \left(0.2 + \frac{5.3}{\alpha^{2m}} + \frac{1.3}{\alpha^{2m}} \right) < \frac{9.83}{\alpha^{2m(n-1)}},$$

which implies in (3) (we used that $0.76/\alpha^{2mn-4} = 0.76\alpha^4/\alpha^{2mn}$). Moreover, in particular, $A_{m,n}$ tends to 0 as $n \to \infty$, then (4) holds, i.e.,

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2}\right)^{-1} \sim F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn} C_m.$$

Moreover, C_m is positive, because by (14) one has

$$C_m := \frac{2}{5} + (-1)^m G_m \geq \frac{2}{5} - |G_m| > \frac{2}{5} - \frac{1.2}{\alpha^{2m}} > 0,$$

where we used that $1.2/\alpha^{2m} \leq 1.2/\alpha^4 < 0.18$ for all $m \geq 2$. Additionally, we combine (12) and (13) to obtain

$$\begin{aligned} C_m = \frac{2}{5}\biggl(1 &+ \frac{1}{\alpha^{4m} - (-1)^m} - \frac{2(-1)^m}{\alpha^{2m}} + \frac{(-1)^m}{\alpha^{4m}} \\ &- \frac{2}{\alpha^{2m}(\alpha^{4m} - (-1)^m)} + \frac{1}{\alpha^{4m}(\alpha^{4m} - (-1)^m)}\biggr). \end{aligned} \tag{20}$$

To obtain the formula in (5), we shall rationalize every fraction in (20). First, we observe that $(\alpha\beta)^{im} = 1$ for $i \in \{2, 4\}$ and

$$\left(\alpha^{4m} - (-1)^m\right)\left(\beta^{4m} - (-1)^m\right) = (\alpha\beta)^{4m} - (-1)^m\left(\alpha^{4m} + \beta^{4m}\right) + 1 = 2 - (-1)^m L_{4m}.$$

Now, let us work with the expression

$$\frac{1}{\alpha^{4m} - (-1)^m} - \frac{2(-1)^m}{\alpha^{2m}} + \frac{(-1)^m}{\alpha^{4m}} - \frac{2}{\alpha^{2m}(\alpha^{4m} - (-1)^m)} + \frac{1}{\alpha^{4m}(\alpha^{4m} - (-1)^m)}. \tag{21}$$

By rationalizing, one has

$$\begin{aligned} &\frac{1}{\alpha^{4m} - (-1)^m} = \frac{\beta^{4m} - (-1)^m}{2 - (-1)^m L_m} \quad \text{and} \quad \frac{2(-1)^m}{\alpha^{2m}} = 2(-1)^m \beta^{2m}, \\ &\frac{(-1)^m}{\alpha^{4m}} = (-1)^m \beta^{4m} \quad \text{and} \quad \frac{2}{\alpha^{2m}(\alpha^{4m} - (-1)^m)} = \frac{2\beta^{2m}(\alpha^{4m} - (-1)^m)}{2 - (-1)^m L_{4m}}, \end{aligned}$$

and finally

$$\frac{1}{\alpha^{4m}(\alpha^{4m} - (-1)^m)} = \frac{\beta^{4m}(\beta^{4m} - (-1)^m)}{2 - (-1)^m L_{4m}}.$$

By putting all this information together and after a straightforward computation, we deduce that

$$C_m = \frac{2}{5}\left(1 + \frac{B_m}{(2 - (-1)^m L_{4m})}\right),$$

where

$$\begin{aligned} B_m := {}&\beta^{4m} - (-1)^m - 4(-1)^m \beta^{2m} + 2\beta^{2m} L_{4m} + 2(-1)^m \beta^{4m} \\ &- \beta^{4m} L_{4m} - 2\beta^{6m} + 2(-1)^m \beta^{2m} + \beta^{8m} - (-1)^m \beta^{4m}. \end{aligned}$$

We then use $\beta^s = \beta F_s + F_{s-1}$ (Lemma 3) to write B_m in the form $r_m + \beta s_m$ (where r_m and s_m belong to $\mathbb{Q}$ and they are called the rational and irrational parts of B_m, respectively). After some manipulations (by using $F_{2\ell m} = F_{\ell m}L_{\ell m}$, where ℓ is a positive integer, and identity $F_{4m-1} - 1 = L_{2m-1}F_{2m}$, see [7]), we arrive at

$$r_m := \bigl(1-(-1)^m\bigr)L_{2m} + 5\bigl(1+(-1)^m\bigr)F_{m-1}F_mF_{2m} \tag{22}$$

and

$$s_m := \bigl(1+(-1)^m\bigr)F_{2m}(L_{2m}-2) \tag{23}$$

as desired. The proof is then complete.

4 The proof of Corollary 1

Now, our goal is to provide a simpler characterization of C_m depending on the parity of m (and to show how this affects its arithmetic nature). By Theorem 1, we have that

$$C_m = \frac{2}{5}\left(1 + \frac{r_m + \beta s_m}{2-(-1)^m L_{4m}}\right),$$

where

$$r_m := \bigl(1-(-1)^m\bigr)L_{2m} + 5\bigl(1+(-1)^m\bigr)F_{m-1}F_mF_{2m}$$

and

$$s_m := \bigl(1+(-1)^m\bigr)F_{2m}(L_{2m}-2).$$

Thus, we only need to work with r_m and s_m for the case in which m is odd and m is even. Therefore, the proof conveniently splits into two cases as follows.

4.1 The proof of item (i)

When m is even, we start by noting that, by Lemma 2(iii), one has

$$2-(-1)^m L_{4m} = 2-(-1)^m\bigl(5F_{2m}^2 + 2(-1)^{2m}\bigr) = 2-\bigl(5F_{2m}^2+2\bigr) = -5F_{2m}^2.$$

Furthermore, it holds

$$s_m = 2F_{2m}(L_{2m}-2) > 0.$$

For r_m, we infer that

$$r_m = 10F_{m-1}F_mF_{2m} = 2(F_{4m-1}+F_{2m}-1).$$

Thus,

$$C_m = \frac{2}{5}\left(1 + \frac{10F_{m-1}F_mF_{2m} + 2\beta F_{2m}(L_{2m}-2)}{-5F_{2m}^2}\right)$$

$$= \frac{2}{5}\frac{5(F_{2m} - 2F_{m-1}F_m) - 2(L_{2m} - 2)\beta}{5F_{2m}}$$
$$= \frac{2}{5}\frac{(L_{2m} - 2) - 2(L_{2m} - 2)\beta}{5F_{2m}}$$
$$= \frac{2(L_{2m} - 2)}{25F_{2m}}(1 - 2\beta) = \frac{2(L_{2m} - 2)}{25F_{2m}}\sqrt{5},$$

where we used Lemma 2(i) and that $1 - 2\beta = \sqrt{5}$.

4.2 The proof of item (ii)

When m is odd, we note that, by Lemma 2(iv), one has

$$2 - (-1)^m L_{4m} = 2 - (-1)^m \left(L_{2m}^2 - 2(-1)^{2m}\right) = L_{2m}^2.$$

Clearly, we get by (22) and (23) that

$$s_m = 0 \quad \text{and} \quad r_m = 2L_{2m}.$$

Thus,

$$C_m = \frac{2}{5}\left(1 + \frac{2L_{2m}}{L_{2m}^2}\right) = \frac{2}{5}\left(1 + \frac{2}{L_{2m}}\right) = \frac{2(L_{2m} + 2)}{5L_{2m}}$$

for any odd $m \geq 1$. The proof is then complete.

5 The proof of Corollary 2

First, for $m = 1$, we observe from (1) that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2}\right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ is even}, n \geq 2; \\ F_{n-1}F_n, & \text{if } n \text{ is odd}, n \geq 1. \end{cases}$$

However, since

$$F_n^2 - F_{n-1}^2 = (F_n - F_{n-1})(F_n + F_{n-1}) = F_{n-2}F_{n+1},$$

we can use Lemma 2(v) to deduce that

$$F_{n-2}F_{n+1} - 1 = F_{n-1}F_n, \quad \text{if } n \text{ is odd}$$

and

$$F_{n-2}F_{n+1} = F_{n-1}F_n - 1, \quad \text{if } n \text{ is even}.$$

Thus, the formula in Corollary 2(ii) holds for $m = 1$.

To deal with $m \geq 2$, we use Corollary 1 to obtain better bounds for C_m. If m is odd, then

$$\frac{2}{5} < C_m < \frac{2}{5}\left(1 + \frac{2}{L_2}\right) = \frac{2}{3}.$$

In the case in which m is even, one has

$$0.28 < \frac{2\sqrt{5}}{25}\left(\frac{L_{2m}}{F_{2m}} - \frac{2}{F_{2m}}\right) < C_m < \frac{2}{5},$$

where we combine items (iii) and (iv) of Lemma 2 to get $L_{2m}/F_{2m} > \sqrt{5}$.

To simplify our notation, we shall denote $X_{m,n}$ as

$$X_{m,n} := \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2}\right)^{-1} - \left(F_{mn}^2 - F_{m(n-1)}^2\right).$$

Thus, inequality (3) yields

$$(-1)^{mn} C_m - \frac{9.83}{\alpha^{2m(n-1)}} < X_{m,n} < (-1)^{mn} C_m + \frac{9.83}{\alpha^{2m(n-1)}}. \tag{24}$$

To prove items (i) and (ii), we may split the proof into two cases as follows.

5.1 The case *mn* even

In this case, we have $C_m \in (0.28, 2/3)$, $2m(n-1) \geq 8$, and so (24) becomes

$$0.07 < 0.28 - \frac{9.83}{\alpha^8} < C_m + \frac{9.83}{\alpha^{2m(n-1)}} < X_{m,n} < C_m + \frac{9.83}{\alpha^8} < \frac{2}{3} + \frac{9.83}{\alpha^8} < 0.88.$$

Thus, $\lfloor X_{m,n} \rfloor = 0$ and so

$$0 = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2}\right)^{-1} - \left(F_{mn}^2 - F_{m(n-1)}^2\right) \right\rfloor = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2}\right)^{-1} \right\rfloor - \left(F_{mn}^2 - F_{m(n-1)}^2\right).$$

Hence, if mn is an even integer, we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2}\right)^{-1} \right\rfloor = F_{mn}^2 - F_{m(n-1)}^2.$$

5.2 The case *mn* odd

In this case, m and n are odd integers. Thus $C_m \in (2/5, 2/3)$ and (24) implies

$$\begin{aligned} -0.7 < -\frac{2}{3} - \frac{9.83}{\alpha^{12}} < -C_m - \frac{9.83}{\alpha^{2m(n-1)}} < X_{m,n} \\ < -C_m + \frac{9.83}{\alpha^{2m(n-1)}} < -\frac{2}{5} + \frac{9.83}{\alpha^{12}} < -0.36, \end{aligned}$$

where we used that $2m(n-1) \geq 12$ (since $m \geq 3$, because ≥ 2 is odd). Thus, $\lfloor X_{m,n} \rfloor = -1$, and so

$$-1 = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2}\right)^{-1} - \left(F_{mn}^2 - F_{m(n-1)}^2\right) \right\rfloor = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2}\right)^{-1} \right\rfloor - \left(F_{mn}^2 - F_{m(n-1)}^2\right).$$

Therefore, if mn is an odd integer, we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} \right\rfloor = F_{mn}^2 - F_{m(n-1)}^2 - 1.$$

This finishes the proof.

6 Conclusions

In this paper, for any $m \geq 1$, we provide an explicit constant $C_m > 0$ for which

$$\left| \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} - \left(F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn} C_m \right) \right| < \frac{9.83}{\alpha^{2m(n-1)}},$$

where $(F_n)_n$ is the Fibonacci sequence and $\alpha = (1 + \sqrt{5})/2$ is the golden number. Moreover, we show that the estimate $|C_m - 2/5| < 1.2/\alpha^{2m}$ holds for all $m \geq 1$. These results solve effectively (and quantitatively) some questions proposed by Lee and Park [8]. As an application of the previous facts, we find the closed formula

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{mn}^2 - F_{m(n-1)}^2, & \text{if } mn \text{ is even;} \\ F_{mn}^2 - F_{m(n-1)}^2 - 1, & \text{if } mn \text{ is odd.} \end{cases}$$

The proof combines several estimates, inequalities, properties of Fibonacci and Lucas numbers as well as some facts about the convergence of series. The computations in this work were performed with MATHEMATICA software.

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Availability of data and materials
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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MD dealt with the conceptualization, supervision, methodology, investigation, and writing—original draft preparation. PT made the formal analysis, writing—review and editing, project administration, and funding acquisition. Both authors read and approved the final manuscript.

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References

1. Basbuk, M., Yazlik, Y.: On the sum of reciprocal of generalized bi-periodic Fibonacci numbers. Miskolc Math. Notes **17**, 35–41 (2016)
2. Choi, G., Choo, Y.: On the reciprocal sums of products of Fibonacci and Lucas numbers. Filomat **32**(8), 2911–2920 (2018)
3. Choo, Y.: On the reciprocal sums of generalized Fibonacci numbers. Int. J. Math. Anal. **10**, 1365–1373 (2016)
4. Choo, Y.: On the finite sums of reciprocal Lucas numbers. Int. J. Math. Anal. **11**, 519–529 (2017)
5. Holliday, S., Komatsu, T.: On the sum of reciprocal generalized Fibonacci numbers. Integers **11**(A), Article ID 11 (2011)
6. Kiliç, E., Arican, T.: More on the infinite sum of reciprocal Fibonacci, Pell and higher order recurrences. Appl. Math. Comput. **219**, 7783–7788 (2013)
7. Koshy, T.: Fibonacci and Lucas Numbers with Applications. Wiley, New York (2001)
8. Lee, H.-H., Park, J.-D.: Asymptotic behavior of reciprocal sum of two products of Fibonacci numbers. J. Inequal. Appl. **2020**(1), 91 (2020)
9. Lee, H.-H., Park, J.-D.: Asymptotic behavior of reciprocal sum of subsequential Fibonacci numbers. Submitted
10. Lin, X., Li, X.: A reciprocal sum related to the Riemann ζ – function. J. Math. Inequal. **11**(1), 209–215 (2017)
11. Ohtsuka, H., Nakamura, S.: On the sum of reciprocal Fibonacci numbers. Fibonacci Q. **46/47**, 153–159 (2008)
12. Yuan, P., He, Z., Zhou, J.: On the sum of reciprocal generalized Fibonacci numbers. Abstr. Appl. Anal. **2014**, Article ID 402540 (2014)
13. Zhang, H., Wu, Z.: On the reciprocal sums of the generalized Fibonacci sequences. Adv. Differ. Equ. **2013**, Article ID 377 (2013)