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Some set-valued and multi-valued contraction results in fuzzy cone metric spaces

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Abstract
This paper aims to present the concept of multi-valued mappings in fuzzy cone metric spaces and prove some basic lemmas, a Hausdorff metric, and fixed point results for set-valued fuzzy cone-contraction and for multi-valued fuzzy cone-contraction mappings. We prove a fixed point theorem for multi-valued rational type fuzzy cone-contractions in fuzzy cone metric spaces. Our results extend and improve some results given in the literature.

MSC: 47H10; 54H25

Keywords: Fixed point; Fuzzy cone metric space; Hausdorff metric; Contraction conditions

1 Introduction

Huang et al. [1] introduced the concept of cone metric spaces by using an ordered Banach space instead of a real number set and proved some fixed point results under cone contraction conditions. After the publication of this article, a number of researchers contributed their ideas to the problems on cone metric spaces by using different contractive type mappings and spaces (see, e.g., [2–11] and the references therein).

Kramosil et al. [12] introduced a fuzzy metric space (FM-space) by using the notion of a fuzzy set and some more notions derived from the one in ordered. These researchers have compared the fuzzy metric notion with the statistical metric space and proved that both conceptions are equivalent in some cases. Later on, the modified form of the metric fuzziness was given by George et al. in [13] by using the continuous t-norm. After that, a number of authors have studied and contributed their ideas to the problems on FM-spaces. Some of their results can be found in [14–25] and the references therein.

Lopez et al. [26] introduced the Hausdorff fuzzy metric on a compact set for a given FM-space and proved some properties for a Hausdorff fuzzy metric. Kiany et al. [19] proved some fixed point results for set-valued mappings and an endpoint theorem in FM-spaces by using contraction conditions. Some other properties and fixed theorems on multi-valued mappings in FM-spaces can be found in [27–29].

The concept of a fuzzy cone metric space (FCM-space) was given by Oner et al. in [30]. They established some properties and a fuzzy cone Banach principle theorem. Some more topological properties, fixed point theorems, and common fixed point theorems in FCM-spaces can be found in [31–37].

In this paper, we introduce the concept of multi-valued mappings in FCM-spaces and prove some basic lemmas and a Hausdorff metric in FCM-spaces. Our result extends and improves the result of Kiany et al. [19] and presents a set-valued fuzzy cone contraction theorem in FCM-spaces. Moreover, we present some fixed point results via multi-valued fuzzy cone contractions in FCM-spaces by extending and improving the result of Ali et al. [27] and a rational type multi-valued fuzzy cone contraction theorem.

2 Preliminaries

Definition 2.1 ([38]) A binary operation $*:[0,1]^2 \to [0,1]$ is called a continuous t-norm if:

(i) $*$ is associative, commutative, and continuous;
(ii) $\forall a_0, a_1, b_0, b_1 \in [0,1]$, then $1 * a_0 = a_0$, while $a_0 * a_1 \leq b_0 * b_1$, whenever $a_0 \leq b_0$ and $a_1 \leq b_1$.

The basic continuous t-norms are minimum, the product and the Lukasiewicz t-norms are defined, respectively, as follows (see [38]):

$$a_0 * b_0 = \min\{a_0, b_0\}, \qquad a_0 * b_0 = a_0 b_0, \quad \text{and} \quad a_0 * b_0 = \max\{a_0 + b_0 - 1, 0\}.$$

Throughout this paper, a set of natural numbers is denoted by $\mathbb{N}$ and a real Banach space is denoted by $\mathbb{E}$. θ represents the zero element of $\mathbb{E}$.

Definition 2.2 ([1]) A subset $P \subset \mathbb{E}$ is known as a cone if

(i) $P \neq \emptyset$, closed, and $P \neq \{\theta\}$;
(ii) If $a_0, b_0 \geq 0$ and $\mu, \nu \in P$, then $a_0\mu + b_0\nu \in P$;
(iii) If both $-\mu, \mu \in P$, then $\mu = \theta$.

A partial ordering "$\preceq$" on $P \subset \mathbb{E}$ is defined by $\mu \preceq \nu$ if and only if $\nu - \mu \in P$. $\mu \prec \nu$ stands for $\mu \preceq \nu$ and $\mu \neq \nu$, while $\mu \ll \nu$ stands for $\nu - \mu \in \text{int}(P)$ and all cones have nonempty interior.

Definition 2.3 ([30]) A 3-tuple $(U, F_m, *)$ is known as an FCM-space if $P \subset \mathbb{E}$ is a cone, U is an arbitrary set, $*$ is a continuous t-norm, and F_m is a fuzzy set on $U \times U \times \text{int}(P)$ satisfying the following;

(i) $F_m(\mu, \nu, t) > 0$, and $F_m(\mu, \nu, t) = 1$ if and only if $\mu = \nu$;
(ii) $F_m(\mu, \nu, t) = F_m(\nu, \mu, t)$;
(iii) $F_m(\mu, \omega, t) * F_m(\omega, \nu, s) \leq F_m(\mu, \nu, t + s)$;
(iv) $F_m(\mu, \nu, .) : \text{int}(P) \to [0,1]$ is continuous

for all $\mu, \nu, \omega \in U$ and $t, s \in \text{int}(P)$.

Definition 2.4 ([30]) Let $(U, F_m, *)$ be an FCM-space, $\mu \in U$, and (μ_n) be a sequence in U. Then

(i) (μ_n) is said to converge to μ if, for $t \gg \theta$ and $0 < r < 1$, there exists $n_1 \in \mathbb{N}$ such that

$$F_m(\mu_n, \mu, t) > 1 - r, \quad \forall n \geq n_1.$$

This can be written as $\lim_{n\to\infty} \mu_n = \mu$ or $\mu_n \to \mu$, as $n \to \infty$.

(ii) (μ_n) is said to be a Cauchy sequence if, for $t \gg \theta$ and $0 < r < 1$, there exists $n_1 \in \mathbb{N}$ such that

$$F_m(\mu_m, \mu_n, t) > 1 - r, \quad \forall m, n \geq n_1.$$

(iii) $(U, F_m, *)$ is complete if every Cauchy sequence is convergent in U.

Lemma 2.5 ([30]) *Let $(U, F_m, *)$ be an FCM-space. The following statements hold*:

(1) *Let $\mu \in U$ and (μ_n) be a sequence in U. Then $\mu_n \to \mu$ if and only if* $\lim_{n\to\infty} F_m(\mu_n, \mu, t) = 1$ *for $t \gg \theta$.*

(2) *An open ball $B(\mu_0, r, t)$ with center μ_0 and radius $0 < r < 1$ can be defined as follows for $t \gg \theta$*:

$$B(\mu_0, r, t) = \{\mu \in U : F_m(\mu_0, \mu, t) > 1 - r\}.$$

Let

$$T_{fc} = \{A \subset U : \mu_0 \in A \text{ iff } \exists 0 < r < 1 \text{ and } t \gg \theta \text{ such that } B(\mu_0, r, t) \subset A\}.$$

Then T_{fc} is a topology on U.

We recall the following definitions given in [27].

Definition 2.6 Let $(U, F_m, *)$ be an FCM-space;

(i) A function $g : U \to \mathbb{R}$ is said to be lower semi-continuous if, for any $(\mu_i) \subset U$ and $\mu \in U$, $\mu_i \to \mu$ implies $g(\mu) \leq \limsup_{i\to\infty} g(\mu_i)$.

(ii) A function $g : U \to \mathbb{R}$ is said to be upper semi-continuous if, for any $(\mu_i) \subset U$ and $\mu \in U$, $\mu_i \to \mu$ implies $g(\mu) \geq \limsup_{i\to\infty} g(\mu_i)$.

(iii) A multi-valued mapping $G : U \to 2^U$ (2^U is the collection of all nonempty subsets of a set U) is called upper semi-continuous if, for any $\mu \in U$ and a neighborhood B of $G(\mu)$, there is a neighborhood A of μ such that, for any $\nu \in A$, we have $G(\nu) \subset B$.

(iv) A multi-valued mapping $G : U \to 2^U$ is said to be lower semi-continuous if, for any $\mu \in U$ and a neighborhood B, $G(\mu) \cap B \neq \emptyset$, there is a neighborhood A of μ such that, for any $\nu \in A$, we have $G(\nu) \cap B \neq \emptyset$.

Definition 2.7 Assume that $(U, F_m, *)$ is an FCM-space, $\mu \in U$, and $(\mu_i)_{i\in\mathbb{N}}$ is a sequence in U. Then:

(i) a subset $A \subseteq U$ is closed if, for every convergent sequence (μ_i) in A such that $\mu_i \to \mu$, we have $\mu \in A$.

(ii) a subset $A \subseteq U$ is compact if every sequence in A has a convergent subsequence in A.

Throughout this paper, $\mathbb{K}(U)$ represents the set of all compact subsets of a set U and $\mathbb{P}(U)$ represents the set of all nonempty subsets of a set U.

3 Some properties and a Hausdorff fuzzy metric in FCM-spaces

Proposition 3.1 *Let $(U, F_m, *)$ be an FCM-space. Then F_m is continuous on $U^2 \times \operatorname{int}(P)$ for every $t \gg \theta$ (i.e., $t \in \operatorname{int}(P)$).*

Proof Let $\mu, \nu \in U$, $t \gg \theta$, and $(\mu_i, \nu_i, t_i)_i$ be a sequence in $X^2 \times \operatorname{int}(P)$ converging to (μ, ν, t). Since $(F_m(\mu_i, \nu_i, t_i))_i$ is a sequence in $(0,1]$, there is a sub-sequence $(\mu_{i_n}, \nu_{i_n}, t_{i_n})_n$ of the sequence $(\mu_i, \nu_i, t_i)_i$ such that $(F_m(\mu_{i_n}, \nu_{i_n}, t_{i_n}))_n$ converges to a point in $[0,1]$. Fix any $\varepsilon > 0$ such that $\varepsilon < \frac{t}{2}$, so there is $i_0 \in \mathbb{N}$ such that $|t - t_i| < \varepsilon$ for all $i \geq i_0$. Then we have

$$\begin{aligned} F_m(\mu_{i_n}, \nu_{i_n}, t_{i_n}) &\geq F_m(\mu_{i_n}, \mu, \varepsilon) * F_m(\mu, \nu, t - 2\varepsilon) * F_m(\nu, \nu_{i_n}, \varepsilon) \\ &\to 1 * F_m(\mu, \nu, t - 2\varepsilon) * 1 = F_m(\mu, \nu, t - 2\varepsilon), \quad \text{as } i \to \infty, t \gg \theta, \end{aligned}$$

and

$$\begin{aligned} F_m(\mu, \nu, t + 2\varepsilon) &\geq F_m(\mu, \mu_{i_n}, \varepsilon) * F_m(\mu_{i_n}, \nu_{i_n}, t_{i_n}) * F_m(\nu_{i_n}, \nu, \varepsilon) \\ &\to 1 * F_m(\mu_{i_n}, \nu_{i_n}, t_{i_n}) * 1 = F_m(\mu_{i_n}, \nu_{i_n}, t_{i_n}), \quad \text{as } i \to \infty, t \gg \theta. \end{aligned}$$

Therefore, by the continuity of the function $t \longmapsto F_m(\mu, \nu, t)$, we can deduce that

$$F_m(\mu, \nu, t) = \lim_{i \to \infty} F_m(\mu_{i_n}, \nu_{i_n}, t_{i_n}) \quad \text{for } t \gg \theta.$$

Thus, F_m is continuous on $U^2 \times \operatorname{int}(P)$. □

Lemma 3.2 *Let $(U, F_m, *)$ be an FCM-space such that*

$$*_{j=i}^{\infty} F_m(\mu, \nu, tb^j) \to 1, \quad \text{as } i \to \infty, \tag{3.1}$$

for all $\mu, \nu \in U$, $t \gg \theta$, and $b > 1$. Let (μ_i) be a sequence in U such that

$$F_m(\mu_i, \mu_{i+1}, at) \geq M(\mu_{i-1}, \mu_i, t)$$

for all $i \in \mathbb{N}$ and $a \in (0,1)$. Then (μ_i) is a Cauchy sequence in U.

Proof For every $i \in \mathbb{N}$ and $t \gg \theta$, we have that

$$F_m(\mu_i, \mu_{i+1}, t) \geq F_m\left(\mu_{i-1}, \mu_i, \frac{1}{a} t\right) \geq F_m\left(\mu_{i-2}, \mu_{i-1}, \frac{1}{a^2} t\right) \geq \cdots \geq F_m\left(\mu_0, \mu_1, \frac{1}{a^i} t\right).$$

Thus, for all $i \in \mathbb{N}$ and $t \gg \theta$, we have

$$F_m(\mu_i, \mu_{i+1}, t) \geq F_m\left(\mu_0, \mu_1, \frac{1}{a^i} t\right).$$

Now, we choose a constant $b > 1$ and $l \in \mathbb{N}$ such that $ab < 1$ and $\sum_{j=l}^{\infty} \frac{1}{b^j} = \frac{1/b^l}{1-(1/b)} < 1$. Hence, for $k \geq i$ and $t \gg \theta$, we have that

$$F_m(\mu_i, \mu_k, t)$$

$$\geq F_m\left(\mu_i,\mu_k,\left(\frac{1}{b^l}+\frac{1}{b^{l+1}}+\cdots+\frac{1}{b^{l+k}}\right)t\right)$$
$$\geq F_m\left(\mu_i,\mu_{i+1},\frac{1}{b^l}t\right)*F_m\left(\mu_{i+1},\mu_{i+2},\frac{1}{b^{l+1}}t\right)*\cdots*F_m\left(\mu_{k-1},\mu_k,\frac{1}{b^{l+k}}t\right)$$
$$\geq F_m\left(\mu_0,\mu_1,\frac{1}{a^{i-1}b^l}t\right)*F_m\left(\mu_0,\mu_1,\frac{1}{a^i b^{l+1}}t\right)*\cdots*F_m\left(\mu_0,\mu_1,\frac{1}{a^{k-2}b^{l+k-i-2}}t\right)$$
$$\geq F_m\left(\mu_0,\mu_1,\frac{1}{(ab)^{i-1}}t\right)*F_m\left(\mu_0,\mu_1,\frac{1}{(ab)^i}t\right)*\cdots*F_m\left(\mu_0,\mu_1,\frac{1}{(ab)^{k-2}}t\right)$$
$$\geq *_{j=i}^{\infty}F_m\left(\mu_0,\mu_1,\frac{1}{(ab)^{j-1}}t\right)\to 1,\quad \text{as } i\to\infty.$$

This proves that (μ_i) is a Cauchy sequence in U. □

Lemma 3.3 *Let $(U,F_m,*)$ be an FCM-space. Then, for every $\mu\in U$, $A\in\mathbb{K}(U)$ and $t\gg\theta$, there exists $a_0\in A$ such that*

$$F_m(\mu,A,t)=F_m(\mu,a_0,t).$$

Proof Let $\mu\in U$, $A\in\mathbb{K}(U)$, and $t\gg\theta$. Then, by Proposition 3.1, the function $\nu\longmapsto F_m(\mu,\nu,t)$ is continuous. Thus, by the compactness of A, there exists $a_0\in A$ such that

$$\sup_{a\in A}F_m(\mu,a,t)=F_m(\mu,a_0,t),$$

that is,

$$F_m(\mu,A,t)=F_m(\mu,a_0,t).$$ □

Lemma 3.4 *Let $(U,F_m,*)$ be an FCM-space. Then, for all $\mu\in U$ and $A\in\mathbb{K}(U)$, the function $t\longmapsto F_m(\mu,A,t)$ is continuous on $\operatorname{int}(P)$, where $t\gg\theta$.*

Proof Since $F_m(\mu,A,t)=\sup_{a_0\in A}F_m(\mu,a_0,t)$ and for every $a_0\in A$, the function $t\longmapsto F_m(\mu,a_0,t)$ is continuous on $\operatorname{int}(P)$, it follows that $t\longmapsto F_m(\mu,A,t)$ is lower semi-continuous on $\operatorname{int}(P)$. Now, we prove that $t\longmapsto F_m(\mu,A,t)$ is upper semi-continuous on $\operatorname{int}(P)$.

Let $t\gg\theta$ and $(t_j)_j$ be a sequence in $\operatorname{int}(P)$ which converges to t. By Lemma 3.3, there exists $a_j\in A$ such that, for all $j\in\mathbb{N}$,

$$F_m(\mu,A,t_j)=F_m(\mu,a_j,t_j).$$

Since $A\in\mathbb{K}(U)$, there are a subsequence $(a_{j_n})_n$ of the sequence $(a_j)_j$ and a point $a^*\in A$ such that $a_{j_n}\to a^*$ in $(U,F_m,*)$. Hence,

$$F_m(\mu,a_{j_n},t_{j_n})\to F_m\big(\mu,a^*,t\big),\quad \text{as } n\to\infty,$$

for $t\gg\theta$. Now, by Proposition 3.1, we have that

$$F_m(\mu,A,t_{j_n})\to F_m\big(\mu,a^*,t\big)\leq F_m(\mu,A,t),\quad \text{as } n\to\infty,$$

for $t \gg \theta$. Consequently, the function $t \longmapsto F_m(\mu, A, t)$ is upper semi-continuous on $\operatorname{int}(P)$, which concludes the required proof. □

Lemma 3.5 *Let $(U, F_m, *)$ be an FCM-space. Then, for every $A \in \mathbb{K}(U)$ and $B \in \mathbb{P}(U)$, there exists $a^* \in A$ such that*

$$\inf_{a_0 \in A} F_m(a_0, B, t) = F_m\bigl(a^*, B, t\bigr)$$

for $t \gg \theta$.

Proof By putting $\beta = \inf_{a_0 \in A} F_m(a_0, B, t)$, there is a sequence $(a_j)_j$ in A such that $\beta + \frac{1}{j} > F_m(a_j, B, t)$ for all $j \in \mathbb{N}$. Since $A \in \mathbb{K}(U)$, there are a subsequence $(a_{j_n})_n$ of $(a_j)_j$ and a point $a^* \in A$ such that $a_{j_n} \to a^*$ in $(U, F_m, *)$. Here, we choose an arbitrary point $b_0 \in B$. Now, by Proposition 3.1, we have

$$F_m(a_{j_n}, b_0, t) \to F_m\bigl(a^*, b_0, t\bigr), \quad \text{as } n \to \infty,$$

for $t \gg \theta$. Since for all $n \in \mathbb{N}$ and $\beta + \frac{1}{j_n} > F_m(a_{j_n}, b_0, t)$. Then, by taking the limit $n \to \infty$, we get

$$\beta \geq F_m\bigl(a^*, b_0, t\bigr) \quad \Rightarrow \quad \beta = F_m\bigl(a^*, b_0, t\bigr) \quad \text{for } t \gg \theta.$$ □

Proposition 3.6 *Let $(U, F_m, *)$ be an FCM-space. Then, for every $A, B \in \mathbb{K}(U)$, $t \longmapsto \inf_{a^* \in A} F_m(a^*, B, t)$ is a continuous function in $\operatorname{int}(P)$, where $t \gg \theta$.*

Proof By Lemma 3.4, $t \longmapsto F_m(a^*, B, t)$ is a continuous function in $\operatorname{int}(P)$. Therefore, $t \longmapsto \inf_{a^* \in A} F_m(a^*, B, t)$ is an upper semi-continuous function in $\operatorname{int}(P)$.

Now, we prove that $t \longmapsto \inf_{a^* \in A} F_m(a^*, B, t)$ is lower semi-continuous in $\operatorname{int}(P)$. Let $(t_j)_j$ be any sequence in $\operatorname{int}(P)$ such that $(t_j)_j \to t$ in $\operatorname{int}(P)$, where $t \gg \theta$. By Lemma 3.5, there exists $a_j \in A$ such that, for all $j \in \mathbb{N}$,

$$F_m(a_j, B, t_j) = \inf_{a^* \in A} F_m\bigl(a^*, B, t_j\bigr).$$

Since $A \in \mathbb{K}(U)$, there are a subsequence $(a_{j_n})_n$ of $(a_j)_j$ and a point $a_1 \in A$ such that $a_{j_n} \to a_1$ in $(U, F_m, *)$. Then, by Lemma 3.3, there exists $b_1 \in B$ such that

$$F_m(a_1, b_1, t) = F_m(a_1, B, t) \quad \text{for } t \gg \theta.$$

Now, by Proposition 3.1,

$$F_m(a_{j_n}, b_1, t_{j_n}) \to F_m(a_1, b_1, t), \quad \text{as } n \to \infty.$$

Therefore, for given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$F_m(a_1, b_1, t) < \delta + F_m(a_{j_n}, b_1, t_{j_n}).$$

Hence,

$$\inf_{a^*\in A} F_m\big(a^*,B,t\big) \le F_m(a_1,b_1,t) < \delta + F_m(a_{j_n},B,t_{j_n}) = \delta + \inf_{a^*\in A} F_m\big(a^*,B,t_{j_n}\big)$$

for all $n \ge n_0$. Consequently, $t \longmapsto \inf_{a^*\in A} F_m(a^*,B,t)$ is a lower semi-continuous function in int(P). It completes the proof. □

Remark 3.7 Note that Proposition 3.6 showed that, for any $A,B \in \mathbb{K}(U)$, $t \longmapsto \inf_{b^*\in B} F_m(A,b^*,t)$ is a continuous function in int(P).

Hausdorff fuzzy cone metric on $\mathbb{K}(U)$: Let $(U,F_m,*)$ be an FCM-space. Then we define a function F_H on $\mathbb{K}(U) \times \mathbb{K}(U) \times \text{int}(P)$ by

$$F_H(A,B,t) = \min\Big\{\inf_{b\in B} F_m(A,b,t), \inf_{a\in A} F_m(a,B,t)\Big\} \tag{3.2}$$

for all $A,B \in \mathbb{K}(U)$ and $t \gg \theta$.

Lemma 3.8 *Let* $(U,F_m,*)$ *be an FCM-space,* $\mu \in U$, $A \in \mathbb{K}(U)$, $B \in \mathbb{P}(U)$, *and* $s,t \gg \theta$. *Then*

$$F_m(\mu,B,t+s) \ge F_m(\mu,a_\mu,t) * F_m(a_\mu,b,s),$$

where $a_\mu \in A$ *satisfies* $F_m(\mu,A,t) = F_m(\mu,a_\mu,t)$.

Proof First, we note that an element $a_\mu \in A$ satisfying $F_m(\mu,A,t) = F_m(\mu,a_\mu,t)$ exists by Lemma 3.3. Now, for every $b \in B$, we have that

$$F_m(\mu,B,t+s) \ge F_m(\mu,b,t+s) \ge F_m(\mu,a_\mu,t) * F_m(a_\mu,b,s).$$

Thus, by the continuity of $*$,

$$F_m(\mu,B,t+s) \ge F_m(\mu,a_\mu,t) * F_m(a_\mu,b,s) \quad \text{for } s,t \gg \theta.$$ □

Theorem 3.9 *Assume that* $(U,F_m,*)$ *is an FCM-space. Then* $(\mathbb{K}(U),F_H,*)$ *is an FCM-space.*

Proof Suppose that $A,B,C \in \mathbb{K}(U)$ and $s,t \gg \theta$. Then, by Lemma 3.5, there exist $a^* \in A$ and $b^* \in B$ such that

$$\inf_{a_0\in A} F_m(a_0,B,t) = M\big(a^*,B,t\big)$$

and

$$\inf_{b_0\in B} F_m(A,b_0,t) = F_m\big(A,b^*,t\big)$$

for $t \gg \theta$. Thus, $F_H(A,B,t) > 0$.

In addition, we know that $F_H(A,B,t) = 1$ if and only if $A = B$, and hence F_H is symmetric, that is,

$$F_H(A,B,t) = F_H(B,A,t) \quad \text{for } t \gg \theta.$$

Moreover, we note that, by Lemma 3.8 and by the continuity of $*$, we have that

$$\inf_{a_0\in A} F_m(a_0,C,t+s) \geq \inf_{a_0\in A} F_m(a_0,B,t) * \inf_{a_0\in A} F_m(b_{a_0},C,s)$$

for $s,t \gg \theta$. Since $\{b_{a_0} : a_0 \in A\} \subseteq B$ such that

$$\inf_{a_0\in A} F_m(b_{a_0},C,s) \geq \inf_{b_0\in B} F_m(b_0,C,s)$$

for $s \gg \theta$, we have

$$\inf_{a_0\in A} F_m(a_0,C,t+s) \geq \inf_{a_0\in A} F_m(a_0,B,t) * \inf_{b_0\in B} F_m(b_0,C,s)$$

for $s,t \gg \theta$. Similarly, we get that

$$\inf_{c_0\in C} F_m(A,c_0,t+s) \geq \inf_{b_0\in B} F_m(A,b_0,,t) * \inf_{c_0\in C} F_m(B,c_0,s).$$

It follows that

$$F_H(A,C,t+s) \geq F_H(A,B,t) * F_H(B,C,s).$$

Finally, the continuity of the function $t \longmapsto F_H(A,B,t)$ on the cone is a direct consequence of Proposition 3.6 and Remark 3.7. We conclude that $(\mathbb{K}(U),F_H,*)$ is an FCM-space. □

4 Set-valued mapping results in FCM-spaces

In this section, we prove a fixed point theorem for set-valued mappings in FCM-spaces.

Theorem 4.1 *Let $(U,F_m,*)$ be a complete FCM-space and $G : U \to U$ be a set-valued mapping with nonempty compact values such that, for all $\mu,\nu \in U$ and $t \gg \theta$,*

$$F_H\bigl(G\mu,G\nu,\delta\bigl(d(\mu,\nu,t)\bigr)t\bigr) \geq F_m(\mu,\nu,t) * F_m(\nu,G\mu,t), \tag{4.1}$$

where $\delta : \operatorname{int}(P) \to [0,1)$ satisfies

$$\limsup_{r\to t^+} \delta(r) < 1 \quad \textit{for all } t \in [0,\infty]$$

and $d(\mu,\nu,t) = \frac{t}{F_m(\mu,\nu,t)} - t$. Moreover, we suppose that $(U,F_m,*)$ satisfies (3.1) for some $\mu_0 \in U$ and $\mu_1 \in G\mu_0$. Then G has a fixed point in U.

Proof First, we notice that, if A and B are nonempty compact subsets of a set U and $a \in A$, then by Lemma 3.3, there exists $b \in B$ such that

$$F_H(A,B,t) \leq \sup_{b\in B} F_m(a,b,t) = F_m(a,b,t)$$

for $t \gg \theta$. Thus, given $\delta \leq F_H(A,B,t)$, there exists a point $b \in B$ such that $\delta \leq F_m(a,b,t)$.

Now, let us fix μ_0 in U and $\mu_1 \in G\mu_0$. If $G\mu_0 = G\mu_1$, then $\mu_1 \in G\mu_1$ and μ_1 is a fixed point of G. The proof is completed. Otherwise, we may assume that $G\mu_0 \neq G\mu_1$. Then, from (4.1), we have

$$F_H\bigl(G\mu_0, G\mu_1, \delta\bigl(d(\mu_0,\mu_1,t)\bigr)t\bigr) \geq F_m(\mu_0,\mu_1,t) * F_m(\mu_1, G\mu_0, t) \geq F_m(\mu_0,\mu_1,t)$$

for $t \gg \theta$. Since $\mu_1 \in G\mu_0$ and G is a compact-valued mapping, then again by Lemma 3.3, there exists $\mu_2 \in G\mu_1$ such that

$$\begin{aligned} F_m(\mu_1,\mu_2,t) &\geq F_m\bigl(\mu_1,\mu_2,\delta\bigl(d(\mu_0,\mu_1,t)\bigr)t\bigr) \\ &= \sup_{r\in G\mu_1} F_m\bigl(\mu_1,r,\delta\bigl(d(\mu_0,\mu_1,t)\bigr)t\bigr) \\ &\geq F_H\bigl(G\mu_0,G\mu_1,\delta\bigl(d(\mu_0,\mu_1,t)\bigr)t\bigr) \geq F_m(\mu_0,\mu_1,t) \end{aligned}$$

for $t \gg \theta$. Similarly,

$$F_m(\mu_2,\mu_3,t) \geq F_m(\mu_1,\mu_2,t) \quad \text{for } t \gg \theta.$$

By induction, we choose a sequence $(\mu_n)_{n\geq 0}$ in U such that $\mu_n \in G\mu_{n-1}$. If $G\mu_{n-1} = G\mu_n$ for some n, then $\mu_n \in G\mu_n$, and so μ_n is a fixed point of G. The proof is completed. Otherwise, we may assume that $G\mu_{n-1} \neq G\mu_n$. Then from (4.1) we have

$$\begin{aligned} F_m(\mu_n,\mu_{n+1},t) &\geq F_m\bigl(\mu_n,\mu_{n+1},\delta\bigl(d(\mu_{n-1},\mu_n,t)\bigr)t\bigr) \\ &= \sup_{r\in G\mu_n} F_m\bigl(\mu_n,r,\delta\bigl(d(\mu_{n-1},\mu_n,t)\bigr)t\bigr) \\ &\geq F_H\bigl(G\mu_{n-1},G\mu_n,\delta\bigl(d(\mu_{n-1},\mu_n,t)\bigr)t\bigr) \\ &\geq F_m(\mu_{n-1},\mu_n,t) * F_m(\mu_n,G\mu_{n-1},t) \\ &\geq F_m(\mu_{n-1},\mu_n,t) \quad \text{for } t \gg \theta. \end{aligned}$$

Hence, $(F_m(\mu_n,\mu_{n+1},t))_n$ is a nondecreasing sequence. Thus, $(d(\mu_n,\mu_{n+1},t))_n$ is a positive nonincreasing sequence, and so it is convergent to some constant, say $\xi \geq 0$. Recall that

$$\limsup_{n\to\infty} \delta\bigl(d(\mu_n,\mu_{n+1},t)\bigr) \leq \limsup_{\varepsilon\to t^+} \delta(\varepsilon) < 1. \tag{4.2}$$

Then there are $\beta < 1$ and $n_0 \in \mathbb{N}$ such that

$$\delta\bigl(d(\mu_n,\mu_{n+1},t)\bigr) < \beta, \quad \forall n > n_0, t \gg \theta. \tag{4.3}$$

Since $F_m(\mu,\nu,.)$ is nondecreasing, we have from (4.1) and (4.3) that, for $t \gg \theta$,

$$F_m(\mu_n,\mu_{n+1},\beta t) \geq F_m\big(\mu_n,\mu_{n+1},\delta\big(d(\mu_{n-1},\mu_n,t)\big)t\big) \geq F_m(\mu_{n-1},\mu_n,t).$$

Thus, we get that

$$F_m(\mu_n,\mu_{n+1},\beta t) \geq F_m(\mu_{n-1},\mu_n,t) \quad \text{for } t \gg \theta.$$

Hence, by Lemma 3.2, we conclude that (μ_n) is a Cauchy sequence in U. Since $(U,F_m,*)$ is complete, there exists $u \in U$ such that

$$\lim_{n\to\infty} F_m(\mu_n,u,t) = 1 \quad \text{for } t \gg \theta. \tag{4.4}$$

This implies that

$$\lim_{n\to\infty} d(\mu_n,u,t) = 0 \quad \text{for } t \gg \theta.$$

Therefore,

$$\limsup_{n\to\infty} \delta\big(d(\mu_n,u,t)\big) \leq \limsup_{\varepsilon\to 0^+} \delta(\varepsilon) < 1.$$

Then there exists $\beta < \xi < 1$ such that

$$\limsup_{n\to\infty} \delta\big(d(\mu_n,u,t)\big) < \xi \quad \text{for } t \gg \theta.$$

Now, we have to show that $u \in Gu$. Since $\mu_{n+1} \in G\mu_n$, one writes

$$\begin{aligned}
F_m(\mu_{n+1},Gu,t) &\geq F_H(G\mu_n,Gu,\xi t)\\
&\geq F_H(G\mu_n,Gu,\beta t)\\
&\geq F_H\big(G\mu_n,Gu,\delta\big(d(\mu_n,u,t)\big)t\big)\\
&\geq F_m(\mu_n,u,t) * F_m(u,G\mu_n,t)\\
&\geq F_m(\mu_n,u,t) * F_m(u,\mu_{n+1},t) \to 1 * 1 = 1, \quad \text{as } n \to \infty,
\end{aligned}$$

for $t \gg \theta$. Hence, we get that

$$\lim_{n\to\infty} \sup_{r\in Gu} F_m(\mu_{n+1},r,t) = 1 \quad \text{for } t \gg \theta.$$

Thus, there exists a sequence (r_n) in Gu such that

$$\lim_{n\to\infty} F_m(\mu_n,r_n,t) = 1 \quad \text{for } t \gg \theta. \tag{4.5}$$

Now, by Definition 2.3(iii), we have that

$$F_m(r_n,u,2t) \geq F_m(r_n,\mu_n,t) * F_m(\mu_n,u,t) \quad \text{for } t \gg \theta, \tag{4.6}$$

for each $n \in \mathbb{N}$. By using (4.4), (4.5) together with (4.6), we can get

$$\lim_{n\to\infty} F_m(r_n, u, 2t) = 1 \quad \text{for } t \gg \theta.$$

This implies that $\lim_{n\to\infty} r_n = u$. Since $r_n \to u$ and $r_n \in Gu$, using the fact that Gu is closed and compact, we get $u \in Gu$. □

Without δ mapping directly, we can get the following two corollaries from Theorem 4.1.

Corollary 4.2 *Let $(U, F_m, *)$ be a complete FCM-space and $G : U \to U$ be a set-valued mapping with nonempty compact values such that, for all $\mu, \nu \in U$ and $t \gg \theta$, it satisfies*

$$F_H(G\mu, G\nu, \beta t) \geq F_m(\mu, \nu, t) * F_m(\nu, G\mu, t), \tag{4.7}$$

where $\beta \in (0, 1)$. Furthermore, we assume that $(U, F_m, *)$ satisfies (3.1) for some $\mu_0 \in U$ and $\mu_1 \in G\mu_0$. Then G has a fixed point in U.

Corollary 4.3 *Let $(U, F_m, *)$ be a complete FCM-space and $G : U \to U$ be a set-valued mapping with nonempty compact values such that, for all $\mu, \nu \in U$ and $t \gg \theta$, it satisfies*

$$F_H(G\mu, G\nu, \beta t) \geq F_m(\mu, \nu, t), \tag{4.8}$$

where $\beta \in (0, 1)$. Furthermore, we assume that $(U, F_m, *)$ satisfies (3.1) for some $\mu_0 \in U$ and $\mu_1 \in G\mu_0$. Then G has a fixed point in U.

5 Multi-valued contraction results in FCM-spaces

In this section, we present some fixed point results for multi-valued contractions in FCM-spaces. Further, we present a fixed point theorem for rational type multi-valued contractions. We present some illustrative examples.

Let $G : U \to 2^U$ be a multi-valued map. Consider $g(\mu) = F_m(\mu, G\mu, t)$ for $t \gg \theta$. For $\alpha \in (0, 1)$, we take the set

$$J_\alpha^\mu = \{\nu \in G\mu; F_m(\mu, \nu, t) \geq F_m(\mu, G\mu, \alpha t)\}. \tag{5.1}$$

Theorem 5.1 *Let $(U, F_m, *)$ be a complete FCM-space and $G : U \to \mathbb{K}(U)$ be a multi-valued map. If there exists a constant $\beta \in (0, 1)$ such that, for any $\mu \in U$, there is $\nu \in j_\alpha^\mu$, so that*

$$F_m(\nu, G\nu, \beta t) \geq F_m(\mu, \nu, t) * F_m(\nu, G\mu, t) \tag{5.2}$$

for $t \gg \theta$. Suppose that $(U, M, *)$ verifies (3.1) for some $\mu_0 \in U$. Then G has a fixed point in U, provided $\beta < \alpha$ and g is upper semi-continuous.

Proof Since $G(\mu) \in \mathbb{K}(U)$, by Lemma 3.3, J_α^μ is nonempty for all μ in U and $\alpha \in (0, 1)$. Let us fix μ_0 in U, so there exists $\mu_1 \in J_\alpha^{\mu_0}$, that is, $\mu_1 \in G\mu_0$ such that

$$F_m(\mu_1, G\mu_1, \beta t) \geq F_m(\mu_0, \mu_1, t) * F_m(\mu_1, G\mu_0, t) \geq F_m(\mu_0, \mu_1, t)$$

for $t \gg \theta$. Similarly, for μ_1 in U, there exists $\mu_2 \in J_\alpha^{\mu_1}$, that is, $\mu_2 \in G\mu_1$, which satisfies

$$F_m(\mu_2, G\mu_2, \beta t) \geq F_m(\mu_1, \mu_2, t) * F_m(\mu_2, G\mu_1, t) \geq F_m(\mu_1, \mu_2, t)$$

for $t \gg \theta$. By induction, we obtain a sequence $(\mu_i)_{i\geq 0}$ in U such that there exists $\mu_{i+1} \in J_\alpha^{\mu_i}$, that is, $\mu_{n+1} \in G\mu_i$, which satisfies

$$F_m(\mu_{i+1}, G\mu_{i+1}, \beta t) \geq F_m(\mu_i, \mu_{i+1}, t) * F_m(\mu_{i+1}, G\mu_i, t) \geq F_m(\mu_i, \mu_{i+1}, t) \tag{5.3}$$

for $t \gg \theta$. On the other hand, $\mu_{i+1} \in J_\alpha^{\mu_i}$, which gives that

$$F_m(\mu_i, \mu_{i+1}, t) \geq F_m(\mu_i, G\mu_i, \alpha t) \quad \text{for } t \gg \theta. \tag{5.4}$$

From (5.3) and (5.4), we get that

$$F_m(\mu_{i+1}, G\mu_{i+1}, \beta t) \geq F_m(\mu_i, G\mu_i, \alpha t) \quad \text{for } t \gg \theta,$$

i.e.,

$$F_m(\mu_{i+1}, G\mu_{i+1}, t) \geq F_m\left(\mu_i, G\mu_i, \frac{\alpha}{\beta} t\right) \quad \text{for } t \gg \theta. \tag{5.5}$$

Let $a = \frac{\beta}{\alpha}$, then (5.5) can be expressed as follows:

$$\begin{aligned} F_m(\mu_i, \mu_{i+1}, t) &\geq F_m\left(\mu_{i-1}, \mu_i, \frac{1}{a} t\right) \geq F_m\left(\mu_{i-2}, \mu_{i-1}, \frac{1}{a^2} t\right) \geq \cdots \\ &\geq F_m\left(\mu_0, \mu_1, \frac{1}{a^i} t\right) \end{aligned} \tag{5.6}$$

for $t \gg \theta$, $i \in \mathbb{N}$, and $a \in (0,1)$. Choose a constant $b > 1$ such that $ab < 1$ and $\sum_{n=0}^{\infty} \frac{1}{b^n} < 1$, i.e., $\sum_{n=i}^{j-1} \frac{1}{b^n} < 1$. Then, for all $j > i$, we get that

$$\left(\frac{1}{b^i} + \frac{1}{b^{i+1}} + \cdots + \frac{1}{b^{j-2}} + \frac{1}{b^{j-1}}\right) t < t, \tag{5.7}$$

where $i, j \in \mathbb{N}$. Then we have

$$\begin{aligned} F_m(\mu_i, \mu_j, t) &\geq F_m\left(\mu_i, \mu_j, t\left(\frac{1}{b^i} + \frac{1}{b^{i+1}} + \cdots + \frac{1}{b^{j-2}} + \frac{1}{b^{j-1}}\right)\right) \\ &\geq F_m\left(\mu_i, \mu_{i+1}, \frac{t}{b^i}\right) * F_m\left(\mu_{i+1}, \mu_{i+2}, \frac{t}{b^{i+1}}\right) * \cdots * F_m\left(\mu_{j-1}, \mu_j, \frac{t}{b^{j-1}}\right) \\ &\geq F_m\left(\mu_0, \mu_1, \frac{t}{(ab)^i}\right) * F_m\left(\mu_0, \mu_1, \frac{t}{(ab)^{i+1}}\right) * \cdots * F_m\left(\mu_0, \mu_1, \frac{t}{(ab)^{j-1}}\right) \\ &\geq *_{n=i}^{\infty}\left(F_m\left(\mu_0, \mu_1, \frac{t}{(ab)^n}\right)\right) \to 1, \quad \text{as } i \to \infty, \end{aligned} \tag{5.8}$$

for $t \gg \theta$. By Lemma 3.2, it is proved that (μ_i) is a Cauchy sequence in U. Since U is complete, there exists $\mu \in U$ such that $\mu_i \to \mu$ as $i \to \infty$. In view of (5.3) and (5.4), it is clear that $g(\mu_i) = F_m(\mu_i, G\mu_i, t)$ is an increasing function and converges to 1. Since g is upper semi-continuous, we have

$$1 \geq g(\mu) \geq \limsup_{n\to\infty} g(\mu_n) = 1.$$

This implies that $g(\mu) = 1$, so that $F_m(\mu, G\mu, t) = 1$. Hence, by using Lemma 3.3, we get that $\mu \in G\mu$. Hence, the proof is completed. □

Corollary 5.2 *Let $(U, F_m, *)$ be a complete FCM-space and $G : U \to \mathbb{K}(U)$ be a multi-valued mapping. If there exists a constant $\beta \in (0, 1)$ such that, for any $\mu \in U$, we have $\nu \in j_\alpha^\mu$ with*

$$F_m(\nu, G\nu, \beta t) \geq F_m(\mu, \nu, t) \tag{5.9}$$

for $t \gg \theta$. Suppose that $(U, F_m, *)$ satisfies (3.1) for some $\mu_0 \in U$. Then G has a fixed point in U provided that $\beta < \alpha$ and g is upper semi-continuous.

In a special case, we get the following corollary of Kiany et al. [19].

Corollary 5.3 ([19]) *Let $(U, F_m, *)$ be a complete FCM-space and $G : U \to \mathbb{K}(U)$ be a multi-valued map. Suppose that there exists $\beta \in (0, 1)$ such that*

$$F_H(G\mu, G\nu, \beta t) \geq F_m(\mu, \nu, t) \tag{5.10}$$

for all $\mu, \nu \in U$ and $t \gg \theta$. Moreover, assume that $(U, F_m, *)$ satisfies (3.1) for some $\mu_0 \in U$ and $\mu_1 \in G\mu_0$. Then G has a fixed point in U.

Remark 5.4 Corollary 5.2 is the generalized form of Corollary 5.3. Suppose that G satisfies the conditions of Corollary 5.3, and if g is upper semi-continuous, then from (5.10) we obtain, for any $\mu \in U$, $\nu \in G\mu$, and $t \gg \theta$,

$$F_m(\nu, G\nu, \beta t) \geq F_H(G\mu, G\nu, \beta t) \geq F_m(\mu, \nu, t)$$

for $t \gg \theta$. Hence, G verifies the conditions of Corollary 5.2 and the existence of a fixed point has been proved.

In the following (Example 5.5), we show that Corollary 5.2 is the generalized form of Corollary 5.3.

Example 5.5 Let $U = \{\frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{3^i}, \dots\} \cup \{0, 1\}$ and the fuzzy metric $F_m : U^2 \times (0, \infty) \to [0, 1]$ be defined as

$$F_m(\mu, \nu, t) = \frac{t}{t + d(\mu, \nu)}, \quad \text{where } d(\mu, \nu) = |\mu - \nu|, \forall \mu, \nu \in U, t > 0.$$

Then $(U, F_m, *)$ is a complete FCM-space, where $* : [0, 1]^2 \to [0, 1]$ is defined as $a * b = ab$.

Let the multi-valued mapping $G: U \to \mathbb{K}(U)$ be defined as

$$G\mu = \begin{cases} \{\frac{1}{3^i}, 1\} & \text{if } \mu = \frac{1}{3^i} \text{ for } i \geq 0, \\ \{0, \frac{1}{3}\} & \text{if } \mu = 0. \end{cases}$$

Since

$$\lim_{i\to\infty} *_{j=i}^{\infty} M(\mu, \nu, tb^j) = M\left(\frac{1}{3^i}, 0, tb^j\right) = \lim_{i\to\infty} *_{j=i}^{\infty} \frac{tb^j}{tb^j + \frac{1}{3^i}} = 1,$$

this shows that G satisfies (3.1). Moreover,

$$F_H\left(G\left(\frac{1}{3^i}\right), G(0), \beta t\right) = \frac{\beta t}{\beta t + H(G(\frac{1}{3^i}), G(0))} = \frac{\beta t}{\beta t + \frac{1}{3}}$$

and

$$F_m\left(\frac{1}{3^i}, 0, t\right) = \frac{t}{t + d(\frac{1}{3^i}, 0)} = \frac{t}{t + \frac{1}{3^i}}.$$

There does not exist any $\beta \in (0,1)$ such that Corollary 5.3 is satisfied. If it exists, then we get

$$\frac{t}{t + 1/3^i} \leq \frac{\beta t}{\beta t + 1/3}.$$

This implies that $\beta \geq 3^{i-1}$, which is a contradiction. On the other hand,

$$g(\mu) = F_m(\mu, G(\mu), t) = \frac{t}{t + d(\mu, G(\mu))} = \begin{cases} \frac{1}{3^{i+1}} & \text{if } \mu = \frac{1}{3^i}, \\ 0 & \text{if } \mu = 0, \end{cases}$$

is continuous and so there exists $\nu \in J^{\mu}_{\frac{3}{4}}$ for any μ such that

$$d(\nu, G(\nu)) = \frac{1}{3} d(\mu, \nu) \leq \frac{2}{3} d(\mu, \nu) \quad \Rightarrow \quad \frac{3}{2} d(\nu, G\nu) \leq d(\mu, \nu).$$

Hence, there exists $\beta = \frac{2}{3} < \frac{3}{4}$ such that

$$F_m\left(\nu, G(\nu), \frac{2}{3}t\right) = \frac{\frac{2}{3}t}{\frac{2}{3}t + d(\nu, G(\nu))} = \frac{t}{t + \frac{3}{2}d(\nu, G(\nu))} \geq \frac{t}{t + d(\mu, \nu)} = F_m(\mu, \nu, t).$$

Then, by Corollary 5.2, we can get the existence of a fixed point of G in U.

Now, we will deal with rational type multi-valued contractions in FCM-spaces. For this, let $G: U \to 2^U$ be a multi-valued map. Define $g(\mu) = F_m(\mu, G\mu, t)$ for $t \gg \theta$. For $\alpha \in (0,1)$, define the set

$$J^{\mu}_{\alpha} = \left\{\nu \in G\mu; \frac{1}{F_m(\mu, \nu, t)} - 1 \leq \frac{1}{F_m(\mu, G\mu, \alpha t)} - 1\right\}. \tag{5.11}$$

Theorem 5.6 *Let $(U, F_m, *)$ be a complete FCM-space and $G : U \to \mathbb{K}(U)$ be a multi-valued map. If there exists a constant $\beta \in (0,1)$ such that, for any $\mu \in U$, there is $\nu \in j_\alpha^\mu$ so that*

$$\frac{1}{F_m(\nu, G\nu, \beta t)} - 1 \le \frac{1}{F_H(G\mu, G\nu, \beta t)} - 1 \le \frac{F_m(\mu, \nu, t) * F_m(\nu, G\nu, t)}{F_m(\mu, G\mu, t) * F_m(\mu, G\nu, 2t) * F_m(\nu, G\mu, 2t)} - 1 \tag{5.12}$$

for $t \gg \theta$. Suppose that $(U, F_m, *)$ satisfies (3.1) for some $\mu_0 \in U$. Then G has a fixed point in U provided that $\beta < \alpha$ and g is upper semi-continuous.

Proof Since $G(\mu) \in \mathbb{K}(U)$, by Lemma 3.3, we have that J_α^μ is nonempty for any μ in U and $\alpha \in (0,1)$. Let us fix $\mu_0 \in U$, so there exists $\mu_1 \in J_\alpha^{\mu_0}$. Then, by (5.12), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{F_m(\mu_1, G\mu_1, \beta t)} - 1 &\le \frac{1}{F_H(G\mu_0, G\mu_1, t)} - 1 \\ &\le \frac{F_m(\mu_0, \mu_1, t) * F_m(\mu_1, G\mu_1, t)}{F_m(\mu_0, G\mu_0, t) * F_m(\mu_0, G\mu_1, 2t) * F_m(\mu_1, G\mu_0, 2t)} - 1 \\ &\le \frac{F_m(\mu_1, \mu_2, t)}{F_m(\mu_0, \mu_2, 2t)} - 1 \end{aligned}$$

by using Definition 2.3(iii) and $F_m(\mu_0, \mu_2, 2t) \ge F_m(\mu_0, \mu_1, t) * F_m(\mu_1, \mu_2, t)$ for $t \gg \theta$. After simplification, we get that

$$\frac{1}{F_m(\mu_1, G\mu_1, \beta t)} - 1 \le \frac{1}{F_m(\mu_0, \mu_1, t)} - 1 \quad \text{for } t \gg \theta.$$

Again for $\mu_1 \in U$, there exists $\mu_2 \in J_\alpha^{\mu_1}$. In view of (5.12),

$$\begin{aligned} \frac{1}{F_m(\mu_2, G\mu_2, \beta t)} - 1 &\le \frac{1}{F_H(G\mu_1, G\mu_2, t)} - 1 \\ &\le \frac{F_m(\mu_1, \mu_2, t) * F_m(\mu_2, G\mu_2, t)}{F_m(\mu_1, G\mu_1, t) * F_m(\mu_1, G\mu_2, 2t) * F_m(\mu_2, G\mu_1, 2t)} - 1 \\ &\le \frac{F_m(\mu_2, \mu_3, t)}{F_m(\mu_1, \mu_3, 2t)} - 1. \end{aligned}$$

Again by Definition 2.3(iii), $F_m(\mu_1, \mu_3, 2t) \ge F_m(\mu_1, \mu_2, t) * F_m(\mu_2, \mu_3, t)$ for $t \gg \theta$. After simplification, we get that

$$\frac{1}{F_m(\mu_2, G\mu_2, \beta t)} - 1 \le \frac{1}{F_m(\mu_1, \mu_2, t)} - 1 \quad \text{for } t \gg \theta.$$

Similarly, by induction, we obtain a sequence $(\mu_i)_{i \ge 0}$ in U such that there exists $\mu_{i+1} \in J_\alpha^{\mu_i}$, then by (5.12)

$$\frac{1}{F_m(\mu_{i+1}, G\mu_{i+1}, \beta t)} - 1 \le \frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \quad \text{for } t \gg \theta. \tag{5.13}$$

On the other hand, by (5.11) and $\mu_{i+1} \in J_\alpha^{\mu_i}$,

$$\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \le \frac{1}{F_m(\mu_i, G\mu_i, \alpha t)} - 1 \quad \text{for } t \gg \theta. \tag{5.14}$$

From (5.13) and (5.14), we can obtain

$$\frac{1}{F_m(\mu_{i+1}, G\mu_{i+1}, \beta t)} - 1 \le \frac{1}{F_m(\mu_i, G\mu_i, \alpha t)} - 1$$

for $t \gg \theta$, that is,

$$\frac{1}{F_m(\mu_{i+1}, G\mu_{i+1}, t)} - 1 \le \frac{1}{F_m(\mu_i, G\mu_i, \frac{\alpha}{\beta} t)} - 1 \tag{5.15}$$

for $t \gg \theta$. Let $a = \frac{\beta}{\alpha}$, then (5.15) can be expressed as follows:

$$\begin{aligned}\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 &\le \frac{1}{F_m(\mu_{i-1}, \mu_i, \frac{1}{a} t)} - 1 \\ &\le \cdots \le \frac{1}{F_m(\mu_0, \mu_1, \frac{1}{a^i} t)} - 1 \quad \text{for } t \gg \theta,\end{aligned} \tag{5.16}$$

for all $i \in \mathbb{N}$ and $a \in (0,1)$. Choose a constant $b > 1$ such that $ab < 1$ and $\sum_{n=0}^{\infty} \frac{1}{b^n} < 1$, i.e., $\sum_{n=i}^{j-1} \frac{1}{b^n} < 1$. Then, for all $j > i$, we get

$$\left(\frac{1}{b^i} + \frac{1}{b^{i+1}} + \cdots + \frac{1}{b^{j-2}} + \frac{1}{b^{j-1}}\right) t < t, \tag{5.17}$$

where $i, j \in \mathbb{N}$. Then we have

$$\begin{aligned}\frac{1}{F_m(\mu_i, \mu_j, t)} - 1 &\le \frac{1}{F_m(\mu_i, \mu_j, t(\frac{1}{b^i} + \frac{1}{b^{i+1}} + \cdots + \frac{1}{b^{j-2}} + \frac{1}{b^{j-1}}))} - 1 \\ &\le \frac{1}{F_m(\mu_i, \mu_{i+1}, \frac{t}{b^i}) * F_m(\mu_{i+1}, \mu_{i+2}, \frac{t}{b^{i+1}}) * \cdots * F_m(\mu_{j-1}, \mu_j, \frac{t}{b^{j-1}})} - 1 \\ &\le \frac{1}{F_m(\mu_0, \mu_1, \frac{t}{(ab)^i}) * F_m(\mu_0, \mu_1, \frac{t}{(ab)^{i+1}}) * \cdots * F_m(\mu_0, \mu_1, \frac{t}{(ab)^{j-1}})} - 1 \\ &\le \frac{1}{*_{n=i}^{\infty} F_m(\mu_0, \mu_1, \frac{t}{(ab)^n})} - 1\end{aligned} \tag{5.18}$$

for $t \gg \theta$. By Lemma 3.2, we have that

$$\lim_{i,j \to \infty} F_m(\mu_i, \mu_j, t) = 1 \quad \text{for } t \gg \theta.$$

It is proved that (μ_i) is a Cauchy sequence in U. Since U is complete, there exists $\mu \in U$ such that $\mu_i \to \mu$ as $i \to \infty$. In view of (5.13) and (5.14), it is clear that $g(\mu_i) = F_m(\mu_i, G\mu_i, t)$ is an increasing function and converges to 1. Since g is upper semi-continuous, we have

$$1 = \limsup_{n \to \infty} g(\mu_n) \le g(\mu) \le 1.$$

This implies that $g(\mu) = 1$, so that $F_m(\mu, G\mu, t) = 1$. Hence, by using Lemma 3.3, we get that $\mu \in G\mu$. □

Directly from Theorem 5.6, we get the following corollary.

Corollary 5.7 *Let $(U, F_m, *)$ be a complete FCM-space and $G : U \to \mathbb{K}(U)$ be a multi-valued map. If there exists a constant $\beta \in (0,1)$ such that*

$$\frac{1}{F_m(\nu, G\nu, \beta t)} - 1 \leq \frac{1}{F_m(\mu, \nu, t)} - 1 \tag{5.19}$$

for all $\mu, \nu \in U$ and $t \gg \theta$. Moreover, assume that $(U, F_m, *)$ satisfies (3.1) for some $\mu_0 \in U$ and $\mu_1 \in G\mu_0$. Then G has a fixed point in U.

Example 5.8 Let $U = \{0.4, 0.4^2, \dots, 0.4^i, \dots\} \cup \{0,1\}$. Let $G : U \to \mathbb{K}(U)$ be defined as

$$G\mu = \begin{cases} \{0.4^i, 1\}, & \text{if } \mu = 0.4^i, \text{ for } i \geq 0, \\ \{0, 0.4\}, & \text{if } \mu = 0. \end{cases}$$

Since

$$\lim_{i\to\infty} *_{j=i}^{\infty} M(\mu, \nu, tb^j) = M(0.4^i, 0, tb^j) = \lim_{i\to\infty} *_{j=i}^{\infty} \frac{tb^j}{tb^j + 0.4^i} = 1,$$

which shows that G satisfies (3.1). By a direct calculation as discussed in Example 5.5, we get $\beta \geq 0.4^{i-1}$ which is a contradiction to the fact that $\beta \geq 0.4^{i-1} \to 0$, as $i \to \infty$, where $\beta \in (0,1)$. On the other hand, we define

$$g(\mu) = F_m(\mu, G(\mu), t) = \frac{t}{t + d(\mu, G(\mu))} = \begin{cases} 0.4^{i+1}, & \text{if } \mu = 0.4^i, \\ 0, & \text{if } \mu = 0. \end{cases}$$

It is continuous and so there exists $\nu \in J_{0.3}^{\mu}$ for any μ such that

$$d(\nu, G(\nu)) = 0.3 d(\mu, \nu) \leq 0.5 d(\mu, \nu).$$

That is,

$$\frac{d(\nu, G\nu)}{0.5} \leq d(\mu, \nu). \tag{5.20}$$

Hence, there exists $\beta = 0.5 < 0.8$, and from (5.20) we get, for $t \gg \theta$,

$$\frac{1}{F_m(\nu, G(\nu), 0.5t)} - 1 = \frac{d(\nu, G\nu)}{0.5t} \leq \frac{d(\mu, \nu)}{t} = \frac{1}{F_m(\mu, \nu, t)} - 1.$$

Then the existence of a fixed point follows from Corollary 5.7.

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