Global strong solution to the 2D inhomogeneous incompressible magnetohydrodynamic fluids with density-dependent viscosity and vacuum
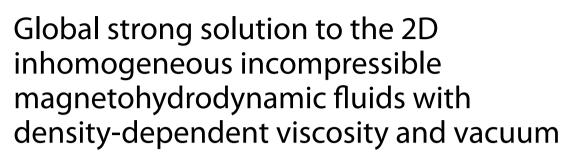
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Abstract

In this paper, we investigate an initial boundary value problem for two-dimensional inhomogeneous incompressible MHD system with density-dependent viscosity. First, we establish a blow-up criterion for strong solutions with vacuum. Precisely, the strong solution exists globally if $\|\nabla\mu(\rho)\|_{L^\infty(0,T;L^p)}$ is bounded. Second, we prove the strong solution exists globally (in time) only if $\|\nabla\mu(\rho_0)\|_{L^p}$ is suitably small, even the presence of vacuum is permitted.

MSC: 35D35; 76D03

1 Introduction

In this paper, we consider the well-posedness of the following inhomogeneous incompressible magnetohydrodynamics (MHD) equations acting as a model on some bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}(\mu(\rho)\nabla u) + H \cdot \nabla H, \\ H_t + u \cdot \nabla H = H \cdot \nabla u + v \triangle H, \\ \operatorname{div} u = \operatorname{div} H = 0, \end{cases} \tag{1}$$

for $(t, x) \in (0, T] \times \Omega$, with the initial value conditions

$$\rho(0, x) = \rho_0(x), \qquad u(0, x) = u_0(x), \qquad H(0, x) = H_0(x), \quad \text{in } \Omega, \tag{2}$$

and the boundary conditions

$$u = H = 0, \quad \text{on } \partial\Omega. \tag{3}$$

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Here ρ, u, H, and P are density, velocity, magnetic field, and pressure, respectively. The viscosity $\mu(\rho)$ is a function of ρ, which is assumed to satisfy

$$\mu(\rho) \in C^1[0,\infty), \quad \text{and} \quad \mu(\rho) \geq \underline{\mu} > 0, \quad \text{on } [0,\infty), \tag{4}$$

for some positive constant $\underline{\mu}$. The constant $\nu > 0$ is the resistivity coefficient. For simplicity, we normalize $\nu = 1$ in the rest of the paper.

Before introducing our main result, let us review some of the results obtained before. The well-posedness on inhomogeneous incompressible flow was started by Kazhikov. Without the effect of magnetic field (i.e. $H = 0$), MHD system turns to be an inhomogeneous incompressible Navier–Stokes system. If $\mu(\rho)$ is a constant and the initial density ρ_0 is bounded away from zero, Kazhikov [24] proved the global existence of weak solutions to the inhomogeneous incompressible Navier–Stokes system in two and three dimensions; see also [3]. After that Antontsev et al. [4] established the first result on local existence and uniqueness of strong solution. Furthermore, the uniqueness of local strong solution was proved to be global one in two dimensions; see also [23, 26, 36].

When the initial density allows vacuum in some subset and $\mu(\rho) \equiv$ Const., Simon [37] established the global existence of weak solution. As for the strong solutions with the vacuum, which may degenerate near vacuum, Choe et al. [9] proposed a compatibility condition, which is similar to (6) below. With such a compatibility condition, they proved the existence and uniqueness of local strong solutions. At the same time, some global solutions in three dimensions with small critical norms have been constructed, we refer the readers to [1, 10, 11, 35] and the references therein. Also, Kim [25] built the blow-up criterion for strong solution with initial vacuum, and she also established a global existence of strong solutions in three dimensions; see also [41]. Very recently, Liang [27] proved the local strong solutions and established a blow-up criterion with vacuum. Soon after that, Lü et al. [33] improved the local solution obtained in [27] to a global one without any small assumption on the initial datum. Liu [29] established the global existence and large time behavior under the small assumption on the L^∞-norm of the density. Recently, Alghamdi, et al. [2] established a new regularity criterion for the 3D density-dependent MHD equations.

If the viscosity $\mu(\rho)$ depends on the density ρ, DePerna et al. established the global weak solution in their pioneer works [13] and [28]. Later, Desjardins [12] improved the regularity of the global weak solution for the two-dimensional case only if the viscosity function $\mu(\rho)$ is a small perturbation of some positive constant in the L^∞-norm. As for the global existence of strong solutions, it was proved by Huang et al. [21] with small assumption on the L^p-norm of $\nabla\mu(\rho_0)$, where they also established the blow-up criterion on L^p-norm of $\nabla\mu(\rho)$. If the strong solution is away from vacuum, Gui et al. [18] established the global well-posedness with ρ_0 is a small perturbation of a constant in H^s, $s \geq 2$. In order to deal with the possible presence of vacuum, Cho et al. [8] generalized the compatibility condition in [9] and constructed the local strong solution in three dimensions. Recently, He et al. [19] considered the global existence and large-time asymptotic behavior of strong solutions to the Cauchy problem of the 3D nonhomogeneous incompressible Navier–Stokes equations with density-dependent viscosity and vacuum, under small assumption on the initial velocity. For more related results, we refer the readers to [5, 7, 15, 31] and the references therein.

Let us come back to the art of inhomogeneous incompressible MHD. Recently, Huang et al. [20] first established the global strong solution to system (1) with $\mu(\rho) \equiv$ Const. and initial vacuum in two dimensions; see also [14]. Recently, Lü et al. [32] established the local strong solutions and then improved the result to a global one in [34] for Cauchy problem on $\mathbb{R}^2$. After that, Chen et al. [6] established the local well-posedness and blow-up criterion to the inhomogeneous incompressible MHD. Later, Gong et al. [16] proved the global existence and uniqueness of strong solutions to an initial-boundary value problem for incompressible MHD equations in three dimensions under some suitable smallness conditions. Soon after that, Gui [17] established global well-posedness of an inhomogeneous incompressible MHD system in the whole space $\mathbb{R}^2$ with $\mu(\rho)$ depending on the density ρ. Very recently and independently, Huang et al. [22] and Zhang [39] obtained the global strong solutions under some suitable small assumptions on the initial datum in three dimensions. And the first author with his co-authors [38] obtained the global strong solutions for initial value problems for (1)–(2) with far-fields density $\tilde{\rho} > 0$, where $\tilde{\rho}$ is some positive constant. In [30], Liu proved the 2D incompressible MHD equations with density-dependent viscosity under the small conditions on $\|\rho\|_{L^\infty} + \|H_0\|_{L^4}$. Zhang [40] consider the 3D system under the small assumption on the initial velocity. And Zhong [42] established the global strong solution to the nonhomogeneous heat conducting MHD with large initial data and vacuum.

Before we state our main result, we first introduce the following result, which can be proved by the methods constructed in [8]. We only list it here without proof.

Theorem 1 *Assume that the initial data* (ρ_0, u_0, H_0) *satisfies the regularity condition*

$$0 \leq \rho_0 \in W^{1,q}, \qquad 2 < q < \infty, \quad u_0, H_0 \in H^1_{0,\sigma} \cap H^2, \tag{5}$$

and the compatibility condition

$$-\operatorname{div}\left(\mu(\rho_0)\nabla u_0\right) + \nabla P_0 - H_0 \cdot \nabla H_0 = \rho_0^{\frac{1}{2}} g \tag{6}$$

for some $(P_0, g) \in H^1 \times L^2$. *Then there exist a time* T^* *and a unique strong solution* (ρ, u, H, P) *to the initial boundary value problem* (1)–(3) *such that*

$$\rho \in C\left([0, T^*]; W^{1,q}\right), \quad \nabla u, P, \nabla H \in C\left([0, T^*]; H^1\right) \cap L^2\left(0, T^*; W^{1,r}\right),$$
$$\rho_t \in C\left([0, T^*]; L^q\right), \quad \sqrt{\rho} u_t, H_t \in L^\infty\left(0, T^*; L^2\right), u_t, H_t \in L^2\left(0, T^*; H^1_0\right)$$

for any r *with* $1 \leq r < q$.

Motivated by [21], we first establish the following blow-up criterion.

Theorem 2 *Assume that the initial data* (ρ_0, u_0, H_0) *satisfy the regularity condition* (5) *and the compatibility condition* (6), *as in Theorem* (1), *and* $0 \leq \rho_0 \leq \bar{\rho}$. *Suppose that* (ρ, u, H, P) *is the unique local strong solution obtained in Theorem* 1, *and* T^* *is the maximal existence time for the solution; then*

$$\sup_{0 \leq t < T^*} \left\|\nabla\mu(\rho)\right\|_{L^p} = \infty \tag{7}$$

for some p *with* $2 < p \leq q$.

Based on the blow-up criterion (7), we can now prove the global strong solution to system (1) under the small assumption on L^p-norm $\nabla\mu(\rho_0)$.

Theorem 3 *Assume that the initial data* (ρ_0, u_0, H_0) *satisfy* (5) *and* (6), *in addition*

$$0 \leq \rho_0 \leq \bar{\rho}, \qquad \|u_0\|_{H^1} + \|H_0\|_{H^1} \leq K, \qquad \underline{\mu} \leq \mu(\rho) \leq \overline{\mu} \quad on \ [0, \overline{\rho}]. \tag{8}$$

Then there exists some small positive constant ε_0, *depending only on* Ω, q, $\underline{\mu}$, $\overline{\mu}$, $\overline{\rho}$ *and* K, *such that if*

$$\left\|\nabla\mu(\rho_0)\right\|_{L^q} \leq \varepsilon_0, \tag{9}$$

then there is a unique global strong solution (ρ, u, H, P) *of the initial boundary value problem* (1)–(3) *with the following regularities*:

$$\rho \in C\big([0,\infty); W^{1,q}\big), \quad \nabla u, P, \nabla H \in C\big([0,\infty); H^1\big) \cap L^2_{\mathrm{loc}}\big(0,\infty; W^{1,r}\big),$$

$$\rho_t \in C\big([0,\infty); L^q\big), \quad \sqrt{\rho} u_t, H_t \in L^\infty_{\mathrm{loc}}\big(0,\infty; L^2\big), \quad u_t, H_t \in L^2_{\mathrm{loc}}\big(0,\infty; H^1_0\big)$$

for any r *with* $1 \leq r < q$.

Let us make some comments on this paper. First, the main difference of the a priori estimates between the classical incompressible Navier–Stokes equations and the inhomogeneous incompressible MHD with density-dependent viscosity and vacuum is the presence of the density and vacuum. It is well known that the vacuum leads to the degeneration and singularity, which cause many troubles in dealing with the a priori estimates. Second, without the effect of magnetic fields, i.e., $H = 0$, system (1) reduces to be the inhomogeneous incompressible Navier–Stokes equations, therefore, Theorems 2 and 3 are the same as those of Huang et al. [21]. Precisely, we generalize the results of [21] to the inhomogeneous incompressible MHD. Third, compared to Gui's [17] global well-posedness result in $\mathbb{R}^2$, with the initial data in critical Besov spaces, our results permit the presence of vacuum. Furthermore, the global well-posedness result obtained in Theorem 3 only if $\|\nabla\mu(\rho_0)\|_{L^p}$ is suitably small, which implies the global strong solution as $\mu(\rho) \equiv \mathrm{Const}$. Recently, such a result was obtained by Huang et al. [20]. Finally, compared with the previous result for inhomogeneous incompressible Navier–Stokes equations in [21], our result is more complicated, and thus more delicate estimates are needed for the analysis of strong solutions.

Finally, we outline the organization for the rest of the paper. In Sect. 2, we present the notions used frequently in this paper and some basic results, while Sect. 3 is devoted to building the blow-up criterion stated in Theorem 2. In the last section, we complete the proof of Theorem 3 for the existence of global strong solution.

2 Preliminaries

In this section, we introduce the notions of this paper and state some auxiliary lemmas, which will be constantly used in the sequel. First, Ω is a smooth bounded domain in $\mathbb{R}^2$. Denote

$$\int f\,dx = \int_\Omega f\,dx.$$

For $1 \le r \le \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in the standard way,

$$
\begin{cases}
L^r = L^r(\Omega), & W^{k,r} = \{f \in L^r : \nabla^k f \in L^r\}, \\
H^k = W^{k,2}, & C_{0,\sigma}^\infty = \{f \in C_0^\infty : \operatorname{div} f = 0 \text{ in } \Omega\}, \\
H_0^1 = \overline{C_0^\infty}, & H_{0,\sigma}^1 = \overline{C_{0,\sigma}^\infty}, \quad \text{closure in the norm of } H^1.
\end{cases}
$$

In order to improve the a priori estimates on u, we need the following regularity results for the Stokes equations, which play an important role in the whole analysis.

Lemma 1 *Assume that $\rho \in W^{1,p}$, $2 < p < \infty$, $0 \le \rho \le \bar{\rho}$, and $\underline{\mu} \le \mu(\rho) \le \bar{\mu}$ on $[0, \bar{\rho}]$. Let $(u, P) \in H_0^1 \times L^2$ be the unique weak solution to the boundary value problem*

$$
-\operatorname{div}\big(\mu(\rho)\nabla u\big) + \nabla P = F, \qquad \operatorname{div} u = 0, \quad \text{in } \Omega, \quad \text{and} \quad \int P \, dx = 0, \tag{10}
$$

and μ satisfies (6). Then we have the following regularity results:
* *If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and*

$$
\begin{cases}
\|u\|_{H^2} \le C\|F\|_{L^2}(1 + \|\nabla\mu(\rho)\|_{L^p})^{\frac{p}{p-2}}, \\
\|P\|_{H^1} \le C\|F\|_{L^2}(1 + \|\nabla\mu(\rho)\|_{L^p})^{\frac{2p-2}{p-2}}.
\end{cases} \tag{11}
$$

* *If $F \in L^r$ for some $r \in (2, p)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and*

$$
\begin{cases}
\|u\|_{W^{2,r}} \le C\|F\|_{L^r}(1 + \|\nabla\mu(\rho)\|_{L^p})^{\frac{pr}{2(p-r)}}, \\
\|P\|_{W^{1,r}} \le C\|F\|_{L^r}(1 + \|\nabla\mu(\rho)\|_{L^p})^{1 + \frac{pr}{2(p-r)}}.
\end{cases} \tag{12}
$$

Here, the constant C in (11) and (12) depends on $\Omega, \overline{\rho}, \underline{\mu}, \overline{\mu}$.

The lemma was proved in [21], hence we omit the details here.
Next, we state the well-known Gagliardo–Nirenberg inequality.

Lemma 2 *If $f \in H^1$, we have*

$$
\|f\|_{L^4}^4 \le C\|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2 \tag{13}
$$

and

$$
\|f\|_{L^\infty}^4 \le C\|f\|_{L^2}^2 \|\nabla f\|_{H^1}^2. \tag{14}
$$

The following important lemma was deduced by Desjardins [12].

Lemma 3 *Suppose that $0 \le \rho \le \bar{\rho}$, $u \in H_0^1$. Then we have*

$$
\|\sqrt{\rho}\, u\|_{L^4}^2 \le C(\bar{\rho}, \Omega)\big(1 + \|\rho u\|_{L^2}\big)\|\nabla u\|_{L^2}\sqrt{\log\big(2 + \|\nabla u\|_{L^2}^2\big)}. \tag{15}
$$

3 Proof of Theorem 2

In this section, we prove the blow-up criterion stated in Theorem 1.2. Let T^* be the maximum time for the existence of strong solution (ρ, u, H, P) to system (1). Suppose that the opposite of (7) holds, that is,

$$\sup_{0 \le t < T^*} \left\| \nabla \mu(\rho) \right\|_{L^p} = M < \infty, \tag{16}$$

with some p satisfying $2 < p \le q$. In this section, C denotes some positive constant which may depend on Ω, $\underline{\mu}$, $\overline{\mu}$, $\overline{\rho}$, the initial data, T^* and M; and it may change line by line.

From now on, under assumption (16), we will derive the following estimates, which can guarantee the extension of local strong solution:

$$\sup_{0 < t < T^*} \left(\left\| \rho(t) \right\|_{W^{1,q}} + \left\| \rho_t(t) \right\|_{L^q} + \left\| \nabla u(t) \right\|_{H^1} + \left\| \nabla H(t) \right\|_{H^1} \right. \tag{17}$$

$$\left. + \left\| \sqrt{\rho} u_t(t) \right\|_{L^2} + \left\| H_t(t) \right\|_{L^2} \right) \le C$$

and

$$\sup_{0 < t < T^*} \left(\int_0^t \left(\| \nabla u \|_{W^{1,p}}^2 + \| \nabla u_t \|_{L^2}^2 \right) ds + \int_0^t \left(\| \nabla H \|_{W^{1,p}}^2 + \| \nabla H_t \|_{L^2}^2 \right) ds \right) \le C \tag{18}$$

for $1 \le p < q$.

First, due to the transport equation $(1)_1$ and the incompressibility condition $\operatorname{div} u = 0$, one easily obtains the following lemma.

Lemma 4 *Suppose that (ρ, u, H, P) is a strong solution to (1) on $[0, T^*)$. Then, for any $t \in [0, T^*)$,*

$$\left\| \rho(t) \right\|_{L^\infty} = \| \rho_0 \|_{L^\infty} \le \overline{\rho}. \tag{19}$$

Next, it follows from the basic energy inequality that

Lemma 5 *Suppose that (ρ, u, H, P) is a strong solution to (1) on $[0, T^*)$. Then, for any $t \in [0, T^*)$,*

$$\left\| \sqrt{\rho} u(t) \right\|_{L^2}^2 + \left\| H(t) \right\|_{L^2}^2 + \int_0^t \left(\left\| \sqrt{\mu(\rho)} \nabla u(s) \right\|^2 + \left\| \nabla H(s) \right\|^2 \right) ds \le C. \tag{20}$$

Proof Multiplying $(1)_2$ and $(1)_3$ by u and H, respectively, and adding them together, and integrating the resultant equations over Ω with respect to x, then using integration by parts and $(1)_1$ and $(1)_4$, one easily obtains (20). □

The following estimate plays a key role for further analysis.

Lemma 6 *Suppose that (ρ, u, H, P) is a strong solution to (1) on $[0, T^*)$. Then, for any $t \in [0, T^*)$,*

$$\left\| H(t) \right\|_{L^4}^4 + \int_0^t \left\| H \cdot \nabla H(s) \right\|_{L^2}^2 ds \le C. \tag{21}$$

Proof Multiplying $(1)_3$ by $4|H|^2H$ and integrating the resultant equation over Ω, we obtain

$$\frac{d}{dt}\int |H|^4\,dx - 4\int \triangle H \cdot H|H|^2\,dx$$
$$= 4\int H \cdot \nabla u \cdot H|H|^2\,dx - 4\int u \cdot \nabla H \cdot H|H|^2\,dx. \tag{22}$$

By integration by parts, the second term on the left-hand side of (22) can be rewritten as

$$-4\int \triangle H \cdot H|H|^2\,dx = 4\int |H|^2|\nabla H|^2\,dx + 2\int \left|\nabla|H|^2\right|^2\,dx. \tag{23}$$

And similarly, the second term on the right-hand side of (22) can be rewritten as

$$-4\int u \cdot \nabla H \cdot H|H|^2\,dx = 2\int \operatorname{div} u|H|^4\,dx = 0, \tag{24}$$

where we have used the incompressibility condition $\operatorname{div} u = 0$.

As for the first term on the right-hand side of (22), we have

$$4\int H \cdot \nabla u \cdot H|H|^2\,dx \leq C\int |\nabla u||H|^4\,dx \tag{25}$$
$$\leq C\|\nabla u\|_{L^2}\|H\|_{L^8}^4$$
$$\leq C\|\nabla u\|_{L^2}\|H\|_{L^4}^2\left\|\nabla|H|^2\right\|_{L^2}$$
$$\leq \left\|\nabla|H|^2\right\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|H\|_{L^4}^4,$$

where we have used the following fact:

$$\|H\|_{L^8}^4 = \left\|H^2\right\|_{L^4}^2 \leq C\left\|H^2\right\|_{L^2}\left\|\nabla|H|^2\right\|_{L^2} = C\|H\|_{L^4}^2\left\|\nabla|H|^2\right\|_{L^2}.$$

Then, substituting (23), (24), and (25) into (22) and integrating the resultant inequality over $(0,t)$, we finally obtain (20). Therefore, we complete the proof of Lemma 6.　□

To proceed, we need the following lemma.

Lemma 7 *Suppose that* (ρ,u,H,P) *is a strong solution to* (1) *on* $[0,T^*)$. *Then, for any* $t \in [0,T^*)$,

$$\|\nabla u\|_{H^1} \leq C\|\sqrt{\rho}u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2} + C\|H \cdot \nabla H\|_{L^2}. \tag{26}$$

Proof It follows from (11) and (13) that

$$\|\nabla u\|_{H^1} \leq C\big(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|H \cdot \nabla H\|_{L^2}\big)\big(1 + \left\|\nabla\mu(\rho)\right\|_{L^p}\big)^{\frac{p}{p-2}}$$
$$\leq C\|\sqrt{\rho}u_t\|_{L^2} + C\|\rho u\|_{L^4}\|\nabla u\|_{L^4} + C\|H \cdot \nabla H\|_{L^2}$$
$$\leq C\|\sqrt{\rho}u_t\|_{L^2} + C\|\rho u\|_{L^4}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}} + C\|H \cdot \nabla H\|_{L^2}$$
$$\leq C\|\sqrt{\rho}u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2} + C\|H \cdot \nabla H\|_{L^2} + \frac{1}{2}\|\nabla u\|_{H^1},$$

which shows (26) directly. Therefore, we finish the proof of Lemma 7.　□

Lemma 8 *Suppose that* (ρ, u, H, P) *is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$\left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\right)(t) + \int_0^t \left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\triangle H\|_{L^2}^2\right)(s)\,ds \leq C. \tag{27}$$

Proof We divide the proof into three steps.

  Step 1. Multiplying $(1)_2$ by u_t and integrating the resultant equation over Ω, and then using integration by parts, we thus obtain

$$\int \rho |u_t|^2 \, dx + \frac{1}{2}\frac{d}{dt}\int \mu(\rho)|\nabla u|^2 \, dx \tag{28}$$

$$= \int \rho u \cdot \nabla u \cdot u_t \, dx + \frac{1}{2}\int \mu_t(\rho)|\nabla u|^2 \, dx$$

$$- \frac{d}{dt}\int H \cdot \nabla u \cdot H \, dx + \int H_t \cdot \nabla u \cdot H \, dx + \int H \cdot \nabla u \cdot H_t \, dx.$$

  Now we consider each term on the right-hand side of (28). First, the first term can be estimated as follows:

$$\left|\int \rho u \cdot \nabla u \cdot u_t \, dx\right| \tag{29}$$

$$\leq C\|\sqrt{\rho}u_t\|_{L^2}\|\rho u\|_{L^4}\|\nabla u\|_{L^4}$$

$$\leq \frac{1}{16}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}$$

$$\leq \frac{1}{16}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho u\|_{L^4}^4\|\nabla u\|_{L^2}^2 + C\|H \cdot \nabla H\|_{L^2}^2.$$

  Then, following from the mass equation $(1)_1$ and incompressibility condition $\operatorname{div} u = 0$, we have

$$\mu_t(\rho) + u \cdot \nabla \mu(\rho) = 0. \tag{30}$$

  Due to (30), we can compute the second term as

$$\left|\frac{1}{2}\int \mu_t(\rho)|\nabla u|^2 \, dx\right| \tag{31}$$

$$= \left|\frac{1}{2}\int u \cdot \nabla \mu(\rho)|\nabla u|^2 \, dx\right|$$

$$\leq C\int \left|\nabla \mu(\rho)\right||u||\nabla u|^2 \, dx$$

$$\leq C\left\|\nabla \mu(\rho)\right\|_{L^p}\|u\|_{L^{\frac{2p}{p-2}}}\|\nabla u\|_{L^4}^2$$

$$\leq C\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}$$

$$\leq \frac{1}{16}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|\rho u\|_{L^4}^4\|\nabla u\|_{L^2}^2 + C\|H\nabla H\|_{L^2}^2.$$

Next, by using $(1)_3$, we rewrite the fourth term on the right-hand side of (28) as

$$\int H_t \cdot \nabla u \cdot H \, dx$$

$$= \int \triangle H \cdot \nabla u \cdot H \, dx + \int H \cdot \nabla u \cdot \nabla u \cdot H \, dx - \int u \cdot \nabla H \cdot \nabla u \cdot H \, dx. \qquad (32)$$

Each term on the right-hand side of (32) can be estimated as follows:

$$\left| \int \triangle H \cdot \nabla u \cdot H \, dx \right|$$

$$\leq C \| \triangle H \|_{L^2} \| \nabla u \|_{L^4} \| H \|_{L^4} \leq C \| \triangle H \|_{L^2} \| \nabla u \|_{L^2}^{\frac{1}{2}} \| \nabla u \|_{H^1}^{\frac{1}{2}}$$

$$\leq \frac{1}{8} \| \triangle H \|_{L^2}^2 + \frac{1}{16} \| \sqrt{\rho} u_t \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^4 + C \| H \cdot \nabla H \|_{L^2}^2,$$

$$\left| \int H \cdot \nabla u \cdot \nabla u \cdot H \, dx \right|$$

$$\leq C \| H \|_{L^4}^2 \| \nabla u \|_{L^4}^2 \leq C \| \nabla u \|_{L^2} \| \nabla u \|_{H^1}$$

$$\leq \frac{1}{16} \| \sqrt{\rho} u_t \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^4 + C \| H \cdot \nabla H \|_{L^2}^2,$$

$$\left| - \int u \cdot \nabla H \cdot \nabla u \cdot H \, dx \right|$$

$$\leq C \| u \|_{L^4} \| \nabla H \|_{L^4} \| \nabla u \|_{L^4} \| H \|_{L^4}$$

$$\leq C \| \nabla u \|_{L^2} \| \nabla H \|_{L^2}^{\frac{1}{2}} \| \nabla H \|_{H^1}^{\frac{1}{2}} \| \nabla u \|_{L^2}^{\frac{1}{2}} \| \nabla H \|_{H^1}^{\frac{1}{2}}$$

$$\leq \frac{1}{8} \| \triangle H \|_{L^2}^2 + \frac{1}{16} \| \sqrt{\rho} u_t \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla H \|_{L^2}^2 + C \| \nabla u \|_{L^2}^4$$

$$+ C \| \rho u \|_{L^4}^4 \| \nabla u \|_{L^2}^2 + C \| H \cdot \nabla H \|_{L^2}^2.$$

Then, substituting the above inequalities into (32), we finally deduce that

$$\left| \int H_t \cdot \nabla u \cdot H \, dx \right| \qquad (33)$$

$$\leq \frac{3}{16} \| \sqrt{\rho} u_t \|_{L^2}^2 + \frac{1}{4} \| \triangle H \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^4$$

$$+ C \| \nabla u \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 + C \| H \cdot \nabla H \|_{L^2}^2 + C \| \rho u \|_{L^4}^4 \| \nabla u \|_{L^2}^2.$$

Similarly, we can also estimate the fifth term on the right-hand side of (28) as follows:

$$\left| \int H \cdot \nabla u \cdot H_t \, dx \right| \qquad (34)$$

$$\leq \frac{3}{16} \| \sqrt{\rho} u_t \|_{L^2}^2 + \frac{1}{4} \| \triangle H \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^4$$

$$+ C \| \nabla u \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 + C \| H \cdot \nabla H \|_{L^2}^2 + C \| \rho u \|_{L^4}^4 \| \nabla u \|_{L^2}^2.$$

Then, substituting (29), (31), (33), and (34) into (28), we finally obtain that

$$\int \rho |u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int \mu(\rho) |\nabla u|^2 \, dx + \frac{d}{dt} \int H \cdot \nabla u \cdot H \, dx \tag{35}$$

$$\leq \frac{1}{2} \|\triangle H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4$$

$$+ C\|\nabla u\|_{L^2}^2 + C\|\nabla H\|_{L^2}^2 + C\|H\nabla H\|_{L^2}^2 + C\|\rho u\|_{L^4}^4 \|\nabla u\|_{L^2}^2.$$

Step 2. Multiplying (1)₃ by $-\triangle H$, and integrating the resultant equation over Ω with respect to x, then using integration by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int |\nabla H|^2 \, dx + \int |\triangle H|^2 \, dx = \int u \cdot \nabla H \cdot \triangle H \, dx - \int H \cdot \nabla u \cdot \triangle H \, dx. \tag{36}$$

Now, we estimate the terms on the right-hand side of (36). First, the first term can be estimated as follows:

$$\left| \int u \cdot \nabla H \cdot \triangle H \, dx \right| = \left| \int u_i \partial_i H_j \partial_{kk}^2 H^j \, dx \right| \tag{37}$$

$$= \left| - \int \partial_k u_i \partial_i H_j \partial_k H_j \, dx - \int u_i \partial_{ik}^2 H_j \partial_k H_j \, dx \right|$$

$$= \int |\nabla u||\nabla H|^2 \, dx \leq C\|\nabla u\|_{L^2} \|\nabla H\|_{L^4}^2$$

$$\leq C\|\nabla u\|_{L^2} \|\nabla H\|_{L^2} \|\triangle H\|_{L^2}$$

$$\leq \frac{1}{4} \|\triangle H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2.$$

Next, the second term can be estimated as follows:

$$\left| \int H \cdot \nabla u \cdot \triangle H \, dx \right| \tag{38}$$

$$\leq C\|H\|_{L^4} \|\nabla u\|_{L^4} \|\triangle H\|_{L^2}$$

$$\leq \frac{1}{4} \|\triangle H\|_{L^2}^2 + C\|\nabla u\|_{L^2} \|\nabla u\|_{H^1}$$

$$\leq \frac{1}{4} \|\triangle H\|_{L^2}^2 + \varepsilon \|\sqrt{\rho} u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|H\nabla H\|_{L^2}^2.$$

Substituting (37) and (38) into (36), we deduce that

$$\frac{d}{dt} \int |\nabla H|^2 \, dx + \frac{1}{2} \int |\triangle H|^2 \, dx \tag{39}$$

$$\leq C\|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + \varepsilon \|\sqrt{\rho} u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|H\nabla H\|_{L^2}^2.$$

Step 3. Notice that

$$\int H \cdot \nabla u \cdot H \, dx \leq C\|\nabla u\|_{L^2} \|H\|_{L^4}^2 \tag{40}$$

$$\leq C\|\nabla u\|_{L^2} \|H\|_{L^2} \|\nabla H\|_{L^2}$$

$$\leq \frac{1}{4} \int \mu(\rho) |\nabla u|^2 \, dx + C_1 \|\nabla H\|_{L^2}^2,$$

and (15) shows that

$$\|\sqrt{\rho} u\|_{L^4}^4 \leq C\big(1 + \|\rho u\|_{L^2}^2\big) \|\nabla u\|_{L^2}^2 \log\big(2 + \|\nabla u\|_{L^2}^2\big) \tag{41}$$

$$\leq C \|\nabla u\|_{L^2}^2 \log\big(2 + \|\nabla u\|_{L^2}^2\big).$$

With the help of (40) and (41), combining (35) and (39) multiplied by $C_1 + 1$, and choosing ε small enough, we finally obtain

$$\frac{d}{dt} \int \big(\mu(\rho)|\nabla u|^2 + |\nabla H|^2\big) \, dx + \int \big(\rho |u_t|^2 + |\triangle H|^2\big) \, dx \tag{42}$$

$$\leq C\|\rho u\|_{L^4}^4 \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4$$

$$+ C\|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + C\|H \cdot \nabla H\|_{L^2}^2$$

$$\leq C\big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)^2 \big(1 + \log\big(2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)\big) + C\|H \cdot \nabla H\|_{L^2}^2,$$

which together with Gronwall's inequality shows (27). Therefore, we finish the proof of Lemma 8. □

It follows from (27), one easily deduces the following result.

Lemma 9 *Suppose that (ρ, u, H, P) is a strong solution to (1) on $[0, T^*)$. Then, for any $t \in [0, T^*)$,*

$$\int_0^t \big\|H_t(s)\big\|_{L^2}^2 \, ds \leq C. \tag{43}$$

Proof It follows from $(1)_3$ that

$$\|H_t\|_{L^2}^2 \leq C\big(\|u \cdot \nabla H\|_{L^2}^2 + \|H \cdot \nabla u\|_{L^2}^2 + \|\triangle H\|_{L^2}^2\big) \tag{44}$$

$$\leq C\|u\|_{L^4}^2 \|\nabla H\|_{L^4}^2 + C\|H\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + C\|\triangle H\|_{L^2}^2$$

$$\leq C\|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2} \|\nabla H\|_{H^1} + C\|\nabla u\|_{L^2} \|\nabla u\|_{H^1} + C\|\triangle H\|_{L^2}^2,$$

which together with (26) and (27) shows (43). Hence, we complete the proof of Lemma 9. □

From now on, we start to derive the higher order derivatives estimates of the density, velocity, and magnetohydrodynamic field.

Lemma 10 *Suppose that (ρ, u, H, P) is a strong solution to (1) on $[0, T^*)$. Then, for any $t \in [0, T^*)$,*

$$\int_0^t \big\|u(s)\big\|_{L^\infty}^4 \, ds \leq C. \tag{45}$$

Proof Due to (14) and (26), we have

$$\int_0^t \left\| u(s) \right\|_{L^\infty}^4 ds$$

$$\leq C \int_0^t \left\| \nabla u(s) \right\|_{L^2}^2 \left\| \nabla u(s) \right\|_{H^1}^2 ds$$

$$\leq C \int_0^t \left(\left\| \nabla u(s) \right\|_{L^2}^2 \left\| \sqrt{\rho} u_t(s) \right\|_{L^2}^2 + \left\| \nabla u(s) \right\|_{L^2}^8 + \left\| \nabla u(s) \right\|_{L^2}^2 \left\| H \nabla H(s) \right\|_{L^2}^2 \right) ds$$

$$\leq C \int_0^t \left(\left\| \nabla u(s) \right\|_{L^2}^2 \left\| \sqrt{\rho} u_t(s) \right\|_{L^2}^2 + \left\| \nabla u(s) \right\|_{L^2}^8 + \left\| \nabla u(s) \right\|_{L^2}^2 \left\| H \right\|_{L^\infty}^2 \left\| \nabla H(s) \right\|_{L^2}^2 \right) ds$$

$$\leq C \int_0^t (\left\| \nabla u(s) \right\|_{L^2}^2 \left\| \sqrt{\rho} u_t(s) \right\|_{L^2}^2 + \left\| \nabla u(s) \right\|_{L^2}^8$$

$$+ \left(\left\| \nabla u(s) \right\|_{L^2}^2 \| H \|_{L^2} \| \nabla H \|_{H^1} \left\| \nabla H(s) \right\|_{L^2}^2 \right) ds$$

$$\leq C,$$

where we have used (13), (14), (26), and (27). Therefore, we complete the proof of Lemma 10. □

To proceed, we first improve the regularity estimates on magnetic field.

Lemma 11 *Suppose that* (ρ, u, H, P) *is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$\left\| \triangle H(t) \right\|_{L^2}^2 + \int_0^t \left\| \nabla H_t(s) \right\|_{L^2}^2 ds \leq C. \tag{46}$$

Proof Multiplying $(1)_3$ by $-\triangle H_t$ and integrating the resultant equation over Ω with respect to x, then using integration by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int |\triangle H|^2 dx + \int |\nabla H_t|^2 dx = \int u \cdot \nabla H \cdot \triangle H_t dx - \int H \cdot \nabla u \cdot \triangle H_t dx. \tag{47}$$

Now, we estimate each term on the right-hand side of (47). For the first term, we have

$$\left| \int u \cdot \nabla H \cdot \triangle H_t dx \right| \tag{48}$$

$$= \left| \int u_i \partial_i H_j \partial_{kk}^2 H_{jt} dx \right|$$

$$= \left| \int \partial_k u_i \partial_i H_j \partial_k H_{jt} dx - \int u_i \partial_{ki}^2 H_j \partial_k H_{jt} dx \right|$$

$$\leq C \| \nabla u \|_{L^4} \| \nabla H \|_{L^4} \| \nabla H_t \|_{L^2} + C \| u \|_{L^\infty} \| \triangle H \|_{L^2} \| \nabla H_t \|_{L^2}$$

$$\leq C \| \nabla u \|_{L^2}^{\frac{1}{2}} \| \nabla u \|_{H^1}^{\frac{1}{2}} \| \nabla H \|_{L^2}^{\frac{1}{2}} \| \triangle H \|_{L^2}^{\frac{1}{2}} \| \nabla H_t \|_{L^2} + C \| u \|_{L^\infty} \| \triangle H \|_{L^2} \| \nabla H_t \|_{L^2}$$

$$\leq \frac{1}{4} \| \nabla H_t \|_{L^2}^2 + C \left(\| \sqrt{\rho} u_t \|_{L^2}^2 + \| H \cdot \nabla H \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| u \|_{L^\infty}^2 \right) \| \triangle H \|_{L^2}^2$$

$$+ C \| \nabla u \|_{L^2}^2 \| \nabla H \|_{L^2}^2 + C \| \nabla u \|_{L^2}^6 \| \nabla H \|_{L^2}^2.$$

Next, the second term can be estimated as follows:

$$
\left| \int H \cdot \nabla u \cdot \triangle H_t \, dx \right| \tag{49}
$$

$$
= \left| \int H_i \partial_i u_j \partial^2_{kk} H_{jt} \, dx \right|
$$

$$
= \left| \int \partial_k H_i \partial_i u_j \partial_k H_{jt} \, dx + \int H_i \partial^2_{ki} u_j \partial_k H_{jt} \, dx \right|
$$

$$
\leq C \|\nabla u\|_{L^4} \|\nabla H\|_{L^4} \|\nabla H_t\|_{L^2} + C \|H\|_{L^\infty} \|\nabla u\|_{H^1} \|\nabla H_t\|_{L^2}
$$

$$
\leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}} \|\nabla H\|_{L^2}^{\frac{1}{2}} \|\triangle H\|_{L^2}^{\frac{1}{2}} \|\nabla H_t\|_{L^2} + C \|H\|_{L^2}^{\frac{1}{2}} \|\triangle H\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1} \|\nabla H_t\|_{L^2}
$$

$$
\leq \frac{1}{4} \|\nabla H_t\|_{L^2}^2 + C \big(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|H \cdot \nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \big) \|\triangle H\|_{L^2}^2
$$

$$
+ C \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^6 \|\nabla H\|_{L^2}^2 + C \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|H \cdot \nabla H\|_{L^2}^2.
$$

Then inserting (48) and (49) into (47), together with (21), (27), (45), and Gronwall's inequality, one easily obtains (46). Therefore, we finish the proof of Lemma 11. □

The next lemma is crucial to improving the regularity of the velocity.

Lemma 12 *Suppose that* (ρ, u, H, P) *is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$
\left\| \sqrt{\rho} u_t(t) \right\|_{L^2}^2 + \int_0^t \left\| \nabla u_t(s) \right\|_{L^2}^2 ds \leq C. \tag{50}
$$

Proof Differentiating $(1)_2$ with respect to x, we obtain

$$
\rho u_{tt} + (\rho u) \cdot \nabla u_t - \mathrm{div}\big(\mu(\rho) \nabla u_t \big) + \nabla P_t \tag{51}
$$

$$
= -\rho_t u_t - (\rho u)_t \cdot \nabla u + \mathrm{div}\big(\mu_t(\rho) \nabla u \big) + H_t \cdot \nabla H + H \cdot \nabla H_t.
$$

Multiplying (51) by u_t and integrating the resultant equation over Ω with respect to x, then due to integration by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 \, dx + \int \mu(\rho) |\nabla u_t|^2 \, dx \tag{52}
$$

$$
= - \int \rho_t |u_t|^2 \, dx - \int (\rho u)_t \cdot \nabla u \cdot u_t \, dx - \int \mu_t(\rho) \nabla u \cdot \nabla u_t \, dx
$$

$$
- \int H_t \cdot u_t \nabla \cdot H \, dx - \int H \cdot \nabla u_t \cdot H_t \, dx = \sum_{i=1}^{5} I_i.
$$

Now we estimate the terms on the right-hand side of (52). First, by using mass equation $(1)_1$, we have

$$
|I_1| = \left| \int \rho_t |u_t|^2 \, dx \right| = 2 \left| \int \rho u \cdot u_t \cdot \nabla u_t \, dx \right| \tag{53}
$$

$$\leq C\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2}$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2.$$

Next, due to mass equation $(1)_1$ and integration by parts, we deduce that

$$|I_2| = \left|\int (\rho u)_t \cdot \nabla u \cdot u_t\,dx\right| \tag{54}$$

$$\leq \left|\int \rho_t u \cdot \nabla u \cdot u_t\,dx\right| + \left|\int \rho u_t \cdot \nabla u \cdot u_t\,dx\right|$$

$$\leq \left|\int \rho u \cdot \nabla u \cdot \nabla u \cdot u_t\,dx\right| + \left|\int \rho u \cdot u \cdot \nabla^2 u \cdot u_t\,dx\right|$$

$$+ \left|\int \rho u \cdot u \cdot \nabla u \cdot \nabla u_t\,dx\right| + \left|\int \rho u_t \cdot \nabla u \cdot u_t\,dx\right| = \sum_{i=1}^4 I_{2i}.$$

Now, we estimate each term on the right-hand side of (54). First, it follows from Sobolev's inequality, (13), and (26) that

$$I_{21} \leq \int \rho|u||\nabla u|^2|u_t|\,dx$$

$$\leq C\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^4}^2$$

$$\leq C\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}$$

$$\leq C\big(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^2}^2\big)\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^8 + C\|\nabla u\|_{L^2}^2\|H\cdot\nabla H\|_{L^2}^2.$$

Similarly, we have

$$I_{22} \leq \int \rho|u|^2|\nabla^2 u||u_t|\,dx$$

$$\leq C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}\|\nabla^2 u\|_{L^2}$$

$$\leq C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}\big(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 + \|H\cdot\nabla H\|_{L^2}\big)$$

$$\leq C\big(\|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^4\big)\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C\|H\cdot\nabla H\|_{L^2}^2$$

and

$$I_{23} \leq \int \rho|u|^2|\nabla u||\nabla u_t|\,dx$$

$$\leq C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|u\|_{L^\infty}^4\|\nabla u\|_{L^2}^2,$$

and

$$I_{24} \leq \int \rho|\nabla u||u_t|^2\,dx$$

$$\leq C\|\sqrt{\rho}u_t\|_{L^2}\|u_t\|_{L^4}\|\nabla u\|_{L^4}$$

$$\leq C\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|\sqrt{\rho}u_t\|_{L^2}^4 + C\|\nabla u\|_{L^2}^4$$

$$\quad + C\|\nabla u\|_{L^2}^4\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|H\cdot\nabla H\|_{L^2}^2.$$

Then, inserting all the above estimates into (54), one easily obtains

$$|I_2| \leq \frac{1}{4}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|\nabla u\|_{L^2}^8 + C\|\nabla u\|_{L^2}^2\|H\cdot\nabla H\|_{L^2}^2 \tag{55}$$

$$\quad + C\big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^4 + \|\sqrt{\rho}u_t\|_{L^2}^2\big)\|\sqrt{\rho}u_t\|_{L^2}^2$$

$$\quad + C\|\nabla u\|_{L^2}^6 + C\|H\cdot\nabla H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|u\|_{L^\infty}^4\|\nabla u\|_{L^2}^2.$$

Then it follows from (30) that

$$|I_3| = \left|\int \mu_t(\rho)\nabla u\cdot\nabla u_t\,dx\right| \leq C\int |u|\big|\nabla\mu(\rho)\big|\,|\nabla u|\,|\nabla u_t|\,dx \tag{56}$$

$$\leq C\|u\|_{L^\infty}\big\|\nabla\mu(\rho)\big\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}}\|\nabla u_t\|_{L^2}$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|u\|_{L^\infty}^2\|\nabla u\|_{H^1}^2$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^6$$

$$\quad + C\|u\|_{L^\infty}^2\|H\|_{L^\infty}^2\|\nabla H\|_{L^2}^2$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^6$$

$$\quad + C\|u\|_{L^\infty}^2\|H\|_{L^2}\|\nabla H\|_{L^2}^2\|\triangle H\|_{L^2}.$$

And, we can also have

$$|I_4| = \left|\int H_t\cdot\nabla u_t\cdot H\,dx\right| \leq C\|H_t\|_{L^4}\|\nabla u_t\|_{L^2}\|H\|_{L^4} \tag{57}$$

$$\leq C\|\nabla H_t\|_{L^2}\|\nabla u_t\|_{L^2}\|H\|_{L^4}$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|H\|_{L^4}^2\|\nabla H_t\|_{L^2}^2,$$

and very similarly we have

$$|I_5| = \left|\int H\cdot\nabla u_t\cdot H_t\,dx\right| \tag{58}$$

$$\leq \frac{1}{8}\int \mu(\rho)|\nabla u_t|^2\,dx + C\|H\|_{L^4}^2\|\nabla H_t\|_{L^2}^2.$$

Then, substituting (53), (55), (56), (57), and (58) into (52), we have

$$
\frac{d}{dt} \int \rho |u_t|^2 \, dx + \int \mu(\rho) |\nabla u_t|^2 \, dx \tag{59}
$$

$$
\leq C \|\nabla u\|_{L^2}^8 + C \|\nabla u\|_{L^2}^2 \|H \cdot \nabla H\|_{L^2}^2 + C \|H\|_{L^4}^2 \|\nabla H_t\|_{L^2}^2
$$

$$
+ C \big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^4 + \|\sqrt{\rho} u_t\|_{L^2}^2 \big) \|\sqrt{\rho} u_t\|_{L^2}^2
$$

$$
+ C \|\nabla u\|_{L^2}^6 + C \|H \cdot \nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 + C \|u\|_{L^\infty}^4 \|\nabla u\|_{L^2}^2
$$

$$
+ C \|u\|_{L^\infty}^2 \|H\|_{L^2} \|\nabla H\|_{L^2}^2 \|\triangle H\|_{L^2}.
$$

Therefore, following from (21), (27), (45), (46), and Gronwall's inequality, we obtain the desired estimate (50). Hence, we complete the proof of Lemma 12. $\qquad \square$

Now, we can obtain the estimate $\|\nabla u\|_{H^1}$.

Lemma 13 *Suppose that (ρ, u, H, P) is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$
\left\| \nabla u(t) \right\|_{H^1} \leq C. \tag{60}
$$

Proof It follows from (26) that

$$
\|\nabla u\|_{H^1} \leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|\nabla u\|_{L^2}^3 + C \|H \nabla H\|_{L^2}
$$

$$
\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|\nabla u\|_{L^2}^3 + C \|H\|_{L^\infty} \|\nabla H\|_{L^2}
$$

$$
\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|\nabla u\|_{L^2}^3 + C \|H\|_{L^2}^{\frac{1}{2}} \|\triangle H\|_{L^2}^{\frac{1}{2}} \|\nabla H\|_{L^2},
$$

which together with (20), (27), (46), and (50) shows (60). Thus, we complete the proof of Lemma 13. $\qquad \square$

Next, we have the following estimate.

Lemma 14 *Suppose that (ρ, u, H, P) is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$
\left\| H_t(t) \right\|_{L^2}^2 \leq C.
$$

Proof The proof of this lemma is directly from (44) together with estimates (46) and (60). $\qquad \square$

To proceed, we need the following result.

Lemma 15 *Suppose that (ρ, u, H, P) is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$
\int_0^t \left\| \nabla u(s) \right\|_{L^\infty} \, ds \leq C. \tag{61}
$$

Proof Choosing some r with $2 < r < \min\{p, 4\}$, we see that

$$\int_0^t \left\| \nabla u(s) \right\|_{L^\infty} ds \tag{62}$$

$$\leq \int_0^t \left\| \nabla u \right\|_{W^{1,r}} ds$$

$$\leq C \int_0^t \left\| \rho u_t(s) \right\|_{L^4} ds + C \int_0^t \left\| \rho u \cdot \nabla u(s) \right\|_{L^4} ds + C \int_0^t \left\| H \cdot \nabla H(s) \right\|_{L^4} ds$$

$$\leq C \int_0^t \left\| \nabla u_t(s) \right\|_{L^2} ds + C \int_0^t \left\| \nabla u(s) \right\|_{H^1}^2 ds + \int_0^t \left\| \triangle H(s) \right\|_{L^2}^2 ds,$$

which together with (27), (50), and (60) shows (62). Therefore, we finish the proof of Lemma 15. □

With the help of (61), we can derive the first order derivative estimates for the density.

Lemma 16 *Suppose that* (ρ, u, H, P) *is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$\left\| \rho(t) \right\|_{W^{1,q}} + \left\| \rho_t(t) \right\|_{L^q} \leq C. \tag{63}$$

Proof Differentiating mass equation $(1)_1$ with respect to x_i $(i = 1, 2)$, we have

$$(\partial_i \rho)_t + u \cdot \nabla(\partial_i \rho) + \partial_i u \cdot \nabla \rho = 0.$$

Then, multiplying the above equality by $q|\partial_i \rho|^{q-2} \partial_i \rho$, then integrating the resultant equation and using integration by parts, one easily obtains

$$\frac{d}{dt} \|\nabla \rho\|_{L^q}^q \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^q}^q, \tag{64}$$

which together with (61) and Gronwall's inequality shows the first part of (63).

Following mass equation $(1)_1$ and the Sobolev embedding theorem, we have

$$\|\rho_t\|_{L^q} \leq \|u \cdot \nabla \rho\|_{L^q} \leq \|u\|_{L^\infty} \|\nabla \rho\|_{L^q} \leq \|\nabla u\|_{H^1} \|\nabla \rho\|_{L^q},$$

from which together with (60) and the first part of (63), we deduce the second part of (63). Hence, we finish the proof of Lemma 16. □

Additionally, we have the following regularity.

Lemma 17 *Suppose that* (ρ, u, H, P) *is a strong solution to* (1) *on* $[0, T^*)$. *Then, for any* $t \in [0, T^*)$,

$$\int_0^t \left(\left\| \nabla u(s) \right\|_{W^{1,p}}^2 + \left\| P(s) \right\|_{W^{1,p}}^2 + \left\| \nabla H(s) \right\|_{W^{1,p}}^2 \right) ds \leq C. \tag{65}$$

Proof Due to (12), (16), (27), (46), (50), (60), and Sobolev's embedding theorem, we deduce that

$$
\int_0^t \left(\left\| \nabla u(s) \right\|_{W^{1,p}} + \left\| P(s) \right\|_{W^{1,p}} \right) ds
$$

$$
\leq C \int_0^t \left(\left\| \rho u_t(s) \right\|_{L^p} + \left\| \rho u \cdot \nabla u(s) \right\|_{L^p} + \left\| H \cdot \nabla H(s) \right\|_{L^p} \right)
$$

$$
\cdot \left(1 + \left\| \nabla \mu(\rho)(s) \right\|_{L^q} \right)^{1 + \frac{qp}{2(q-p)}} ds
$$

$$
\leq C \int_0^t \left(\left\| \nabla u_t(s) \right\|_{L^2} + C \left\| \nabla u(s) \right\|_{H^1}^2 + C \left\| \nabla H(s) \right\|_{H^1}^2 \right) ds \leq C.
$$

Next, following from the $W^{2,p}$-regularity of elliptic system, we have

$$
\int_0^t \left\| \nabla H(s) \right\|_{W^{1,p}}^2 ds \leq C \int_0^t \left(\left\| H_t(s) \right\|_{L^p}^2 + \left\| u \cdot \nabla H(s) \right\|_{L^p}^2 + \left\| H \cdot \nabla u(s) \right\|_{L^p}^2 \right) ds
$$

$$
\leq C \int_0^t \left(\left\| \nabla H_t \right\|_{L^2}^2 + \left\| \nabla H \right\|_{H^1}^2 \left\| \nabla u \right\|_{H^1}^2 \right) ds
$$

$$
\leq C,
$$

where we have used (27), (46), (60), and Sobolev's inequality. Combining the above two estimates, we complete the proof of (65). Thus, we finish the proof of Lemma 17. □

Indeed, following from the a priori estimates obtained in Lemmas 4–17, we complete all the desired estimates in (17), therefore we finish the proof of Theorem 2.

4 Proof of Theorem 3

In this section, we devote ourselves to the proof of Theorem 3. First, supposing that $\| \nabla \mu(\rho) \|_{L^p} \leq 1$, and with the condition to deduce the desired a priori estimates, and then due to the condition $\| \nabla \mu(\rho_0) \|_{L^p}$ small enough to close the condition $\| \nabla \mu(\rho) \|_{L^p} \leq 1$. Furthermore, based on the uniform estimates, we extend the local strong solution to be a global one.

First, it is the same as Lemma 4 that

Lemma 18 *Suppose that (ρ, u, H, P) is the unique local strong solution to (1) on $[0, T]$; with the initial data (ρ_0, u_0, H_0), we have*

$$
\sup_{0 \leq t \leq T} \left\| \rho(t) \right\|_{L^\infty} \leq \overline{\rho}. \tag{66}
$$

Next, the basic energy estimate gives the following result.

Lemma 19 *Suppose that (ρ, u, H, P) is the unique local strong solution to (1) on $[0, T]$; with the initial data (ρ_0, u_0, H_0), we have*

$$
\sup_{0 \leq t \leq T} \left(\| \sqrt{\rho} u \|_{L^2}^2 + \| H \|_{L^2}^2 \right) + \int_0^T \left(\| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 \right) dt \leq C. \tag{67}
$$

Hence, we can also obtain

$$\sup_{0\le t\le T} t\big(\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2\big) + \int_0^T t\big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)\,dt \le C. \tag{68}$$

Proof The proof of (67) is the same as that of (20). We only need to show the proof of (68). First, it follows from the energy equality that

$$\frac{1}{2}\frac{d}{dt}\big(\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2\big) + \big(\big\|\sqrt{\mu(\rho)}\nabla u\big\|^2 + \|\nabla H\|^2\big) = 0. \tag{69}$$

Then, it follows from Poincaré's inequality that

$$\frac{1}{2}\big(\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2\big) \le C\big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big) \le C\big(\big\|\sqrt{\mu(\rho)}\nabla u\big\|^2 + \|\nabla H\|^2\big), \tag{70}$$

where we have used the fact $\mu(\rho) \ge \underline{\mu} > 0$. Hence, (69) and (70) show the following result:

$$\big(\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2\big) \le Ce^{-Ct}\big(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|H_0\|_{L^2}^2\big). \tag{71}$$

Furthermore, multiplying (69) by t and then integrating the resultant equation over Ω with respect to x, we obtain

$$\frac{d}{dt}\left(\frac{t}{2}\|\sqrt{\rho}u\|_{L^2}^2 + \frac{t}{2}\|H\|_{L^2}^2\right) + t\big(\big\|\sqrt{\mu(\rho)}\nabla u\big\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big) = \frac{1}{2}\big(\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2\big).$$

Integrating the above inequality over $(0, t)$, one easily deduces

$$\frac{t}{2}\big(\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2\big) + \int_0^t s\big(\big\|\sqrt{\mu(\rho)}\nabla u(s)\big\|_{L^2}^2 + \big\|\nabla H(s)\big\|_{L^2}^2\big)\,ds$$

$$= \frac{1}{2}\int_0^t \big(\big\|\sqrt{\rho}u(s)\big\|_{L^2}^2 + \big\|H(s)\big\|_{L^2}^2\big)\,ds \le C,$$

where we have used (71) in the last inequality. Therefore, we complete the proof of Lemma 19. □

Next, we improve the regularity on H.

Lemma 20 *Suppose that (ρ, u, H, P) is the unique local strong solution to (1) on $[0, T]$; with the initial data (ρ_0, u_0, H_0), we have*

$$\sup_{0\le t\le T} \|H\|_{L^4}^4 + \int_0^T \|H\cdot\nabla H\|_{L^2}^2\,dt \le C \tag{72}$$

and

$$\sup_{0\le t\le T} t\|H\|_{L^4}^4 + \int_0^T t\|H\cdot\nabla H\|_{L^2}^2\,dt \le C. \tag{73}$$

Proof The proof of (72) is the same as that of (21). It remains to show (73).

First, multiplying $(1)_3$ by $4|H|^2H$, then integrating the resultant equation over Ω, and using integration by parts, after simple calculations, we can obtain

$$\frac{d}{dt}\|H\|_{L^4}^4 + \|H \cdot \nabla H\|_{L^2}^2 \le C\|H\|_{L^4}^4\|\nabla u\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2, \tag{74}$$

where we have used (72) in the last inequality. Multiplying (74) by t, we have

$$\frac{d}{dt}t\|H\|_{L^4}^4 + t\|H \cdot \nabla H\|_{L^2}^2 \le Ct\|\nabla u\|_{L^2}^2 + \|H\|_{L^4}^4.$$

Integrating the above inequality over $(0,t)$, we have

$$t\|H\|_{L^4}^4 + \int_0^t s\|H \cdot \nabla H\|_{L^2}^2\, ds$$
$$\le C\int_0^t s\|\nabla u\|_{L^2}^2\, ds + C\int_0^t \|H\|_{L^2}^2\|\nabla H\|_{L^2}^2\, ds.$$

Therefore, it follows from (67) and (68) that we conclude (73). Hence, we complete the proof of (73) and we finish the proof of Lemma 20.   □

The next result is really the same as (26), which we only write down here without a detailed proof.

Lemma 21 *Suppose that (ρ, u, H, P) is the unique local strong solution to (1) on $[0, T]$ and*

$$\sup_{0 \le t \le T} \left\| \nabla \mu\big(\rho(t)\big) \right\|_{L^q} \le 1.$$

Then we have

$$\|\nabla u\|_{H^1} \le C\|\sqrt{\rho} u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2} + C\|H \cdot \nabla H\|_{L^2}. \tag{75}$$

Next, we deduce some time-weighted estimates for L^2-norms of ∇u and ∇H.

Lemma 22 *Suppose that (ρ, u, H, P) is the unique local strong solution to (1) on $[0, T]$ and satisfies*

$$\sup_{0 \le t \le T} \left\| \nabla \mu\big(\rho(t)\big) \right\|_{L^q} \le 1.$$

Then we have

$$\sup_{0 \le t \le T} t^\alpha \big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big) + \int_0^T t^\alpha \big(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\triangle H\|_{L^2}^2\big)\, dt \le C \tag{76}$$

for every $\alpha \in [0, 2]$.

Proof To prove (76), we only need to verify (76) for $\alpha = 0$ and $\alpha = 2$.

If $\alpha = 0$, the proof is exactly the same as that of (27).

If $\alpha = 2$, multiplying (42) by t^2, we have

$$
\frac{d}{dt} \int t^2 \big(\mu(\rho)|\nabla u|^2 + |\nabla H|^2\big)\, dx + \frac{t^2}{2} \int \big(\rho|u_t|^2 + |\triangle H|^2\big)\, dx
$$

$$
\leq 2t \int \big(\mu(\rho)|\nabla u|^2 + |\nabla H|^2\big)\, dx + Ct^2 \|H \cdot \nabla H\|_{L^2}^2
$$

$$
+ Ct^2 \big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)^2 \big(1 + \log\big(2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)\big)
$$

$$
\leq 2t \int \big(\mu(\rho)|\nabla u|^2 + |\nabla H|^2\big)\, dx + Ct^2 \big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)^2 + \frac{t^2}{4} \|\triangle H\|_{L^2}^2,
$$

where we have used (76) as $\alpha = 0$ and (14). Then the above inequality together with (67), (68), and Gronwall's inequality shows (76) with $\alpha = 2$. Therefore, we complete the proof of Lemma 22. □

That the following result is the same as (43), we only write it down here without proof.

Lemma 23 *Suppose that (ρ, u, H, P) is the unique local strong solution to (1) on $[0, T]$ and satisfies*

$$
\sup_{0 \leq t \leq T} \big\|\nabla \mu\big(\rho(t)\big)\big\|_{L^q} \leq 1.
$$

Then we have

$$
\int_0^T \big\|H_t(t)\big\|_{L^2}^2\, dt \leq C. \tag{77}
$$

To proceed, we need the following result.

Lemma 24 *Suppose that (ρ, u, H, P) is the unique local strong solution to (1) on $[0, T]$ and satisfies*

$$
\sup_{0 \leq t \leq T} \big\|\nabla \mu\big(\rho(t)\big)\big\|_{L^q} \leq 1.
$$

Then we have

$$
\int_0^T \big(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2\big)\, dt \leq C, \tag{78}
$$

and also

$$
\int_0^T t\|u\|_{L^\infty}^4\, dt + \int_0^T t^2 \|u\|_{L^\infty}^4\, dt \leq C. \tag{79}
$$

Proof It follows from (75) and Sobolev's inequality that we have

$$
\int_0^T \big(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2\big)\, dt
$$

$$
\leq C \int_0^T \|\nabla u\|_{H^1}^2\, dt
$$

$$\leq C \int_0^T \|\rho u_t\|_{L^2}^2 \, dt + C \int_0^T \|\rho u\|_{L^4}^4 \|\nabla u\|_{L^2}^2 \, dt + C \int_0^T \|H \cdot \nabla H\|_{L^2}^2 \, dt$$

$$\leq C \int_0^T \|\sqrt{\rho} u_t\|_{L^2}^2 \, dt + C \int_0^T \|\nabla u\|_{L^2}^6 \, dt + C \int_0^T \|\nabla H\|_{L^2}^2 \|\triangle H\|_{L^2}^2 \, dt,$$

which together with (67) and (76) with $\alpha = 0$ shows (78).

Next, we consider the first term on the left-hand side of (79). Due to (14), (75) and Poincaré's inequality, we have

$$\int_0^T t \|u\|_{L^\infty}^4 \, dt \tag{80}$$

$$\leq C \int_0^T t \|u\|_{L^2}^2 \|\nabla u\|_{H^1}^2 \, dt$$

$$\leq C \int_0^T t \|\nabla u\|_{L^2}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 \, dt + C \int_0^T t \|\nabla u\|_{L^2}^8 \, dt + C \int_0^T t \|\nabla u\|_{L^2}^2 \|H \cdot \nabla H\|_{L^2}^2 \, dt$$

$$\leq C \Big(\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \Big) \int_0^T t \|\sqrt{\rho} u_t\|_{L^2}^2 \, dt + C \Big(\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^6 \Big) \int_0^T t \|\nabla u\|_{L^2}^2 \, dt$$

$$+ C \Big(\sup_{0 \leq t \leq T} \big(\|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 \big) \Big) \int_0^T t \|\triangle H\|_{L^2}^2 \, dt \leq C,$$

where we have used (68) and (76).

Similarly, we have

$$\int_0^T t^2 \|u\|_{L^\infty}^4 \, dt \leq C \Big(\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \Big) \int_0^T t^2 \|\sqrt{\rho} u_t\|_{L^2}^2 \, dt \tag{81}$$

$$+ C \Big(\sup_{0 \leq t \leq T} \big(t \|\nabla u\|_{L^2}^2 \big) \cdot \|\nabla u\|_{L^2}^4 \Big) \int_0^T t \|\nabla u\|_{L^2}^2 \, dt$$

$$+ C \Big(\sup_{0 \leq t \leq T} \big(\|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 \big) \Big) \int_0^T t^2 \|\triangle H\|_{L^2}^2 \, dt \leq C,$$

where we have used (68) and (76). This completes the proof of the second term on the left-hand side of (79).

Therefore, combining (80) and (81), one obtains (79). Hence, we finish the proof of Lemma 24. □

The following result is the same as (46). Here we only write it down.

Lemma 25 *Suppose that* (ρ, u, H, P) *is the unique local strong solution to* (1) *on* $[0, T]$ *and satisfies*

$$\sup_{0 \leq t \leq T} \big\| \nabla \mu \big(\rho(t) \big) \big\|_{L^q} \leq 1.$$

Then we have

$$\sup_{0 \leq t \leq T} \|\triangle H\|_{L^2}^2 + \int_0^T \|\nabla H_t\|_{L^2}^2 \, dt \leq C. \tag{82}$$

Next, we obtain some time-weighted estimates for $\|\sqrt{\rho}u_t\|_{L^2}$.

Lemma 26 *Suppose that (ρ, u, H, P) is the unique local strong solution to* (1) *on* $[0, T]$ *and satisfies*

$$\sup_{0 \le t \le T} \left\| \nabla \mu\left(\rho(t)\right) \right\|_{L^q} \le 1.$$

Then we have

$$\sup_{0 \le t \le T} t^\beta \|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t^\beta \|\nabla u_t\|_{L^2}^2 \, dt \le C(\beta) \tag{83}$$

for every $\beta \in [1, 2]$.

Proof It is enough to verify (83) for $\beta = 1$ and $\beta = 2$.

If $\beta = 1$, multiplying (59) by t, then we have

$$\frac{d}{dt} \int t\rho |u_t|^2 \, dx + \int t\mu(\rho) |\nabla u_t|^2 \, dx$$

$$\le \int \rho |u_t|^2 \, dx + Ct\|\nabla u\|_{L^2}^8 + Ct\|\nabla u\|_{L^2}^2 \|H \cdot \nabla H\|_{L^2}^2 + Ct\|H\|_{L^4}^2 \|\nabla H_t\|_{L^2}^2$$

$$+ Ct\left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^4 + \|\sqrt{\rho}u_t\|_{L^2}^2 \right) \|\sqrt{\rho}u_t\|_{L^2}^2$$

$$+ Ct\|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^6 + Ct\|H \cdot \nabla H\|_{L^2}^2 + Ct\|\nabla u\|_{L^2}^4 + Ct\|u\|_{L^\infty}^4 \|\nabla u\|_{L^2}^2$$

$$+ Ct\|H \cdot \nabla H\|_{L^2}^2 + Ct\|u\|_{L^\infty}^2 \|H\|_{L^2} \|\nabla H\|_{L^2}^2 \|\triangle H\|_{L^2} + Ct\|H \cdot \nabla H\|_{L^2}^2,$$

which together with Gronwall's inequality shows

$$\sup_{0 \le t \le T} t\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t\|\nabla u_t\|_{L^2}^2 \, dt \le C, \tag{84}$$

due to (67), (68), (73), (76), (79), and (82).

Furthermore, if $\beta = 2$, multiplying (59) by t^2 shows

$$\frac{d}{dt} \int t^2 \rho |u_t|^2 \, dx + \int t^2 \mu(\rho) |\nabla u_t|^2 \, dx$$

$$\le \int t\rho |u_t|^2 \, dx + Ct^2\|\nabla u\|_{L^2}^8 + Ct\|\nabla u\|_{L^2}^2 \|H \cdot \nabla H\|_{L^2}^2 + Ct^2\|H\|_{L^4}^2 \|\nabla H_t\|_{L^2}^2$$

$$+ Ct^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^4 + \|\sqrt{\rho}u_t\|_{L^2}^2 \right) \|\sqrt{\rho}u_t\|_{L^2}^2$$

$$+ Ct^2\|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^6 + Ct^2\|H \cdot \nabla H\|_{L^2}^2 + Ct^2\|\nabla u\|_{L^2}^4 + Ct^2\|u\|_{L^\infty}^4 \|\nabla u\|_{L^2}^2$$

$$+ Ct^2\|H \cdot \nabla H\|_{L^2}^2 + Ct^2\|u\|_{L^\infty}^2 \|H\|_{L^2} \|\nabla H\|_{L^2}^2 \|\triangle H\|_{L^2} + Ct^2\|H \cdot \nabla H\|_{L^2}^2.$$

With estimates (67), (68), (73), (76), (79), and (82) in hands, we can show the estimate (83) with $\beta = 2$ by Gronwall's inequality. Hence, we finish the proof of Lemma 26.　　　□

The next lemma is crucial to deducing the higher order estimates for the density.

Lemma 27 *Suppose that (ρ, u, H, P) is the unique local strong solution to* (1) *on* $[0, T]$ *and satisfies*

$$\sup_{0 \leq t \leq T} \left\| \nabla \mu \big(\rho(t) \big) \right\|_{L^q} \leq 1.$$

Then we have

$$\int_0^T \|\nabla u\|_{L^\infty} \, dt \leq C. \tag{85}$$

Proof Select some r satisfying $2 < r < \min\{3, q\}$, due to (12),

$$\int_0^T \|\nabla u\|_{L^\infty} \, dt \leq C \int_0^T \|\nabla u\|_{W^{1,r}} \, dt \tag{86}$$

$$\leq C \int_0^T \|\rho u_t\|_{L^3} \, dt + C \int_0^T \|\rho u \cdot \nabla u\|_{L^3} \, dt + C \int_0^T \|H \cdot \nabla H\|_{L^3} \, dt.$$

Next, due to interpolation inequality and Poincaré's inequality, we have

$$\|\rho u_t\|_{L^3} \leq C \|\rho u_t\|_{L^2}^{\frac{1}{2}} \|\rho u_t\|_{L^6}^{\frac{1}{2}} \leq C \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}},$$

from which we have

$$\int_0^T \|\rho u_t\|_{L^3} \, dt \leq C \int_0^T t^{-\frac{3}{8}} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \cdot t^{\frac{3}{8}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \, dt$$

$$\leq C \left[\int_0^T t^{-\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{2}{3}} \, dt \right]^{\frac{3}{4}} \cdot \left[\int_0^T t^{\frac{2}{3}} \|\nabla u_t\|_{L^2}^2 \, dt \right]^{\frac{1}{4}}.$$

If $0 \leq T \leq 1$, taking $\beta = 1$ or $\beta = \frac{3}{2}$ in (83), we can deduce

$$\int_0^T \|\rho u_t\|_{L^3} \, dt \leq C \Big(\sup_{0 \leq t \leq T} t \|\sqrt{\rho} u_t\|_{L^2}^2 \Big)^{\frac{1}{4}} \left[\int_0^T t^{-\frac{5}{6}} \, dt \right]^{\frac{3}{4}} \left[\int_0^T t^{\frac{3}{2}} \|\nabla u_t\|_{L^2}^2 \, dt \right]^{\frac{1}{4}} \leq C.$$

As for $T > 1$, taking $\beta = 2$ in (83), one can also obtain

$$\int_0^T \|\rho u_t\|_{L^3} \, dt \leq C \int_0^1 \|\rho u_t\|_{L^3} \, dt + \int_1^T \|\rho u_t\|_{L^3} \, dt$$

$$\leq C \left[\int_0^1 t^{-\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{2}{3}} \, dt \right]^{\frac{3}{4}} \cdot \left[\int_0^1 t^{\frac{2}{3}} \|\nabla u_t\|_{L^2}^2 \, dt \right]^{\frac{1}{4}}$$

$$+ C \left[\int_1^T t^{-\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{2}{3}} \, dt \right]^{\frac{3}{4}} \cdot \left[\int_1^T t^{\frac{2}{3}} \|\nabla u_t\|_{L^2}^2 \, dt \right]^{\frac{1}{4}}$$

$$\leq C \left[\int_0^1 t^{-\frac{5}{6}} \, dt \right]^{\frac{3}{4}} \cdot \left[\int_0^1 t^{\frac{2}{3}} \|\nabla u_t\|_{L^2}^2 \, dt \right]^{\frac{1}{4}}$$

$$+ C \Big(\sup_{1 < t \leq T} t^2 \|\sqrt{\rho} u_t\|_{L^2}^2 \Big)^{\frac{1}{4}} \left[\int_1^T t^{-\frac{7}{6}} \, dt \right]^{\frac{3}{4}} \cdot \left[\int_1^T t^{\frac{2}{3}} \|\nabla u_t\|_{L^2}^2 \, dt \right]^{\frac{1}{4}}$$

$$\leq C.$$

Therefore, no matter $0 \leq T \leq 1$ or $T \geq 1$, combining the above two inequalities, we show that

$$\int_0^T \|\rho u_t\|_{L^3}\, dt \leq C, \tag{87}$$

and we emphasize that C is independent of T.

Then, utilizing (78) and Hölder's inequality, we have

$$\int_0^T \|\rho u \cdot \nabla u\|_{L^3}\, dt \leq C \int_0^T \|u\|_{L^\infty} \|\nabla u\|_{L^3}\, dt \tag{88}$$

$$\leq C \left(\int_0^T \|u\|_{L^\infty}^2\, dt \right)^{\frac{1}{2}} \left(\int_0^T \|\nabla u\|_{L^3}^2\, dt \right)^{\frac{1}{2}}$$

$$\leq C.$$

Finally, we have

$$\int_0^T \|H \cdot \nabla H\|_{L^3}\, dt \leq C \int_0^T \|\triangle H\|_{L^2}^2\, dt \leq C, \tag{89}$$

where we have used Sobolev's inequality and (76) with $\alpha = 0$.

Thus, inserting (87), (88), and (89) into (86), we complete the proof of (85). Hence, we finish the proof of Lemma 27.　　　　　　　　　　　　　　　　　　　　　　　□

Now, we close the estimates for $\nabla \mu(\rho)$.

Lemma 28 *Suppose that (ρ, u, H, P) is the unique local strong solution to* (1) *on* $[0, T]$ *and*

$$\sup_{0 \leq t \leq T} \left\| \nabla \mu(\rho) \right\|_{L^q} \leq 1.$$

There exists some positive number ε_0 depending only on Ω, q, $\overline{\rho}$, $\underline{\mu}$, $\overline{\mu}$, $\|u_0\|_{H^1}$ and $\|H_0\|_{H^1}$ such that if

$$\left\| \nabla \mu(\rho_0) \right\|_{L^q} \leq \varepsilon_0,$$

then we have

$$\sup_{0 \leq t \leq T} \left\| \nabla \mu(\rho) \right\|_{L^q} \leq \frac{1}{2}, \tag{90}$$

where ε_0 is independent of the time T.

Proof Taking the operator ∂_{x_i} $(i = 1, 2)$ to the renormalized mass equation (30), we have

$$\left(\partial_i \mu(\rho) \right)_t + \left(\partial_i u \cdot \nabla \right) \mu(\rho) + u \cdot \nabla \left(\partial_i \mu(\rho) \right) = 0.$$

Then, multiplying the above equality by $|\partial_i \mu(\rho)|^{q-2} \partial_i \mu(\rho)$, then integrating the resultant equation, and using integration by parts, we can obtain

$$\frac{d}{dt} \left\| \nabla \mu(\rho) \right\|_{L^q}^q \leq C \|\nabla u\|_{L^\infty} \left\| \nabla \mu(\rho) \right\|_{L^q}^q,$$

from which together with Gronwall's inequality and (85), we have

$$\left\|\nabla\mu(\rho)(t)\right\|_{L^q} \leq C\left\|\nabla\mu(\rho_0)\right\|_{L^q} \cdot \exp\left\{\int_0^T \|\nabla u\|_{L^\infty}\,dt\right\} \leq C_2\left\|\nabla\mu(\rho_0)\right\|_{L^q},$$

where C_2 is independent of T.

Therefore, let $\varepsilon_0 = 1/C_2$, then we conclude (90). Hence we complete the proof of Lemma 28. □

At last, we have the following higher order estimates, which can be obtained similarly as those in Sect. 3. Hence, we only write them down here without details.

Lemma 29 *Suppose that* (ρ, u, H, P) *is the unique local strong solution to* (1) *on* $[0, T]$ *and*

$$\sup_{0 \leq t \leq T}\left\|\nabla\mu(\rho)\right\|_{L^q} \leq 1.$$

Then we have

$$\sup_{0 \leq t \leq T}\left(\left\|\rho(t)\right\|_{W^{1,q}} + \left\|\rho_t(t)\right\|_{L^q} + \left\|u(t)\right\|_{H^2} + \left\|H(t)\right\|_{H^2} + \left\|\sqrt{\rho}u_t\right\|_{L^2}\right.$$
$$\left. + \left\|H_t(t)\right\|_{L^2}\right) \leq \overline{C} \tag{91}$$

and

$$\int_0^T \left(\|\nabla u\|_{W^{1,p}}^2 + \|\nabla H\|_{W^{1,p}}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2\right) dt \leq \overline{C}.$$

Here we emphasize that $\overline{C}$ *may depend on* T, μ *and the initial data.*

Proof of Theorem 3　With the above estimates obtained in hand, we can complete the proof of Theorem 3. Due to Theorem 1, there exists a positive time $T^* > 0$ such that the inhomogeneous incompressible MHD system (1) has a unique local strong solution (ρ, u, H, P) on $[0, T^*]$, and T^* depends on $\|\rho_0\|_{W^{1,q}}$, $\|\nabla u_0\|_{H^1}$, $\|\nabla H_0\|_{H^1}$, $\|g\|_{L^2}$, and μ, where g is the function showed in (6). Our aim is to extend the local strong solution to be a global one.

Because of $\|\nabla\mu(\rho_0)\|_{L^q} \leq \varepsilon_0 \leq 1/2$ and the continuity of $\nabla\mu(\rho)$ in L^q, there is $T_1 \in (0, T^*)$ such that

$$\sup_{0 \leq t \leq T_1}\left\|\nabla\mu(\rho)(t)\right\|_{L^q} \leq 1.$$

Set

$$T^* = \sup\left\{T \mid (\rho, u, H, P) \text{ is a strong solution to (1) on } [0, T]\right\} \tag{92}$$

and

$$T_1^* = \sup\left\{T \mid (\rho, u, H, P) \text{ is a strong solution to (1) on } [0, T]\right.$$
$$\left. \text{and } \sup_{0 \leq t \leq T}\left\|\nabla\mu(\rho)\right\|_{L^q} \leq 1\right\}.$$

Then $T_1^* \geq T_1 > 0$. Noticing the result obtained in Lemma 28, one can easily confirm that

$$T^* = T_1^*.$$

Next, we claim that

$$T^* = \infty. \tag{93}$$

Otherwise, if $T^* < \infty$, then by (83) and (91) we have $\sqrt{\rho} u_t + \sqrt{\rho} u \cdot \nabla u \in L^2$. Thus, Theorem 1 implies that there is some $T^{**} > T^*$ such that the solution (ρ, u, H, P) exists on $[0, T^{**}]$, which contradicts (92). Therefore, (93) holds. Hence, we complete the proof of Theorem 3. □

5 Conclusion

In this paper, we mainly prove the global existence of nonhomogeneous incompressible MHD in two dimensions with the density-dependent viscosity in the bounded domain. Meanwhile, similar results could also be obtained by the same method to the periodic domain and Cauchy problem with the positive constant density at far-field behavior. The Cauchy problem with vacuum at far-field behavior will be a little more complicated due to the lack of $\|u\|_{L^p(\mathbb{R}^2)}$, for more details one can refer to [32–34], where the viscosity is positive constant. It should be pointed out that we borrow some ideas from [21] on the nonhomogeneous incompressible Navier–Stokes equation to obtain our results. However, compared with the previous results [21], the presence of H introduced in this paper causes many troubles. Here we only mention two of them. First, in order to control H^2-norm of u by using (11), we need to require that the term $H \cdot \nabla H$ appears on the right-hand side and should be bounded in $L^2(0, T; L^2)$. These requirements bring us many troubles. To meet the requirements, we multiply $(1)_3$ by $4|H|^2 H$, perform integration by parts, then it leads to (21) after delicate estimates. Second, to complete the proof of (27), the terms $H_t \cdot \nabla u \cdot H$ and $H \cdot \nabla u \cdot H_t$ need to be bounded due to the lack of any L^p bound of H_t. The requirements mentioned above also bring us troubles. To overcome the difficulties, we use $(1)_3$ and finally get desired a priori estimates.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by MLS. MLS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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