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Existence results for coupled nonlinear fractional differential equations of different orders with nonlocal coupled boundary conditions

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Abstract

This paper is concerned with the solvability of coupled nonlinear fractional differential equations of different orders supplemented with nonlocal coupled boundary conditions on an arbitrary domain. The tools of the fixed point theory are applied to obtain the criteria ensuring the existence and uniqueness of solutions of the problem at hand. Examples illustrating the main results are presented.

MSC: 34A08; 34B15

Keywords: Fractional differential equations; Caputo fractional derivative; System; Existence; Fixed point theorems

1 Introduction

We introduce and study a new class of coupled systems of mixed-order fractional differential equations equipped with nonlocal multi-point coupled boundary conditions. In precise terms, we consider the following fully coupled system:

$$\begin{cases} {}^CD^{\xi}_{a^+}x(t) = \varphi(t,x(t),y(t)), & 1<\xi\le 2, t\in[a,b],\\ {}^CD^{\zeta}_{a^+}y(t) = \psi(t,x(t),y(t)), & 2<\zeta\le 3, t\in[a,b],\\ x(a)=0, \quad x(b)=p_1y(\theta_3), \\ y(\theta_1)=0, \quad y(\theta_2)=0, \quad y(b)=p_2x(\theta_3), & a<\theta_1<\theta_2<\theta_3<b, \end{cases} \tag{1.1}$$

where ${}^CD^{\chi}$ is the Caputo fractional derivative of order $\chi\in\{\xi,\zeta\}$, $\varphi,\psi:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given functions, and $p_i,\in\mathbb{R}$, $i=1,2,3$.

The tools of fractional calculus are found to be of great help in modeling several real-world problems appearing in scientific and technical disciplines. For examples and details, see financial economics [1], ecology [2], immune systems [3], chaotic synchronization [4, 5], etc. The widespread interest in this branch of mathematical analysis motivated many researchers to explore it further. In particular, the area of fractional order boundary value problems has been extensively studied. For some recent works on nonlocal nonlinear and integral boundary value problems involving different types of fractional differential equations, for instance, see [6–17]. On the other hand, fractional differential systems equipped with a variety of boundary conditions also received great attention in view of the occurrence of such systems in the mathematical modeling of several physical and engineering processes [18–20]. Concerning the theoretical development of these systems, one can find the details in the articles [21–31].

Recently, in [32], the authors studied a new class of coupled systems of mixed-order fractional differential equations equipped with nonlocal multi-point coupled boundary conditions of the form:

$$\begin{cases} D^{\xi}x(t) = \varphi(t,x(t),y(t)), & t \in [a,b], 0 < \xi < 1, \\ D^{\zeta}y(t) = \psi(t,x(t),y(t)), & t \in [a,b], 1 < \zeta < 2, \\ px(a) + qy(b) = x_0, \\ y(a) = 0, \quad y'(b) = \sum_{i=1}^{m} \delta_i x(\sigma_i), & a < \sigma_i < b, \end{cases} \tag{1.2}$$

where D^{χ} is the Caputo fractional derivative of order $\chi \in \{\xi, \zeta\}$, $\varphi, \psi : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions $p, q, \delta_i \in \mathbb{R}$, $i = 1, 2, \dots, m$. In [33], the existence and uniqueness of solutions for the following system were investigated by using the Leray–Schauder alternative and the contraction mapping principle:

$$\begin{cases} {}^{c}D^{\xi}_{a^+}x(t) = \varphi(t,x(t),y(t)), & 0 < \xi \le 1, t \in [a,b], \\ {}^{c}D^{\zeta}_{a^+}y(t) = \psi(t,x(t),y(t)), & 1 < \zeta \le 2, t \in [a,b], \\ px(a) + qy(b) = y_0 + x_0 \int_a^b (x(s) + y(s))\, ds, \\ y(a) = 0, \quad y'(b) = \sum_{i=1}^{m} \delta_i x(\sigma_i) + \lambda \int_{\tau}^{b} x(s)\, ds, \\ a < \sigma_1 < \sigma_2 < \cdots < \sigma_m < \tau < b, \end{cases} \tag{1.3}$$

where ${}^{c}D^{\chi}$ is the Caputo fractional derivative of order $\chi \in \{\xi, \zeta\}$, $\varphi, \psi : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions, $p, q, \delta_i, x_0, y_0 \in \mathbb{R}$, $i = 1, 2, \dots, m$.

In the present research, inspired by the published articles [32] and [33], we consider a coupled system (1.1) consisting of fractional differential equations of two different fractional-orders: $(1,2]$ and $(2,3]$ on an arbitrary domain supplemented with a new set of coupled nonlocal multi-point boundary conditions. We emphasize that the present study is novel and more general, and contributes significantly to the existing literature on the topic. Moreover, several new results follow as special cases of the results presented in this work (see Sect. 5).

The rest of the paper is organized as follows: In Sect. 2 we recall some definitions and prove a basic lemma helping us to transform system (1.1) into equivalent integral equations. The main results are established in Sect. 3. An existence result is proved via the Leray–Schauder alternative, and the existence of a unique solution is established by using Banach's contraction mapping principle. Examples illustrating the obtained results are also constructed in Sect. 4.

2 Preliminaries

Let us begin this section with some definitions related to our study [34].

Definition 2.1 The Riemann–Liouville fractional integral of order $\omega \in \mathbb{R}$ ($\omega > 0$) for a locally integrable real-valued function h defined on $-\infty \le a < t < b \le +\infty$, denoted by $I^{\omega}_{a^+}h$, is defined by

$$I^{\omega}_{a^+}h(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1} h(s)\, ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2 Let $h, h^{(m)} \in L^1[a,b]$ for $-\infty \le a < t < b \le +\infty$. The Riemann–Liouville fractional derivative $D^{\omega}_{a^+}h$ of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$, is defined as

$$D^{\omega}_{a^+}h(t) = \frac{1}{\Gamma(m-\omega)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\omega} h(s)\, ds,$$

while the Caputo fractional derivative ${}^C D^{\omega}_{a^+}h$ of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$, is defined as

$${}^C D^{\omega}_{a^+}h(t) = D^{\omega}_{a^+}\left[h(t) - h(a) - h'(a)\frac{(t-a)}{1!} - \cdots - h^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!} \right].$$

Remark 2.3 The Caputo fractional derivative of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$ for a continuous function $h : (0, \infty) \to \mathbb{R}$ such that $h \in C^m[a,b]$, existing almost everywhere on $[a,b]$, is defined by

$${}^C D^{\omega}h(t) = \frac{1}{\Gamma(m-\omega)} \int_a^t (t-s)^{m-\omega-1} h^{(m)}(s)\, ds.$$

Now we present an important result to analyze problem (1.1).

Lemma 2.4 *Let $\Phi, \Psi \in C([a,b], \mathbb{R})$ and $\Lambda \neq 0$. Then the unique solution of the system*

$$\begin{cases} {}^C D^{\xi}_{a^+}x(t) = \Phi(t)), & 1 < \xi \le 2, t \in [a,b], \\ {}^C D^{\zeta}_{a^+}y(t) = \Psi(t), & 2 < \zeta \le 3, t \in [a,b], \\ x(a) = 0, \quad x(b) = p_1 y(\theta_3), \\ y(\theta_1) = 0, \quad y(\theta_2) = 0, \quad y(b) = p_2 x(\theta_3), \quad a < \theta_1 < \theta_2 < \theta_3 < b, \end{cases} \tag{2.1}$$

is given by a pair of integral equations

$$\begin{aligned} x(t) = {} & \int_a^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\Phi(s)\, ds + \frac{t-a}{\Lambda}\left\{ \epsilon_3 \int_a^{\theta_1} \frac{(\theta_1-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\, ds \right. \\ & + \epsilon_4 \int_a^{\theta_2} \frac{(\theta_2-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\, ds + A_2\left(p_1 \int_a^{\theta_3} \frac{(\theta_3-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\, ds \right. \\ & \left. - \int_a^b \frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\Phi(s)\, ds \right) - A_1\left(p_2 \int_a^{\theta_3} \frac{(\theta_3-s)^{\xi-1}}{\Gamma(\xi)}\Phi(s)\, ds \right. \\ & \left.\left. - \int_a^b \frac{(b-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\, ds \right) \right\}, \end{aligned} \tag{2.2}$$

$$y(t) = \int_a^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\, ds + b_1(t) \int_a^{\theta_1} \frac{(\theta_1-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\, ds$$

$$
\begin{aligned}
&+ b_2(t)\int_a^{\theta_2}\frac{(\theta_2-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\,ds + b_3(t)\int_a^{\theta_3}\frac{(\theta_3-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\,ds \\
&+ b_4(t)\int_a^{b}\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\Phi(s)\,ds + b_5(t)\int_a^{\theta_3}\frac{(\theta_3-s)^{\xi-1}}{\Gamma(\xi)}\Phi(s)\,ds \\
&+ b_6(t)\int_a^{b}\frac{(b-s)^{\zeta-1}}{\Gamma(\zeta)}\Psi(s)\,ds,
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
&b_1(t) = \epsilon_5 + \epsilon_6(t-a) + \frac{\epsilon_1}{\Lambda}(t-a)^2, \qquad b_2(t) = \epsilon_7 + \epsilon_8(t-a) + \frac{\epsilon_2}{\Lambda}(t-a)^2, \\
&b_3(t) = \frac{\epsilon_9}{\theta_1-\theta_2}\big(a_2 + a_1(t-a) + (\theta_1-\theta_2)(t-a)^2\big), \\
&b_4(t) = \frac{\epsilon_{10}}{\theta_1-\theta_2}\big(a_2 + a_1(t-a) + (\theta_1-\theta_2)(t-a)^2\big), \\
&b_5(t) = \frac{\epsilon_{11}}{\theta_1-\theta_2}\big(a_2 + a_1(t-a) + (\theta_1-\theta_2)(t-a)^2\big), \\
&b_6(t) = \frac{\epsilon_{12}}{\theta_1-\theta_2}\big(a_2 + a_1(t-a) + (\theta_1-\theta_2)(t-a)^2\big), \\
&\epsilon_1 = \frac{p_1p_2(\theta_3-a)(\theta_2-\theta_3) - (b-a)(\theta_2-b)}{\theta_1-\theta_2}, \\
&\epsilon_2 = \frac{p_1p_2(\theta_3-a)(\theta_3-\theta_1) + (b-a)(\theta_1-b)}{\theta_1-\theta_2}, \\
&\epsilon_3 = \frac{A_2p_1(\theta_2-\theta_3) + A_1(\theta_2-b)}{\theta_1-\theta_2}, \\
&\epsilon_4 = \frac{A_2p_1(\theta_3-\theta_1) + A_1(b-\theta_1)}{\theta_1-\theta_2}, \\
&\epsilon_5 = \frac{\theta_2 - a + \frac{a_2\epsilon_1}{\Lambda}}{\theta_1-\theta_2}, \qquad \epsilon_6 = \frac{\frac{a_1\epsilon_1}{\Lambda} - 1}{\theta_1-\theta_2}, \qquad \epsilon_7 = \frac{a - \theta_1 + \frac{a_2\epsilon_2}{\Lambda}}{\theta_1-\theta_2}, \qquad \epsilon_8 = \frac{\frac{a_1\epsilon_2}{\Lambda} + 1}{\theta_1-\theta_2}, \\
&\epsilon_9 = \frac{p_1p_2(\theta_3-a)}{\Lambda}, \qquad \epsilon_{10} = \frac{p_2(a-\theta_3)}{\Lambda}, \qquad \epsilon_{11} = \frac{p_2(b-a)}{\Lambda}, \qquad \epsilon_{12} = \frac{a-b}{\Lambda}, \\
&a_1 = (\theta_2-a)^2 - (\theta_1-a)^2, \qquad a_2 = (\theta_2-a)(\theta_1-a)(\theta_1-\theta_2), \\
&A_1 = \frac{-p_1a_2 - p_1a_1(\theta_3-a)}{\theta_1-\theta_2} - p_1(\theta_3-a)^2, \\
&A_2 = \frac{a_2 + (b-a)a_1}{\theta_1-\theta_2} + (b-a)^2, \\
&\Lambda = A_1p_2(\theta_3-a) + A_2(b-a).
\end{aligned}
\tag{2.4}
$$

Proof The solution of system (2.1) can be written as

$$
x(t) = I_{a^+}^{\xi}\Phi(t) + c_1 + c_2(t-a), \tag{2.5}
$$

$$
y(t) = I_{a^+}^{\zeta}\Psi(t) + c_3 + c_4(t-a) + c_5(t-a)^2, \tag{2.6}
$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \dots, 5)$ are unknown constants. Using the condition $x(a) = 0$ in (2.5), we get $c_1 = 0$, while making use of the conditions $y(\theta_1) = 0$, $y(\theta_2) = 0$ in (2.6) leads to the equations

$$I^{\zeta}_{a^+}\Psi(\theta_1) + c_3 + c_4(\theta_1 - a) + c_5(\theta_1 - a)^2 = 0, \tag{2.7}$$

$$I^{\zeta}_{a^+}\Psi(\theta_2) + c_3 + c_4(\theta_2 - a) + c_5(\theta_2 - a)^2 = 0. \tag{2.8}$$

Using the conditions $x(b) = p_1 y(\theta_3)$ and $y(b) = p_2 x(\theta_3)$ with $c_1 = 0$ yields

$$I^{\xi}_{a^+}\Phi(b) + c_2(b - a) = p_1\big(I^{\zeta}_{a^+}\Psi(\theta_3) + c_3 + c_4(\theta_3 - a) + c_5(\theta_3 - a)^2\big), \tag{2.9}$$

$$I^{\zeta}_{a^+}\Psi(b) + c_3 + c_4(b - a) + c_5(b - a)^2 = p_2\big(I^{\xi}_{a^+}\Phi(\theta_3) + c_2(\theta_3 - a)\big). \tag{2.10}$$

Subtracting (2.8) from (2.7), we get

$$c_4 = \frac{1}{\theta_1 - \theta_2}\big(a_1 c_5 + I^{\zeta}_{a^+}\Psi(\theta_2) - I^{\zeta}_{a^+}\Psi(\theta_1)\big), \tag{2.11}$$

where $a_1 = (\theta_2 - a)^2 - (\theta_1 - a)^2$. Inserting the value of c_4 in (2.7), we find that

$$c_3 = \frac{1}{\theta_1 - \theta_2}\big((\theta_2 - a)I^{\zeta}_{a^+}\Psi(\theta_1) - (\theta_1 - a)I^{\zeta}_{a^+}\Psi(\theta_2) + a_2 c_5\big). \tag{2.12}$$

Substituting the values of c_3 and c_4 in (2.9) and (2.10), we obtain

$$\begin{aligned}
(b - a)c_2 + A_1 c_5 &= p_1\left\{I^{\zeta}_{a^+}\Psi(\theta_3) + \frac{\theta_2 - \theta_3}{\theta_1 - \theta_2}I^{\zeta}_{a^+}\Psi(\theta_1) + \frac{\theta_3 - \theta_1}{\theta_1 - \theta_2}I^{\zeta}_{a^+}\Psi(\theta_2)\right\} \\
&\quad - I^{\xi}_{a^+}\Phi(b), \\
-p_2(\theta_3 - a)c_2 + A_2 c_5 &= p_2 I^{\xi}_{a^+}\Phi(\theta_3) + \frac{b - \theta_2}{\theta_1 - \theta_2}I^{\zeta}a^+\Psi(\theta_1) + \frac{\theta_1 - b}{\theta_1 - \theta_2}I^{\zeta}a^+\Psi(\theta_2) \\
&\quad - I^{\zeta}_{a^+}\Psi(b).
\end{aligned}$$

Solving the above system, we get

$$\begin{aligned}
c_5 &= \frac{1}{\Lambda}\big(\epsilon_1 I^{\zeta}_{a^+}\Psi(\theta_1) + \epsilon_2 I^{\zeta}_{a^+}\Psi(\theta_2) + p_1 p_2(\theta_3 - a)I^{\zeta}_{a^+}\Psi(\theta_3) - p_2(\theta_3 - a)I^{\xi}_{a^+}\Phi(b) \\
&\quad + p_2(b - a)I^{\xi}_{a^+}\Phi(\theta_3) - (b - a)I^{\zeta}_{a^+}\Psi(b)\big), \\
c_2 &= \frac{1}{\Lambda}\big(\epsilon_3 I^{\zeta}_{a^+}\Psi(\theta_1) + \epsilon_4 I^{\zeta}_{a^+}\Psi(\theta_2) + A_2 p_1 I^{\zeta}_{a^+}\Psi(\theta_3) - A_2 I^{\xi}_{a^+}\Phi(b) - A_1 p_2 I^{\xi}_{a^+}\Phi(\theta_3) \\
&\quad + A_1 I^{\zeta}_{a^+}\Psi(b)\big).
\end{aligned}$$

Now substituting the value of c_5 in (2.11) and (2.12), we find that

$$\begin{aligned}
c_3 &= \frac{1}{\theta_1 - \theta_2}\left\{\left(\theta_2 - a + \frac{a_2\epsilon_1}{\Lambda}\right)I^{\zeta}_{a^+}\Psi(\theta_1) - \left(\theta_1 - a - \frac{a_2\epsilon_2}{\Lambda}\right)I^{\zeta}_{a^+}\Psi(\theta_2)\right. \\
&\quad + \frac{a_2}{\Lambda}\big(p_1 p_2(\theta_3 - a)I^{\zeta}_{a^+}\Psi(\theta_3) - p_2(\theta_3 - a)I^{\xi}_{a^+}\Phi(b) + p_2(b - a)I^{\xi}_{a^+}\Phi(\theta_3)\big) \\
&\quad \left. - (b - a)I^{\zeta}_{a^+}\Psi(b)\right\},
\end{aligned}$$

$$c_4=\frac{1}{\theta_1-\theta_2}\left\{\left(\frac{a_1\epsilon_1}{\Lambda}-1\right)I^{\zeta}_{a^+}\Psi(\theta_1)+\left(\frac{a_1\epsilon_2}{\Lambda}+1\right)I^{\zeta}_{a^+}\Psi(\theta_2)+\frac{a_1}{\Lambda}\big(p_1p_2(\theta_3-a)I^{\zeta}_{a^+}\Psi(\theta_3)\right.$$
$$\left.-p_2(\theta_3-a)I^{\xi}_{a^+}\Phi(b)+p_2(b-a)I^{\xi}_{a^+}\Phi(\theta_3)-(b-a)I^{\zeta}_{a^+}\Psi(b)\big)\right\}.$$

Finally, inserting the values of the constants c_i, $i=1,2,\dots,5$, into (2.5) and (2.6) yields equations (2.2) and (2.3). This completes the proof. We can prove the converse by direct computation. The proof is finished. □

3 Main results

Let $X=C([a,b],\mathbb{R})$ be a Banach space endowed with the norm $\|x\|=\sup|x(t)|$, $t\in[a,b]$.

In view of Lemma 2.4, we define an operator $T:X\times X\to X$ by

$$T\big(x(t),y(t)\big)=\big(T_1\big(x(t),y(t)\big),T_2\big(x(t),y(t)\big)\big),$$

where

$$\begin{aligned}
&T_1\big(x(t),y(t)\big)\\
&=\int_a^t\frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,x(s),y(s)\big)\,ds+\frac{t-a}{\Lambda}\left\{\epsilon_3\int_a^{\theta_1}\frac{(\theta_1-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds\right.\\
&\quad+\epsilon_4\int_a^{\theta_2}\frac{(\theta_2-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds+A_2\left(p_1\int_a^{\theta_3}\frac{(\theta_3-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds\right.\\
&\quad\left.-\int_a^b\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,x(s),y(s)\big)\,ds\right)-A_1\left(p_2\int_a^{\theta_3}\frac{(\theta_3-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,x(s),y(s)\big)\,ds\right.\\
&\quad\left.\left.-\int_a^b\frac{(b-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds\right)\right\},\\
&T_2\big(x(t),y(t)\big)\\
&=\int_a^t\frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds+b_1(t)\int_a^{\theta_1}\frac{(\theta_1-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds\\
&\quad+b_2(t)\int_a^{\theta_2}\frac{(\theta_2-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds+b_3(t)\int_a^{\theta_3}\frac{(\theta_3-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds\\
&\quad+b_4(t)\int_a^{b}\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,x(s),y(s)\big)+b_5(t)\int_a^{\theta_3}\frac{(\theta_3-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,x(s),y(s)\big)\,ds\\
&\quad+b_6(t)\int_a^{b}\frac{(b-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,x(s),y(s)\big)\,ds.
\end{aligned}$$

Here $(X\times X,\|(x,y)\|)$ is a Banach space equipped with the norm $\|(x,y)\|=\|x\|+\|y\|$, $x,y\in X$.

In our first result, we establish the existence of a solution for system (1.1) by applying the Leray–Schauder alternative [35].

Lemma 3.1 (Leray–Schauder alternative) : *Let $\mathfrak{J}:\mathcal{U}\longrightarrow\mathcal{U}$ be a completely continuous operator (i.e., a map restricted to any bounded set in $\mathcal{U}$ is compact). Let $\mathcal{Q}(\mathfrak{J})=\{x\in\mathcal{U}:x=\eta\mathfrak{J}(x)$ for some $0<\eta<1\}$. Then either the set $\mathcal{Q}(\mathfrak{J})$ is unbounded or $\mathfrak{J}$ has at least one fixed point.*

For computational convenience, we set

$$
\begin{aligned}
L_1 &= \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{b-a}{|\Lambda|\Gamma(\xi+1)}\big(|A_2|(b-a)^{\xi} + |A_1 p_2|(\theta_3-a)^{\xi}\big),\\
M_1 &= \frac{b-a}{|\Lambda|\Gamma(\zeta+1)}\big(|\epsilon_3|(\theta_1-a)^{\zeta} + |\epsilon_4|(\theta_2-a)^{\zeta} + |A_2 p_1|(\theta_3-a)^{\zeta} + |A_1|(b-a)^{\zeta}\big),\\
L_2 &= \frac{1}{\Gamma(\xi+1)}\big(\delta_4(b-a)^{\xi} + \delta_5(\theta_3-a)^{\xi}\big),\\
M_2 &= \frac{1}{\Gamma(\zeta+1)}\big(\delta_1(\theta_1-a)^{\zeta} + \delta_2(\theta_2-a)^{\zeta} + \delta_3(\theta_3-a)^{\zeta} + (\delta_6+1)(b-a)^{\zeta}\big),
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\delta_1 &= |\epsilon_5| + |\epsilon_6|(b-a) + \frac{|\epsilon_1|}{|\Lambda|}(b-a)^2,\\
\delta_2 &= |\epsilon_7| + |\epsilon_8|(b-a) + \frac{|\epsilon_2|}{|\Lambda|}(b-a)^2,\\
\delta_3 &= \frac{|\epsilon_9|}{|\theta_1-\theta_2|}\big(|a_2| + |a_1|(b-a) + |\theta_1-\theta_2|(b-a)^2\big),\\
\delta_4 &= \frac{|\epsilon_{10}|}{|\theta_1-\theta_2|}\big(|a_2| + |a_1|(b-a) + |\theta_1-\theta_2|(b-a)^2\big),\\
\delta_5 &= \frac{|\epsilon_{11}|}{|\theta_1-\theta_2|}\big(|a_2| + |a_1|(b-a) + |\theta_1-\theta_2|(b-a)^2\big),\\
\delta_6 &= \frac{|\epsilon_{12}|}{|\theta_1-\theta_2|}\big(|a_2| + |a_1|(b-a) + |\theta_1-\theta_2|(b-a)^2\big).
\end{aligned}
$$

Theorem 3.2 *Let* $\Lambda \neq 0$ *(*Λ *is defined by* (2.4)*). In addition, we assume that*:

(H_1) $\varphi, \psi : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist real constants* $k_i, \gamma_i \geq 0$ $(i = 1,2)$ *and* $k_0 > 0$, $\gamma_0 > 0$ *such that, for all* $t \in [a,b]$ *and* $x, y \in \mathbb{R}$,

$$
\begin{aligned}
\big|\varphi(t,x,y)\big| &\leq k_0 + k_1|x| + k_2|y|,\\
\big|\psi(t,x,y)\big| &\leq \gamma_0 + \gamma_1|x| + \gamma_2|y|.
\end{aligned}
$$

Then system (1.1) *has at least one solution on* $[a,b]$ *provided that*

$$
(L_1+L_2)k_1 + (M_1+M_2)\gamma_1 < 1 \quad \textit{and} \quad (L_1+L_2)k_2 + (M_1+M_2)\gamma_2 < 1,
\tag{3.2}
$$

where L_1, M_1, L_2, M_2 *are given in* (3.1).

Proof Observe that the continuity of the operator $T : X \times X \to X \times X$ follows that of the functions φ and ψ. Next, let $\Omega \subset X \times X$ be bounded such that

$$
\big|\varphi\big(t,x(t),y(t)\big)\big| \leq K_1, \qquad \big|\psi\big(t,x(t),y(t)\big)\big| \leq K_2, \quad \forall (x,y) \in \Omega,
$$

for positive constants K_1 and K_2. Then, for any $(x,y) \in \Omega$, we have

$$\begin{aligned}
\left|T_1(x,y)(t)\right| \leq{}& \int_a^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)} \left|\varphi\left(s,x(s),y(s)\right)\right| ds \\
&+ \frac{b-a}{|\Lambda|}\left(|\epsilon_3| \int_a^{\theta_1} \frac{(\theta_1-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \right.\\
&+ |\epsilon_4| \int_a^{\theta_2} \frac{(\theta_2-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \\
&+ |A_2 p_1| \int_a^{\theta_3} \frac{(\theta_3-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \\
&+ |A_2| \int_a^{b} \frac{(b-s)^{\xi-1}}{\Gamma(\xi)} \left|\varphi\left(s,x(s),y(s)\right)\right| ds \\
&+ |A_1 p_2| \int_a^{\theta_3} \frac{(\theta_3-s)^{\xi-1}}{\Gamma(\xi)} \left|\varphi\left(s,x(s),y(s)\right)\right| ds \\
&\left.+ |A_1| \int_a^{b} \frac{(b-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \right) \\
\leq{}& \left\{ \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{b-a}{|\Lambda|}\left(|A_2| \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + |A_1 p_2| \frac{(\theta_3-a)^{\xi}}{\Gamma(\xi+1)} \right) \right\} K_1 \\
&+ \left\{ \frac{b-a}{|\Lambda|}\left(|\epsilon_3| \frac{(\theta_1-a)^{\zeta}}{\Gamma(\zeta+1)} + |\epsilon_4| \frac{(\theta_2-a)^{\zeta}}{\Gamma(\zeta+1)} + |A_2 p_1| \frac{(\theta_3-a)^{\zeta}}{\Gamma(\zeta+1)} \right.\right.\\
&\left.\left.+ |A_1| \frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)} \right) \right\} K_2 \\
={}& L_1 K_1 + M_1 K_2,
\end{aligned}$$

which implies that

$$\left\| T_1(x,y) \right\| \leq L_1 K_1 + M_1 K_2.$$

In a similar way, in view of notation (3.1), we have

$$\begin{aligned}
\left|T_2(x,y)(t)\right| \leq{}& \int_a^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \\
&+ \left|b_1(t)\right| \int_a^{\theta_1} \frac{(\theta_1-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \\
&+ \left|b_2(t)\right| \int_a^{\theta_2} \frac{(\theta_2-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \\
&+ \left|b_3(t)\right| \int_a^{\theta_3} \frac{(\theta_3-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds \\
&+ \left|b_4(t)\right| \int_a^{b} \frac{(b-s)^{\xi-1}}{\Gamma(\xi)} \left|\varphi\left(s,x(s),y(s)\right)\right| ds \\
&+ \left|b_5(t)\right| \int_a^{\theta_3} \frac{(\theta_3-s)^{\xi-1}}{\Gamma(\xi)} \left|\varphi\left(s,x(s),y(s)\right)\right| ds \\
&+ \left|b_6(t)\right| \int_a^{b} \frac{(b-s)^{\zeta-1}}{\Gamma(\zeta)} \left|\psi\left(s,x(s),y(s)\right)\right| ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{\delta_4 \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \delta_5 \frac{(\theta_3-a)^{\xi}}{\Gamma(\xi+1)}\right\} K_1 \\
&\quad + \left\{\frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)} + \delta_1 \frac{(\theta_1-a)^{\zeta}}{\Gamma(\zeta+1)} + \delta_2 \frac{(\theta_2-a)^{\zeta}}{\Gamma(\zeta+1)} + \delta_3 \frac{(\theta_3-a)^{\zeta}}{\Gamma(\zeta+1)} \right. \\
&\quad \left. + \delta_6 \frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)}\right\} K_2 \\
&= L_2 K_1 + M_2 K_2,
\end{aligned}
$$

which yields

$$
\left\| T_2(x,y) \right\| \leq L_2 K_1 + M_2 K_2.
$$

From the above argument, we deduce that the operator T is uniformly bounded, as

$$
\left\| T(x,y) \right\| \leq (L_1 + L_2) K_1 + (M_1 + M_2) K_2.
$$

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Then we have

$$
\begin{aligned}
&\left| T_1\big(x(t_2), y(t_2)\big) - T_1\big(x(t_1), y(t_1)\big) \right| \\
&\leq K_1 \left| \frac{1}{\Gamma(\xi)} \int_a^{t_1} \left[(t_2-s)^{\xi-1} - (t_1-s)^{\xi-1}\right] ds + \frac{1}{\Gamma(\xi)} \int_{t_1}^{t_2} (t_1-s)^{\xi-1} \, ds \right| \\
&\quad + \left\{\frac{t_2-t_1}{|\Lambda|} \left(|A_2| \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + |A_1 p_2| \frac{(\theta_3-a)^{\xi}}{\Gamma(\xi+1)} \right)\right\} K_1 \\
&\quad + \left\{\frac{t_2-t_1}{|\Lambda|} \left(|\epsilon_3| \frac{(\theta_1-a)^{\zeta}}{\Gamma(\zeta+1)} + |\epsilon_4| \frac{(\theta_2-a)^{\zeta}}{\Gamma(\zeta+1)} + |A_2 p_1| \frac{(\theta_3-a)^{\zeta}}{\Gamma(\zeta+1)} + |A_1| \frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)} \right)\right\} K_2 \\
&\leq \frac{K_1}{\Gamma(\xi+1)} \left[2(t_2-t_1)^{\xi} + \left| t_2^{\xi} - t_1^{\xi} \right|\right] \\
&\quad + \left\{\frac{t_2-t_1}{|\Lambda|} \left(|A_2| \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + |A_1 p_2| \frac{(\theta_3-a)^{\xi}}{\Gamma(\xi+1)} \right)\right\} K_1 \\
&\quad + \left\{\frac{t_2-t_1}{|\Lambda|} \left(|\epsilon_3| \frac{(\theta_1-a)^{\zeta}}{\Gamma(\zeta+1)} + |\epsilon_4| \frac{(\theta_2-a)^{\zeta}}{\Gamma(\zeta+1)} + |A_2 p_1| \frac{(\theta_3-a)^{\zeta}}{\Gamma(\zeta+1)} \right.\right. \\
&\quad \left.\left. + |A_1| \frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)} \right)\right\} K_2.
\end{aligned}
\tag{3.3}
$$

Analogously, we can obtain

$$
\begin{aligned}
&\left| T_2\big(x(t_2), y(t_2)\big) - T_2\big(x(t_1), y(t_1)\big) \right| \\
&\leq \frac{K_2}{\Gamma(\zeta+1)} \left[2(t_2-t_1)^{\zeta} + \left| t_2^{\zeta} - t_1^{\zeta} \right|\right] \\
&\quad + \frac{t_2-t_1}{|\theta_1-\theta_2|} \left\{ \left(\frac{|a_1 p_2|}{|\Lambda|} (\theta_3-a) \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{|a_1 p_2|}{|\Lambda|} (b-a) \frac{(\theta_3-a)^{\xi}}{\Gamma(\xi+1)} \right) K_1 \right. \\
&\quad + \left(\left| \frac{a_1 \epsilon_1}{\Lambda} - 1 \right| \frac{(\theta_1-a)^{\xi}}{\Gamma(\xi+1)} + \left| \frac{a_1 \epsilon_2}{\Lambda} + 1 \right| \frac{(\theta_2-a)^{\zeta}}{\Gamma(\zeta+1)} + \frac{|a_1 p_1 p_2|}{|\Lambda|} \frac{(\theta_3-a)^{\zeta+1}}{\Gamma(\zeta+1)} \right. \\
&\quad \left.\left. + \frac{|a_1|}{|\Lambda|} \frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+1)} \right) K_2 \right\} + \frac{|(t_2^2 - t_1^2) - 2a(t_2-t_1)|}{|\Lambda|} \left\{ \left(|p_2| (\theta_3-a) \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} \right.\right.
\end{aligned}
$$

$$+ |p_2|(b-a)\frac{(\theta_3-a)^{\xi}}{\Gamma(\xi+1)}\bigg)K_1 + \bigg(|\epsilon_1|\frac{(\theta_1-a)^{\zeta}}{\Gamma(\zeta+1)} + |\epsilon_2|\frac{(\theta_2-a)^{\zeta}}{\Gamma(\zeta+1)}$$
$$+ |p_1p_2|\frac{(\theta_3-a)^{\zeta+1}}{\Gamma(\zeta+1)} + \frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+1)}\bigg)K_2\bigg\}. \tag{3.4}$$

Clearly the right-hand sides of inequalities (3.3) and (3.4) tend to zero independently of x and y as $t_1 \to t_2$. This shows that the operator $T(x,y)$ is equicontinuous. In consequence, we deduce that the operator $T(x,y)$ is completely continuous.

Finally, we consider the set $\mathcal{P} = \{(x,y) \in X \times X : (x,y) = \nu T(x,y), 0 \leq \nu \leq 1\}$ and show that it is bounded.

Let $(x,y) \in \mathcal{P}$ with $(x,y) = \nu T(x,y)$. For any $t \in [a,b]$, we have $x(t) = \nu T_1(x,y)(t)$, $y(t) = \nu T_2(x,y)(t)$. Then, by (H_1), we have

$$\begin{aligned} \big|x(t)\big| &\leq L_1\big(k_0 + k_1|x| + k_2|y|\big) + M_1\big(\gamma_0 + \gamma_1|x| + \gamma_2|y|\big) \\ &= L_1k_0 + M_1\gamma_0 + (L_1k_1 + M_1\gamma_1)|x| + (L_1k_2 + M_1\gamma_2)|y|, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \big|y(t)\big| &\leq L_2\big(k_0 + k_1|x| + k_2|y|\big) + M_2\big(\gamma_0 + \gamma_1|x| + \gamma_2|y|\big) \\ &= L_2k_0 + M_2\gamma_0 + (L_2k_1 + M_2\gamma_1)|x| + (L_2k_2 + M_2\gamma_2)|y|. \end{aligned} \tag{3.6}$$

In consequence of the above arguments, we deduce that

$$\|x\| \leq L_1k_0 + M_1\gamma_0 + (L_1k_1 + M_1\gamma_1)\|x\| + (L_1k_2 + M_1\gamma_2)\|y\|$$

and

$$\|y\| \leq L_2k_0 + M_2\gamma_0 + (L_2k_1 + M_2\gamma_1)\|x\| + (L_2k_2 + M_2\gamma_2)\|y\|,$$

which imply that

$$\begin{aligned} \|x\| + \|y\| \leq{}& (L_1+L_2)k_0 + (M_1+M_2)\gamma_0 \\ &+ \big[(L_1+L_2)k_1 + (M_1+M_2)\gamma_1\big]\|x\| \\ &+ \big[(L_1+L_2)k_2 + (M_1+M_2)\gamma_2\big]\|y\|. \end{aligned} \tag{3.7}$$

Thus

$$\big\|(x,y)\big\| \leq \frac{1}{M_0}\big[(L_1+L_2)k_0 + (M_1+M_2)\gamma_0\big],$$

where $M_0 = \min\{1 - [(L_1+L_2)k_1 + (M_1+M_2)\gamma_1], 1 - [(L_1+L_2)k_2 + (M_1+M_2)\gamma_2]\}$. Hence the set $\mathcal{P}$ is bounded. Thus, by the Leray–Schauder alternative, we deduce that the operator T has at least one fixed point, which corresponds to the fact that problem (1.1) has at least one solution on $[a,b]$. The proof is completed. □

In the next theorem we prove the existence of a unique solution of system (1.1) by using the contraction mapping principle due to Banach.

Theorem 3.3 *Let $\Lambda \neq 0$ (Λ is defined by (2.4)). In addition, we assume that:*

(H_2) *$\varphi, \psi : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist positive constants l_1 and l_2 such that, for all $t \in [a,b]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$, we have*

$$\begin{aligned} \left|\varphi(t,x_1,x_2) - \varphi(t,y_1,y_2)\right| &\le l_1\big(|x_1 - y_1| + |x_2 - y_2|\big), \\ \left|\psi(t,x_1,x_2) - \psi(t,y_1,y_2)\right| &\le l_2\big(|x_1 - y_1| + |x_2 - y_2|\big). \end{aligned}$$

If

$$(L_1 + L_2)l_1 + (M_1 + M_2)l_2 < 1, \tag{3.8}$$

where L_1, M_1 and L_2, M_2 are given by (3.1), then system (1.1) has a unique solution on $[a,b]$.

Proof Define $\sup_{t\in[a,b]} \varphi(t,0,0) = N_1 < \infty$, $\sup_{t\in[a,b]} \psi(t,0,0) = N_2 < \infty$, and $r > 0$ such that

$$r > \frac{(L_1 + L_2)N_1 + (M_1 + M_2)N_2}{1 - (L_1 + L_2)l_1 - (M_1 + M_2)l_2}.$$

In the first step, we show that $TB_r \subset B_r$, where $B_r = \{(x,y) \in X \times X : \|(x,y)\| \le r\}$. By assumption ($H_2$), for $(x,y) \in B_r$, $t \in [a,b]$, we have

$$\begin{aligned} \left|\varphi\big(t,x(t),y(t)\big)\right| &\le \left|\varphi\big(t,x(t),y(t)\big) - \varphi(t,0,0)\right| \\ &\le l_1\big(\left|x(t)\right| + \left|y(t)\right|\big) + N_1 \\ &\le l_1\big(\|x\| + \|y\|\big) + N_1 \le l_1 r + N_1. \end{aligned}$$

Similarly, we get

$$\left|\psi\big(t,x(t),y(t)\big)\right| \le l_2\big(\|x\| + \|y\|\big) + N_2 \le l_2 r + N_2.$$

Then, we obtain

$$\begin{aligned} \left|T_1(x,y)(t)\right| \le{}& \left\{\frac{(b-a)^\xi}{\Gamma(\xi+1)} + \frac{b-a}{|\Lambda|}\left(|A_2|\frac{(b-a)^\xi}{\Gamma(\xi+1)} + |A_1 p_2|\frac{(\theta_3-a)^\xi}{\Gamma(\xi+1)}\right)\right\}(l_1 r + N_1) \\ &+ \left\{\frac{b-a}{|\Lambda|}\left(|\epsilon_3|\frac{(\theta_1-a)^\zeta}{\Gamma(\zeta+1)} + |\epsilon_4|\frac{(\theta_2-a)^\zeta}{\Gamma(\zeta+1)} + |A_2 p_1|\frac{(\theta_3-a)^\zeta}{\Gamma(\zeta+1)}\right.\right. \\ &\left.\left. + |A_1|\frac{(b-a)^\zeta}{\Gamma(\zeta+1)}\right)\right\}(l_2 r + N_2) \\ ={}& L_1(l_1 r + N_1) + M_1(l_2 r + N_2) \\ ={}& (L_1 l_1 + M_1 l_2)r + L_1 N_1 + M_1 N_2. \end{aligned}$$

Taking the norm, we get

$$\left\|T_1(x,y)\right\| \le (L_1 l_1 + M_1 l_2)r + L_1 N_1 + M_1 N_2.$$

Likewise, we can find that

$$\|T_2(x,y)\| \le (L_2 l_1 + M_2 l_2)r + L_2 N_1 + M_2 N_2.$$

Consequently,

$$\|T(x,y)\| \le \big[(L_1+L_2)l_1 + (M_1+M_2)l_2\big]r + (L_1+L_2)N_1 + (M_1+M_2)N_2 \le r.$$

Now, for $(x_1,y_1),(x_2,y_2) \in X \times X$ and for any $t \in [a,b]$, we get

$$\begin{aligned}
&\big|T_1(x_2,y_2)(t) - T_1(x_1,y_1)(t)\big| \\
&\quad\le \left\{\frac{(b-a)^\xi}{\Gamma(\xi+1)} + \frac{b-a}{|\Lambda|}\left(|A_2|\frac{(b-a)^\xi}{\Gamma(\xi+1)} + |A_1 p_2|\frac{(\theta_3-a)^\xi}{\Gamma(\xi+1)}\right)\right\} l_1\big(\|x_2-x_1\| + \|y_2-y_1\|\big) \\
&\qquad + \left\{\frac{b-a}{|\Lambda|}\left(|\epsilon_3|\frac{(\theta_1-a)^\zeta}{\Gamma(\zeta+1)} + |\epsilon_4|\frac{(\theta_2-a)^\zeta}{\Gamma(\zeta+1)} + |A_2 p_1|\frac{(\theta_3-a)^\zeta}{\Gamma(\zeta+1)}\right.\right. \\
&\qquad \left.\left. + |A_1|\frac{(b-a)^\zeta}{\Gamma(\zeta+1)}\right)\right\} l_2\big(\|x_2-x_1\| + \|y_2-y_1\|\big) \\
&\quad\le (L_1 l_1 + M_1 l_2)\big(\|x_2-x_1\| + \|y_2-y_1\|\big),
\end{aligned}$$

which implies that

$$\|T_1(x_2,y_2) - T_1(x_1,y_1)\| \le (L_1 l_1 + M_1 l_2)\big(\|x_2-x_1\| + \|y_2-y_1\|\big). \tag{3.9}$$

Similarly, we find that

$$\|T_2(x_2,y_2) - T_2(x_1,y_1)\| \le (L_2 l_1 + M_2 l_2)\big(\|x_2-x_1\| + \|y_2-y_1\|\big). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\|T(x_2,y_2) - T(x_1,y_1)\| \le \big[(L_1+L_2)l_1 + (M_1+M_2)l_2\big]\big(\|x_2-x_1\| + \|y_2-y_1\|\big).$$

From the above inequality and (3.8), we deduce that T is a contraction. Hence it follows by Banach's fixed point theorem that there exists a unique fixed point for the operator T, which corresponds to a unique solution of problem (1.1) on $[a,b]$. This completes the proof. □

4 Examples

Let us consider the following mixed-type coupled fractional differential systems:

$$\begin{cases}
D_{a^+}^{3/2} x(t) = \varphi(t,x(t),y(t)), & t \in [2,3], \\
D_{a^+}^{5/2} y(t) = \psi(t,x(t),y(t)), & t \in [2,3], \\
x(2) = 0, \qquad x(3) = \frac{1}{100} y(\frac{14}{5}), \\
y(\frac{11}{5}) = 0, \qquad y(\frac{11}{4}) = 0, \qquad y(3) = \frac{1}{50} x(\frac{14}{5}).
\end{cases} \tag{4.1}$$

Here $\xi = 3/2$, $\zeta = 5/2$, $\theta_1 = 11/5$, $\theta_2 = 11/4$, $\theta_3 = 14/5$, $p_1 = 1/100$, $p_2 = 1/50$. With the given data, it is found that $L_1 \simeq 1.5045$, $L_2 \simeq 0.23941$, $M_1 \simeq 1.2806 \times 10^{-3}$, $M_2 \simeq 5.3193$.

(1) In order to illustrate Theorem 3.2, we take

$$\begin{aligned} \varphi(t,x,y) &= \sqrt{t^3+1} + \frac{1}{40} x \sin y + \frac{1}{5e} y \cos x, \\ \psi(t,x,y) &= \frac{1}{\sqrt{t^2+1}} + \frac{1}{10} e^{-t/2} x + \frac{1}{100} y \cos y. \end{aligned} \tag{4.2}$$

It is easy to check that condition (H_1) is satisfied with $k_0 = \sqrt{28}$, $k_1 = 1/40$, $k_2 = 1/(5e)$, $\gamma_0 = 1/\sqrt{5}$, $\gamma_1 = 1/(10e)$, $\gamma_2 = 1/100$.

Furthermore, $(L_1 + L_2)k_1 + (M_1 + M_2)\gamma_1 \simeq 0.20009 < 1$ and $(L_1 + L_2)k_2 + (M_1 + M_2)\gamma_2 \simeq 0.18152 < 1$. Clearly, the hypotheses of Theorem 3.2 are satisfied, and hence the conclusion of Theorem 3.2 applies to problem (4.1) with φ and ψ given by (4.2).

(2) In order to illustrate Theorem 3.3, we take

$$\varphi(t,x,y) = \frac{1}{4+t}\big(\sin x + |y|\big) + \cos t, \qquad \psi(t,x,y) = \frac{1}{5e^{\frac{t}{2}}}\big(\cos x + |y|\big) + \tan t, \tag{4.3}$$

which clearly satisfy condition (H_2) with $l_1 = 1/6$ and $l_2 = 1/(5e)$.

Moreover, $(L_1 + L_2)l_1 + (M_1 + M_2)l_2 \simeq 0.6811 < 1$. Thus the hypotheses of Theorem 3.3 hold true, and consequently there exists a unique solution of problem (4.1) with φ and ψ given by (4.3) on $[2, 3]$.

5 Conclusions

In this paper, we have studied the existence of solution for a boundary value problem consisting of a coupled system of nonlinear fractional differential equations of different orders and five-point nonlocal coupled boundary conditions on an arbitrary domain. The given problem is transformed into an equivalent fixed point problem, which is solved by applying the standard tools of the modern functional analysis to obtain the existence and uniqueness results for the original problem. Our results are not only new in the given setting, but also reduce to some new results as special cases by fixing the parameters involved in the boundary conditions. For example, if we take $p_1 = 0 = p_2$ in the obtained results, we get the ones associated with four-point nonlocal boundary conditions: $x(a) = 0$, $x(b) = 0$, $y(\theta_1) = 0$, $y(\theta_2) = 0$, $y(b) = 0$.

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Authors' contributions
Each of the authors, AA, SH, BA, and SKN, contributed equally to each part of this work. All authors read and approved the final manuscript.

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