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L^p harmonic 1-forms on conformally flat Riemannian manifolds

Jing Li[1], Shuxiang Feng[2] and Peibiao Zhao[1*]

*Correspondence:
pbzhao@njust.edu.cn
[1] School of Science, Nanjing University of Science and Technology, 210094, Nanjing, China
Full list of author information is available at the end of the article

Abstract
In this paper, we establish a finiteness theorem for L^p harmonic 1-forms on a locally conformally flat Riemannian manifold under the assumptions on the Schrödinger operators involving the squared norm of the traceless Ricci form. This result can be regarded as a generalization of Han's result on L^2 harmonic 1-forms.

MSC: 53C21; 53C25

Keywords: L^p harmonic 1-forms; Conformally flat; Finite index

1 Introduction

Investigating the relationship between the geometry and topology of a Riemannian manifold M and the spaces of harmonic forms is one of the most important problems in differential geometry. Thanks to the Hodge theory, it is known that when M is compact, the space of harmonic 1-forms on M is isomorphic to its first de Rham cohomology group. When M is noncompact, the Hodge theory is no longer applicable, and it is natural to consider L^2 harmonic forms. Furthermore, L^2 Hodge theory holds for complete noncompact manifolds (see, e.g., [1, 9]), just like classical Hodge theory works on the compact case. Particularly, Li and Tam [21] showed that the theory of L^2 harmonic 1-forms can be used for understanding the topology at infinity of a complete Riemannian manifold.

Recall that a Riemannian manifold (M^m, g) of dimension m is said to be locally conformally flat if it admits a coordinate covering $(U_\alpha, \varphi_\alpha)$ such that the map $\varphi_\alpha : (U_\alpha, g_\alpha) \to (S^m, g_0)$ is a conformal map, where g_0 is the standard metric on S^m. A conformally flat Riemannian manifold may be regarded as a generalization of a Riemannian surface because every two-dimensional Riemannian manifold is locally conformally flat. However, not all higher-dimensional manifolds have locally conformally flat structure, and giving classification of locally conformally flat manifolds is important as well as difficult. However, under various geometric conditions, there are substantial research results on the classification of conformally flat Riemannian manifolds (see [2, 5, 6, 14, 18, 22, 29] for details).

For L^2 harmonic forms, Lin [23] proved some vanishing and finiteness theorems for L^2 harmonic 1-forms on a locally conformally flat Riemannian manifold that satisfies an integral pinching condition on the traceless Ricci tensor and for which the scalar curvature is nonpositive or satisfies some integral pinching conditions. Dong et al. [10] proved vanishing theorems for L^2 harmonic p-forms on a complete noncompact locally conformally flat Riemannian manifold under suitable conditions. Similarly, Han [16] obtained some vanishing and finiteness theorems for L^2 harmonic 1-forms on a locally conformally flat Riemannian manifold under the assumptions on the Schrödinger operators involving the squared norm of the traceless Ricci form. Moreover, many results showed that there is a close correlation between the topologies of the submanifolds and L^2 harmonic 1-forms; see [3, 11, 12, 24, 30, 31] and the references therein.

The results of L^2 harmonic forms make L^2 theory on manifolds clearer and easier to understand as compared to general L^p theory (see [26]). For L^p harmonic 1-forms, Han et al. [18] obtained some vanishing and finiteness theorems for L^p p-harmonic 1-forms on a locally conformally flat Riemannian manifold with some assumptions. Analogously, there is substantial research indicating that the topologies of the submanifolds is closely associated with L^p harmonic 1-forms; see [4, 7, 8, 15, 17, 19, 22] and the references therein.

Meanwhile, Lin [24] studied the relations between the index of the Schrödinger operator $L = \Delta + \frac{m-1}{m}|B|^2$ and the topology of M^m, where B is the second fundamental form of M^m, and M^m is a complete noncompact minimal submanifold of dimension m immersed in R^{m+n}. In particular, when M^m is an m-dimensional locally conformally flat Riemannian manifold, Han [16] focused on the Schrödinger operator $L = \Delta + |T|$ and investigated the relations between it and the topological structure of M^m, where Ric, R, and $T = Ric - \frac{R}{m}g$ are the Ricci curvature tensor, the scalar curvature, and the traceless Ricci tensor of (M^m, g), respectively.

Inspired by Han's work [16] and the above mentioned aspects, in this paper, we investigate relations between the index of a Schrödinger operator $L = \Delta + |T|$ of the locally conformally flat manifold M^m and the space of L^p harmonic 1-forms on M^m. We prove that if the index of a Schrödinger operator is finite and $\int_M |R|^{\frac{m}{2}}\, dv < \infty$, then for certain $p > 0$, the dimension of $H^1(L^{2p}(M))$ is finite. This can be regarded as a generalization of Han's results [16] for the space of L^2 harmonic 1-forms.

In this paper, we obtain the following finiteness theorem for the space of L^p harmonic 1-forms.

Theorem 1.1 *Let (M^m, g), $m \geq 3$, be an m-dimensional complete, simply connected, and locally conformally flat Riemannian manifold. Assume that the index of the operator $\Delta + |T|$ is finite. If $\int_M |R|^{\frac{m}{2}}\, dv < \infty$ and $p \in (1 - \sqrt{\frac{1}{m-1}}, 1 + \sqrt{\frac{1}{m-1}})$, then*

$$\dim H^1\big(L^{2p}(M)\big) < \infty,$$

where $H^1(L^{2p}(M))$ denotes the space of L^{2p} harmonic 1-forms on M.

2 Preliminaries

Consider an elliptic operator $L = \Delta + \tilde{Q}$ on M^m, where $\tilde{Q}$ is the smooth potential of it. Let D be a relatively compact domain of M^m, and let $\mathrm{ind}(L_D)$ be the number of negative eigenvalues of L with Dirichlet boundary condition: $Lf + \lambda f = 0, f|_{\partial D} = 0$. The index $\mathrm{ind}(L)$ of L is defined by

$$\mathrm{ind}(L) = \sup\big\{\mathrm{ind}(L_D) | D \in M \text{ rel. comp.}\big\}.$$

Let (M^m, g) be complete locally flat Riemannian manifold of dimension m, and let Δ be the Hodge Laplace–Beltrami operator of M^m that acts on the space of differential $\tilde{p}$-forms. From the Weitzenböck formula [28] we know that

$$\Delta = \nabla^2 - K_{\tilde{p}},$$

where ∇^2 is the Bochner Laplacian, and $K_{\tilde{p}}$ is an endomorphism depending on the curvature of M^m. By choosing an orthonormal basis $\{\theta^1, \dots, \theta^m\}$ dual to $\{e_1, \dots, e_m\}$, we can express $K_{\tilde{p}}$ as

$$\langle K_{\tilde{p}}(\omega), \omega \rangle = \left\langle \sum_{j,k=1}^{m} \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega \right\rangle$$

for $\tilde{p}$-forms ω. In particular, when ω is a 1-form and $\omega^\sharp$ expresses the vector field dual to ω, we have

$$\langle K_1(\omega), \omega \rangle = Ric(\omega^\sharp, \omega^\sharp).$$

We also need the following lemmas, which are important tools in proving our result.

Lemma 2.1 ([23]) *Let (M^m, g) be an m-dimensional complete Riemannian manifold. Then*

$$Ric \geq -|T|g - \frac{|R|}{\sqrt{m}} g$$

in the sense of quadratic forms, where Ric, R, and $T = Ric - \frac{R}{m}g$ are the Ricci curvature tensor, the scalar curvature, and the traceless Ricci tensor of (M^m, g), respectively.

A simply connected and locally conformally flat manifold M^m ($m \geq 3$) has a conformal immersion into S^m. From [13] we know that the Yamabe constant of M^m satisfies $Q(M^m) = Q(S^m) = \frac{m(m-2)\omega_m^{\frac{2}{m}}}{4}$, where ω_m is the volume of the unit sphere in R^m. Hence we have the inequality

$$Q(S^m)\left(\int_M f^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \int_M |\nabla f|^2 + \frac{m-2}{4(m-1)} \int_M Rf^2 \tag{1}$$

for any $f \in C_0^\infty(M)$. By using (1) Lin [23] obtained the following result.

Lemma 2.2 ([23]) *Let (M^m, g) ($m \geq 3$) be an m-dimensional complete, simply connected, and locally conformally flat Riemannian manifold with $R \leq 0$ or $\int_M |R|^{\frac{m}{2}}\, dv < \infty$. Then we have the Sobolev inequality*

$$\left(\int_M f^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq C(m) \int_M |\nabla f|^2 \tag{2}$$

for some constant $C(m) > 0$, which is equal to $Q(S^m)^{-1}$ in the case of $R \leq 0$, where $f \in C_0^\infty(M)$. In particular, M has infinite volume.

Lemma 2.3 ([20, 25]) *Let E be a finite-dimensional subspace of L^{2p} harmonic $\bar{q}$-forms on an m-dimensional complete noncompact Riemannian manifold M for any $p > 0$. Then there exists $\eta \in E$ such that*

$$(\dim E)^{\min\{1,p\}} \int_{B_x(r)} |\eta|^{2p} \leq \operatorname{Vol}\bigl(B_x(r)\bigr) \min\left\{ \binom{m}{\bar{q}}, \dim E \right\}^{\min\{1,p\}} \cdot \sup_{B_x(r)} |\eta|^{2p}$$

for any $x \in M$ and $r > 0$.

3 Proof of Theorem 1.1

Let ω be a nontrivial L^{2p} harmonic 1-form on M, that is,

$$\Delta\omega = 0 \quad \text{and} \quad \int_M |\omega|^{2p} < \infty.$$

By Lemma 2.1 and the Weitzenböck formula [28] we have

$$\frac{1}{2}\Delta|\omega|^2 = \langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + Ric\bigl(\omega^\sharp, \omega^\sharp\bigr) \geq |\nabla\omega|^2 - |T||\omega|^2 - \frac{|R|}{\sqrt{m}}|\omega|^2. \tag{3}$$

Moreover,

$$\frac{1}{2}\Delta|\omega|^2 = |\omega|\Delta|\omega| + \bigl|\nabla|\omega|\bigr|^2. \tag{4}$$

From (3), (4), and the refined Kato inequality $|\nabla\omega|^2 \geq \frac{m}{m-1}|\nabla|\omega||^2$ (see [27]) it follows that

$$|\omega|\Delta|\omega| \geq \frac{1}{m-1}\bigl|\nabla|\omega|\bigr|^2 - |T||\omega|^2 - \frac{|R|}{\sqrt{m}}|\omega|^2. \tag{5}$$

Furthermore $|\nabla|\omega|^p|^2 = p^2|\omega|^{2p-2}|\nabla|\omega||^2$. Combining with (5), we have

$$\begin{aligned}
&|\omega|^p\Delta|\omega|^p \\
&\quad = |\omega|^p p(p-1)|\omega|^{p-2}\bigl|\nabla|\omega|\bigr|^2 + p|\omega|^p|\omega|^{p-1}\Delta|\omega| \\
&\quad = p(p-1)\frac{1}{p^2}\bigl|\nabla|\omega|^p\bigr|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega| \\
&\quad \geq \frac{p-1}{p}\bigl|\nabla|\omega|^p\bigr|^2 + p|\omega|^{2p-2}\left[\frac{1}{m-1}\bigl|\nabla|\omega|\bigr|^2 - |T||\omega|^2 - \frac{|R|}{\sqrt{m}}|\omega|^2\right] \\
&\quad = \left(1 - \frac{1}{p} + \frac{1}{p(m-1)}\right)\bigl|\nabla|\omega|^p\bigr|^2 - p|T||\omega|^{2p} - \frac{p|R|}{\sqrt{m}}|\omega|^{2p},
\end{aligned}$$

that is,

$$|\omega|^p\Delta|\omega|^p \geq \left(1 - \frac{1}{p} + \frac{1}{p(m-1)}\right)\bigl|\nabla|\omega|^p\bigr|^2 - p|T||\omega|^{2p} - \frac{p|R|}{\sqrt{m}}|\omega|^{2p}. \tag{6}$$

Since the operator $\Delta + |T|$ has finite index, there exists a large enough $r_0 > 0$ such that

$$\int_{M\backslash B_{x_0}(r_0)} |T|u^2 \le \int_{M\backslash B_{x_0}(r_0)} |\nabla u|^2 \tag{7}$$

for any fixed point x_0 and any $u \in C_0^\infty(M\backslash B_{x_0}(r_0))$. Choose $r > r_0 + 1$ and $\eta \in C_0^\infty(M\backslash B_{x_0}(r_0))$ such that

$$\eta = \begin{cases} 0 & \text{on } B_{x_0}(r_0) \cup (M\backslash B_{x_0}(2r)), \\ \rho(x_0, x) - r_0 & \text{on } B_{x_0}(r_0+1)\backslash B_{x_0}(r_0), \\ 1 & \text{on } B_{x_0}(r)\backslash B_{x_0}(r_0+1), \\ \frac{2r-\rho(x_0,x)}{r} & \text{on } B_{x_0}(2r)\backslash B_{x_0}(r), \end{cases}$$

where $\rho(x_0, x)$ denotes the geodesic distance from x_0 to x on M. Then, choosing $u = \eta|\omega|^p$ in (7) and noting that $\omega \in H^1(L^{2p}(M))$, we get

$$\int_{M\backslash B_{x_0}(r_0)} |T|\eta^2|\omega|^{2p} \le \int_{M\backslash B_{x_0}(r_0)} \left|\nabla\left(\eta|\omega|^p\right)\right|^2. \tag{8}$$

Multiplying inequality (6) by η^2 and integrating over $M\backslash B_{x_0}(r_0)$, we get

$$\begin{aligned} &\int_{M\backslash B_{x_0}(r_0)} \eta^2|\omega|^p\Delta|\omega|^p \\ &\quad\ge \int_{M\backslash B_{x_0}(r_0)} \eta^2\left(1 - \frac{1}{p} + \frac{1}{p(m-1)}\right)\left|\nabla|\omega|^p\right|^2 - \int_{M\backslash B_{x_0}(r_0)} p\eta^2|T||\omega|^{2p} \\ &\qquad - \int_{M\backslash B_{x_0}(r_0)} \eta^2\frac{p|R|}{\sqrt{m}}|\omega|^{2p}, \end{aligned}$$

that is,

$$\begin{aligned} &\int_{M\backslash B_{x_0}(r_0)} \eta^2|\omega|^p\Delta|\omega|^p + p\int_{M\backslash B_{x_0}(r_0)} \eta^2|T||\omega|^{2p} \\ &\qquad + \frac{p}{\sqrt{m}}\int_{M\backslash B_{x_0}(r_0)} |R|\eta^2|\omega|^{2p} \\ &\quad\ge \left(1 - \frac{1}{p} + \frac{1}{p(m-1)}\right)\int_{M\backslash B_{x_0}(r_0)} \eta^2\left|\nabla|\omega|^p\right|^2. \end{aligned} \tag{9}$$

From integration by parts and (9) we get

$$\begin{aligned} &-2\int_{M\backslash B_{x_0}(r_0)} \eta|\omega|^p\langle\nabla|\omega|^p, \nabla\eta\rangle + p\int_{M\backslash B_{x_0}(r_0)} \eta^2|T||\omega|^{2p} \\ &\qquad + \frac{p}{\sqrt{m}}\int_{M\backslash B_{x_0}(r_0)} |R|\eta^2|\omega|^{2p} \\ &\quad\ge \left(2 - \frac{1}{p} + \frac{1}{p(m-1)}\right)\int_{M\backslash B_{x_0}(r_0)} \eta^2\left|\nabla|\omega|^p\right|^2. \end{aligned} \tag{10}$$

From (2), (8), (10), and Hölder's inequality we can deduce that

$$\begin{aligned}
&\left(2-\frac{1}{p}+\frac{1}{p(m-1)}\right)\int_{M\backslash B_{x_0}(r_0)}\eta^2\left|\nabla|\omega|^p\right|^2\\
&\leq -2\int_{M\backslash B_{x_0}(r_0)}\eta|\omega|^p\left\langle\nabla|\omega|^p,\nabla\eta\right\rangle+p\int_{M\backslash B_{x_0}(r_0)}\eta^2|T||\omega|^{2p}\\
&\quad+\frac{p}{\sqrt{m}}\int_{M\backslash B_{x_0}(r_0)}|R|\eta^2|\omega|^{2p}\\
&\leq -2\int_{M\backslash B_{x_0}(r_0)}\eta|\omega|^p\left\langle\nabla|\omega|^p,\nabla\eta\right\rangle+p\int_{M\backslash B_{x_0}(r_0)}\eta^2|T||\omega|^{2p}\\
&\quad+\frac{p}{\sqrt{m}}\left(\int_{\operatorname{supp}\eta}|R|^{\frac{m}{2}}\right)^{\frac{2}{m}}\left(\int_{M\backslash B_{x_0}(r_0)}\left(\eta|\omega|^p\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}\\
&\leq -2\int_{M\backslash B_{x_0}(r_0)}\eta|\omega|^p\left\langle\nabla|\omega|^p,\nabla\eta\right\rangle+p\int_{M\backslash B_{x_0}(r_0)}\eta^2|T||\omega|^{2p}\\
&\quad+\frac{p}{\sqrt{m}}C(m)\varphi(\eta)\int_{M\backslash B_{x_0}(r_0)}\left|\nabla\left(\eta|\omega|^p\right)\right|^2\\
&\leq 2\int_{M\backslash B_{x_0}(r_0)}\eta|\omega|^p\left|\nabla|\omega|^p\right||\nabla\eta|+p\int_{M\backslash B_{x_0}(r_0)}\left|\nabla\left(\eta|\omega|^p\right)\right|^2\\
&\quad+\frac{p}{\sqrt{m}}C(m)\varphi(\eta)\int_{M\backslash B_{x_0}(r_0)}\left|\nabla\left(\eta|\omega|^p\right)\right|^2\\
&=2\int_{M\backslash B_{x_0}(r_0)}\eta|\omega|^p\left|\nabla|\omega|^p\right||\nabla\eta|\\
&\quad+\left(p+\frac{p}{\sqrt{m}}C(m)\varphi(\eta)\right)\int_{M\backslash B_{x_0}(r_0)}\left|\nabla\left(\eta|\omega|^p\right)\right|^2, \qquad (11)
\end{aligned}$$

where $\varphi(\eta)=(\int_{\operatorname{supp}\eta}|R|^{\frac{m}{2}})^{\frac{2}{m}}$. By the Cauchy–Schwarz inequality and (11) we get

$$\begin{aligned}
&\left(2-\frac{1}{p}+\frac{1}{p(m-1)}\right)\int_{M\backslash B_{x_0}(r_0)}\eta^2\left|\nabla|\omega|^p\right|^2\\
&\leq \varepsilon\int_{M\backslash B_{x_0}(r_0)}\eta^2\left|\nabla|\omega|^p\right|^2+\frac{1}{\varepsilon}\int_{M\backslash B_{x_0}(r_0)}|\omega|^{2p}|\nabla\eta|^2\\
&\quad+\left(p+\frac{p}{\sqrt{m}}C(m)\varphi(\eta)\right)\left\{(1+\varepsilon)\int_{M\backslash B_{x_0}(r_0)}\eta^2\left|\nabla|\omega|^p\right|^2\right.\\
&\quad\left.+\left(1+\frac{1}{\varepsilon}\right)\int_{M\backslash B_{x_0}(r_0)}|\omega|^{2p}|\nabla\eta|^2\right\}\\
&=\left[\varepsilon+\left(p+\frac{p}{\sqrt{m}}C(m)\varphi(\eta)\right)(1+\varepsilon)\right]\int_{M\backslash B_{x_0}(r_0)}\eta^2\left|\nabla|\omega|^p\right|^2\\
&\quad+\left[\frac{1}{\varepsilon}+\left(p+\frac{p}{\sqrt{m}}C(m)\varphi(\eta)\right)\left(1+\frac{1}{\varepsilon}\right)\right]\int_{M\backslash B_{x_0}(r_0)}|\omega|^{2p}|\nabla\eta|^2
\end{aligned}$$

for any $\varepsilon > 0$. Together with this inequality, we have the inequality

$$A \int_{M \setminus B_{x_0}(r_0)} \eta^2 \big|\nabla |\omega|^p\big|^2 \leq B \int_{M \setminus B_{x_0}(r_0)} |\omega|^{2p} |\nabla \eta|^2, \tag{12}$$

where A and B are two constants defined as

$$A = \left(2 - \frac{1}{p} + \frac{1}{p(m-1)}\right) - \left(p + \frac{p}{\sqrt{m}} C(m)\varphi(\eta)\right)(1+\varepsilon) - \varepsilon \tag{13}$$

and

$$B = \frac{1}{\varepsilon} + \left(p + \frac{p}{\sqrt{m}} C(m)\varphi(\eta)\right)\left(1 + \frac{1}{\varepsilon}\right).$$

According to the hypothetical condition $\int_M |R|^{\frac{m}{2}} < \infty$, there exists a large enough r_0 such that

$$\int_{M \setminus B_{x_0}(r_0)} |R|^{\frac{m}{2}} < \left(\frac{\sqrt{m}[1-(m-1)(p-1)^2]}{p^2(m-1)C(m)}\right)^{\frac{m}{2}}, \tag{14}$$

where $p \in (1 - \sqrt{\frac{1}{m-1}}, 1 + \sqrt{\frac{1}{m-1}})$ by (13) and (14). From (14) and the definition of η it follows that

$$\varphi(\eta) = \left(\int_{\operatorname{supp}\eta} |R|^{\frac{m}{2}}\right)^{\frac{2}{m}} < \frac{\sqrt{m}[1-(m-1)(p-1)^2]}{p^2(m-1)C(m)}. \tag{15}$$

By (13) and (15) we have

$$A = \left(2 - \frac{1}{p} + \frac{1}{p(m-1)}\right) - \left(p + \frac{p}{\sqrt{m}} C(m)\varphi(\eta)\right)(1+\varepsilon) - \varepsilon > 0,$$

for ε small enough. Therefore from (12) we have

$$\int_{M \setminus B_{x_0}(r_0)} \eta^2 \big|\nabla |\omega|^p\big|^2 \leq D(m,p) \int_{M \setminus B_{x_0}(r_0)} |\omega|^{2p} |\nabla \eta|^2, \tag{16}$$

where $D(m,p)$ is a positive constant depending only on m and p.

On the other hand, applying the Sobolev inequality (2) to the term $\eta|\omega|^p$, we have

$$\begin{aligned}
&\left(\int_{M \setminus B_{x_0}(r_0)} \left(\eta|\omega|^p\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \\
&\quad \leq C(m) \int_{M \setminus B_{x_0}(r_0)} \big|\nabla\left(\eta|\omega|^p\right)\big|^2 \\
&\quad \leq 2C(m) \int_{M \setminus B_{x_0}(r_0)} \left[\eta^2 \big|\nabla |\omega|^p\big|^2 + |\omega|^{2p}|\nabla\eta|^2\right],
\end{aligned} \tag{17}$$

where $C(m) > 0$ is the Sobolev constant. From (16) and (17) we have

$$\begin{aligned}&\left(\int_{M\setminus B_{x_0}(r_0)}\left(\eta|\omega|^p\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}\\&\quad\leq 2C(m)D(m,p)\int_{M\setminus B_{x_0}(r_0)}|\omega|^{2p}|\nabla\eta|^2+2C(m)\int_{M\setminus B_{x_0}(r_0)}|\omega|^{2p}|\nabla\eta|^2\\&\quad= C_1(m,p)\int_{M\setminus B_{x_0}(r_0)}|\omega|^{2p}|\nabla\eta|^2,\end{aligned}\tag{18}$$

where $C_1(m,p) = C(m)(1 + D(m,p))$ is a positive constant depending only on m and p. From here the proof mainly follows by the standard techniques (e.g., see [3]). Applying the definition of η to inequality (18), we get

$$\begin{aligned}&\left(\int_{B_{x_0}(r)\setminus B_{x_0}(r_0+1)}\left(|\omega|^p\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}\\&\quad\leq C_2(m,p)\int_{B_{x_0}(r_0+1)\setminus B_{x_0}(r_0)}|\omega|^{2p}+\frac{C_2(m,p)}{r^2}\int_{B_{x_0}(2r)\setminus B_{x_0}(r)}|\omega|^{2p},\end{aligned}$$

where $C_2(m,p) = C(m)(1 + D(m,p))$ is a positive constant depending only on m and p.

Then, letting $r \to \infty$ and noting that $|\omega| \in L^{2p}(M)$, we have

$$\left(\int_{M\setminus B_{x_0}(r_0+1)}\left(|\omega|^p\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}\leq C_2(m,p)\int_{B_{x_0}(r_0+1)\setminus B_{x_0}(r_0)}|\omega|^{2p}.\tag{19}$$

By Hölder's inequality we conclude that

$$\begin{aligned}&\int_{B_{x_0}(r_0+2)\setminus B_{x_0}(r_0+1)}|\omega|^{2p}\\&\quad\leq\left[\operatorname{Vol}\left(B_{x_0}(r_0+2)\right)\right]^{\frac{2}{m}}\left(\int_{B_{x_0}(r_0+2)\setminus B_{x_0}(r_0+1)}\left(|\omega|^p\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}.\end{aligned}\tag{20}$$

From (19) and (20) we have

$$\begin{aligned}&\int_{B_{x_0}(r_0+2)}|\omega|^{2p}\\&\quad\leq\left[\operatorname{Vol}\left(B_{x_0}(r_0+2)\right)\right]^{\frac{2}{m}}\left(\int_{B_{x_0}(r_0+2)\setminus B_{x_0}(r_0+1)}\left(|\omega|^p\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}\\&\qquad+\int_{B_{x_0}(r_0+2)}|\omega|^{2p}\\&\quad\leq\left[\operatorname{Vol}\left(B_{x_0}(r_0+2)\right)\right]^{\frac{2}{m}}C_2(m,p)\int_{B_{x_0}(r_0+1)}|\omega|^{2p}+\int_{B_{x_0}(r_0+2)}|\omega|^{2p}\\&\quad=\left[1+\left[\operatorname{Vol}\left(B_{x_0}(r_0+2)\right)\right]^{\frac{2}{m}}C_2(m,p)\right]\int_{B_{x_0}(r_0+1)}|\omega|^{2p},\end{aligned}$$

that is,

$$\int_{B_{x_0}(r_0+2)} |\omega|^{2p} \le C_3 \int_{B_{x_0}(r_0+1)} |\omega|^{2p}, \tag{21}$$

where C_3 is a positive constant depending only on $\mathrm{Vol}(B_{x_0}(r_0+2))$, m, and p.

Let $F : M \to [0,\infty)$ be the function defined by $F = p|T| + p\frac{|R|}{\sqrt{m}}$. From (6) we have

$$\begin{aligned} |\omega|^p \Delta|\omega|^p &\ge \left(1 - \frac{1}{p} + \frac{1}{p(m-1)}\right)\left|\nabla|\omega|^p\right|^2 - p|T||\omega|^{2p} - \frac{p|R|}{\sqrt{m}}|\omega|^{2p} \\ &= \left(1 - \frac{1}{p} + \frac{1}{p(m-1)}\right)\left|\nabla|\omega|^p\right|^2 - F|\omega|^{2p}. \end{aligned} \tag{22}$$

Fix $x \in M$ and choose $\mu \in C_0^\infty(B_x(1))$. Multiplying (22) by $\mu^2|\omega|^{p(q-2)}$ with $q \ge 2$ and integrating by parts, we have

$$\begin{aligned} &\left(1 - \frac{1}{p} + \frac{1}{p(m-1)}\right) \int_{B_x(1)} \mu^2|\omega|^{p(q-2)}\left|\nabla|\omega|^p\right|^2 - \int_{B_x(1)} F\mu^2|\omega|^{pq} \\ &\quad\le -(q-1)\int_{B_x(1)} \mu^2|\omega|^{p(q-2)}\left|\nabla|\omega|^p\right|^2 - 2\int_{B_x(1)} |\omega|^{p(q-1)}\mu\langle\nabla|\omega|^p, \nabla\mu\rangle \\ &\quad\le \left[-(q-1) + \frac{1}{p(m-1)}\right]\int_{B_x(1)} \mu^2|\omega|^{p(q-2)}\left|\nabla|\omega|^p\right|^2 \\ &\qquad + p(m-1)\int_{B_x(1)} |\omega|^{pq}|\nabla\mu|^2, \end{aligned} \tag{23}$$

where the second inequality follows from

$$\begin{aligned} &-2\int_{B_x(1)} |\omega|^{p(q-1)}\mu\langle\nabla|\omega|^p, \nabla\mu\rangle \\ &\quad\le \frac{1}{p(m-1)}\int_{B_x(1)} \mu^2|\omega|^{p(q-2)}\left|\nabla|\omega|^p\right|^2 + p(m-1)\int_{B_x(1)} |\omega|^{pq}|\nabla\mu|^2. \end{aligned}$$

Then from (23) we have

$$\begin{aligned} &\left(q - \frac{1}{p}\right)\int_{B_x(1)} \mu^2|\omega|^{p(q-2)}\left|\nabla|\omega|^p\right|^2 \\ &\quad\le \int_{B_x(1)} F\mu^2|\omega|^{pq} + p(m-1)\int_{B_x(1)} |\nabla\mu|^2|\omega|^{pq}. \end{aligned} \tag{24}$$

By the Cauchy–Schwarz inequality we have

$$\begin{aligned} &\int_{B_x(1)} \left|\nabla\left(\mu\left(|\omega|^p\right)^{\frac{q}{2}}\right)\right|^2 \\ &\quad\le (1+q)\int_{B_x(1)} |\omega|^{pq}|\nabla\mu|^2 + \left(1 + \frac{1}{q}\right)\frac{q^2}{4}\int_{B_x(1)} |\omega|^{p(q-2)}\mu^2\left|\nabla|\omega|^p\right|^2. \end{aligned} \tag{25}$$

From (24) and (25) we have

$$\begin{aligned}
&\int_{B_x(1)} \bigl|\nabla\bigl(\mu\bigl(|\omega|^p\bigr)^{\frac{q}{2}}\bigr)\bigr|^2 \\
&\leq \frac{pq(1+q)}{4(pq-1)}\biggl[\int_{B_x(1)} F\mu^2|\omega|^{pq} + p(m-1)\int_{B_x(1)} |\nabla\mu|^2|\omega|^{pq}\biggr] \\
&\quad + (1+q)\int_{B_x(1)} |\omega|^{pq}|\nabla\mu|^2 \\
&= \biggl[\frac{p^2q(1+q)(m-1)}{4(pq-1)} + q + 1\biggr]\int_{B_x(1)} |\nabla\mu|^2|\omega|^{pq} \\
&\quad + \frac{pq(1+q)}{4(pq-1)}\int_{B_x(1)} F\mu^2|\omega|^{pq} \\
&\leq 4m^2p^2q\int_{B_x(1)} F\mu^2|\omega|^{pq} + 4m^2p^2q\int_{B_x(1)} |\nabla\mu|^2|\omega|^{pq}.
\end{aligned} \tag{26}$$

Using (26) and applying the Sobolev inequality (2) to $\mu|\omega|^{\frac{pq}{2}}$, we have

$$\begin{aligned}
&\biggl(\int_{B_x(1)} \bigl|\mu|\omega|^{\frac{pq}{2}}\bigr|^{\frac{2m}{m-2}}\biggr)^{\frac{m-2}{m}} \\
&\leq C(m)\int_{B_x(1)} \bigl|\nabla\bigl(\mu|\omega|^{\frac{pq}{2}}\bigr)\bigr|^2 \\
&\leq C(m)\biggl[4m^2p^2q\int_{B_x(1)} F\mu^2|\omega|^{pq} + 4m^2p^2q\int_{B_x(1)} |\nabla\mu|^2|\omega|^{pq}\biggr] \\
&\leq 4m^2p^2qC_4\int_{B_x(1)} \bigl(\mu^2 + |\nabla\mu|^2\bigr)|\omega|^{pq},
\end{aligned} \tag{27}$$

where C_4 is a positive constant depending only on m, p, and $\sup_{B_x(1)} F$.

Let $q_k = \frac{2m^k}{(m-2)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$ for $k = 0, 1, 2, \ldots$. Choose a function $\mu_k \in C_0^\infty(B_x(\rho_k))$ as follows:

$$\begin{cases}
0 \leq \mu_k \leq 1, \\
\mu_k = 1 \quad \text{on } B_x(\rho_{k+1}), \\
|\nabla\mu_k| \leq 2^{k+3}.
\end{cases}$$

By choosing $q = q_k$ and $\mu = \mu_k$ in (27) we get

$$\begin{aligned}
&\biggl(\int_{B_x(\rho_{k+1})} \bigl||\omega|^{\frac{pq_k}{2}}\bigr|^{\frac{2m}{m-2}}\biggr)^{\frac{m-2}{m}} \\
&\leq 4m^2p^2q_kC_4\int_{B_x(\rho_k)} \bigl[1 + |\nabla\mu|^2\bigr]|\omega|^{pq_k} \\
&\leq 4m^2p^2q_kC_4\int_{B_x(\rho_k)} \bigl[1 + 4^{k+3}\bigr]|\omega|^{pq_k} \\
&\leq 4^kq_k\bigl(4m^2p^2C_4 + 4^4m^2p^2C_4\bigr)\int_{B_x(\rho_k)} |\omega|^{pq_k}.
\end{aligned}$$

Then we have

$$\left(\int_{B_x(\rho_{k+1})}|\omega|^{pq_{k+1}}\right)^{\frac{1}{q_{k+1}}}\leq\left(q_k\cdot 4^{k+k_0}\right)^{\frac{1}{q_k}}\left(\int_{B_x(\rho_k)}|\omega|^{pq_k}\right)^{\frac{1}{q_k}},$$

where k_0 is a positive integer such that $4m^2p^2C_4(1+4^3)\leq 4^{k_0}$. From the above inequality and the Morse iteration we conclude that

$$\left(\int_{B_x(\rho_{k+1})}|\omega|^{pq_{k+1}}\right)^{\frac{1}{q_{k+1}}}\leq\prod_{i=0}^{k}q_i^{\frac{1}{q_i}}4^{\frac{i+4}{q_i}}\left(\int_{B_x(1)}|\omega|^{2p}\right)^{\frac{1}{2}}. \tag{28}$$

Letting $k\to\infty$ in (28), we have

$$\left\||\omega|^p\right\|_{L^\infty(B_x(\frac{1}{2}))}\leq C_5\left\||\omega|^p\right\|_{L^2(B_x(1))}, \tag{29}$$

where C_5 is a positive constant depending only on m, p, and $\sup_{B_x(1)}F$. Now take $y\in B_{x_0}(r_0+1)$ such that

$$|\omega|^{2p}(y)=\sup_{B_{x_0}(r_0+1)}|\omega|^{2p}. \tag{30}$$

Note that $B_y(1)\subset B_{x_0}(r_0+2)$. Then (29) and (30) imply that

$$\sup_{B_{x_0}(r_0+1)}|\omega|^{2p}\leq C_5\int_{B_{x_0}(r_0+2)}|\omega|^{2p}. \tag{31}$$

From (21) and (31) we have

$$\sup_{B_{x_0}(r_0+1)}|\omega|^{2p}\leq C_6\int_{B_{x_0}(r_0+1)}|\omega|^{2p}, \tag{32}$$

where C_6 is a positive constant depending only on m, p, $\mathrm{Vol}(B_{x_0}(r_0+2))$, and $\sup_{B_{x_0}(r_0+2)}F$.

To show the finiteness of the dimension of $H^1(L^{2p}(M))$, we only need to prove that the dimension of any finite-dimensional subspace of $H^1(L^{2p}(M))$ is upper bounded by a fixed constant. Let $\mathcal{W}$ be any finite-dimensional subspace of $H^1(L^{2p}(M))$. By Lemma 2.3 there exists $\omega\in\mathcal{W}$ such that

$$\begin{aligned}&(\dim\mathcal{W})^{\min\{1,p\}}\int_{B_{x_0}(r_0+1)}|\omega|^{2p}\\&\quad\leq\mathrm{Vol}\big(B_{x_0}(r_0+1)\big)\min\{m,\dim\mathcal{W}\}^{\{\min\{1,p\}}\cdot\sup_{B_{x_0}(r_0+1)}|\omega|^{2p}.\end{aligned}$$

This, together with (32), yields that $\dim\mathcal{W}$ is upper bounded by a fixed constant, that is, $\dim\mathcal{W}\leq C_7$, where C_7 depends only on m, p, $\mathrm{Vol}(B_{x_0}(r_0+2))$, and $\sup_{B_{x_0}(r_0+2)}F$. Then we obtain that $\dim H^1(L^{2p}(M))<\infty$. This completes the proof of Theorem 1.1.

4 Conclusions

In this paper, we study the dimension of the space of L^p harmonic 1-forms on a locally conformally flat Riemannian manifold M^m. The key to our research is Lemma 2.3 for L^{2p}

harmonic $\bar{q}$-forms [20, 25] and the geometric analysis techniques. With their help, we prove that the dimension of the space of L^p harmonic 1-forms must be finite for certain p under the assumptions on the Schrödinger operators involving the squared norm of the traceless Ricci form. This result can be regarded as a generalization of Han's result [16] for L^2 harmonic 1-forms.

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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]School of Science, Nanjing University of Science and Technology, 210094, Nanjing, China. [2]School of Mathematics and Statistics, Southwest University, 400715, Chongqing, China.

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