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Conditions for the validity of a class of optimal Hilbert type multiple integral inequalities with nonhomogeneous kernels

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Abstract

For the Hilbert type multiple integral inequality

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K(\|x\|_{m,\rho},\|y\|_{n,\rho})f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \le M\|f\|_{p,\alpha}\|g\|_{q,\beta}$$

with a nonhomogeneous kernel $K(\|x\|_{m,\rho},\|y\|_{n,\rho}) = G(\|x\|^{\lambda_1}_{m,\rho}/\|y\|^{\lambda_2}_{n,\rho})$ $(\lambda_1\lambda_2 > 0)$, in this paper, by using the weight function method, necessary and sufficient conditions that parameters p, q, λ_1, λ_2, α, β, m, and n should satisfy to make the inequality hold for some constant M are established, and the expression formula of the best constant factor is also obtained. Finally, their applications in operator boundedness and operator norm are also considered, and the norms of several integral operators are discussed.

MSC: 26D15; 47A07

Keywords: Nonhomogeneous kernel; Hilbert type multiple integral inequality; Parameter condition; Boundedness of operator; Operator norm

1 Introduction and preparatory knowledge

Let $k \in \mathbb{N} = \{1,2,3,\ldots\}$, $\rho > 0$, $x = (x_1,x_2,\ldots,x_k)$, $\mathbb{R}_+^k = \{x = (x_1,x_2,\ldots,x_k) : x_i > 0, i = 1,2,\ldots,k\}$, $\|x\|_{k,\rho} = (x_1^\rho + x_2^\rho + \cdots + x_k^\rho)^{1/\rho}$. Define

$$L_p^\alpha(\mathbb{R}_+^k) = \left\{f(x) \ge 0 : \|f\|_{p,\alpha} = \left(\int_{\mathbb{R}_+^k} \|x\|^\alpha_{k,\rho} f^p(x)\,\mathrm{d}x\right)^{1/p} < +\infty\right\}.$$

In this paper, for a class of nonhomogeneous kernels $K(\|x\|_{m,\rho},\|y\|_{n,\rho}) = G(\|x\|^{\lambda_1}_{m,\rho}/\|y\|^{\lambda_2}_{n,\rho})$ $(\lambda_1\lambda_2 > 0)$, we discuss the equivalent parameter conditions for the validity of Hilbert type multiple integral inequality

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \le M\|f\|_{p,\alpha}\|g\|_{q,\beta}. \tag{1}$$

That is, what conditions do parameters λ_1, λ_2, p, q, α, β satisfy to make (1) hold? On the contrary, what conditions do the parameters satisfy when (1) holds? Meanwhile, the best constant factor and its application in operator theory are also considered.

In [1], we studied the necessary and sufficient conditions for the validity of Hilbert type multiple integral inequalities with kernel $K(\|x\|_{m,\rho}, \|y\|_{n,\rho}) = G(\|x\|_{m,\rho}^{\lambda_1}\|y\|_{n,\rho}^{\lambda_2})$ $(\lambda_1\lambda_2 > 0)$. The present paper is a supplement and improvement of [1], more relevant research can be referred to [2–20].

Lemma 1.1 ([21]) *Let $p_i > 0$, $a_i > 0$, $\alpha_i > 0(i = 1,2,\dots,n)$, $\psi(u)$ be measurable. Then*

$$\int_{(\frac{x_1}{a_1})^{\alpha_1}+\cdots+(\frac{x_n}{a_n})^{\alpha_n}\le 1;x_i>0} \psi\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\cdots+\left(\frac{x_n}{a_n}\right)^{\alpha_n}\right)x_1^{p_1-1}\cdots x_n^{p_n-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$
$$= \frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma(\frac{p_1}{\alpha_1})\cdots\Gamma(\frac{p_n}{\alpha_n})}{\alpha_1\cdots\alpha_n\Gamma(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n})}\int_0^1 \psi(u)u^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1}\,\mathrm{d}u,$$

where $\Gamma(t)$ represents the gamma function.

By using Lemma 1.1, under the same conditions, it is not difficult to obtain: Let $\varphi(u)$ be measurable, $\rho > 0$, $n \ge 1$, $x = (x_1,x_2,\dots,x_n) \in \mathbb{R}_+^n$. Then

$$\begin{aligned}\int_{\|x\|_{n,\rho}\le r}\varphi\left(\|x\|_{n,\rho}\right)\mathrm{d}x &= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_0^r\varphi(u)u^{n-1}\,\mathrm{d}u,\\ \int_{\|x\|_{n,\rho}\ge r}\varphi\left(\|x\|_{n,\rho}\right)\mathrm{d}x &= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_r^{+\infty}\varphi(u)u^{n-1}\,\mathrm{d}u.\end{aligned} \tag{2}$$

Suppose that $K(u,v) = G(u^{\lambda_1}/v^{\lambda_2})$, then obviously $K(u,v)$ satisfies the following property:

$$K(u,v) = K\left(1,u^{-\lambda_1/\lambda_2}v\right) = K\left(v^{-\lambda_2/\lambda_1}u,1\right).$$

Lemma 1.2 *Let $\frac{1}{p}+\frac{1}{q} = 1(p > 1)$, $\rho > 0$, $m,n \in \mathbb{N}$, $K(\|x\|_{m,\rho}, \|y\|_{n,\rho}) = G(\|x\|_{m,\rho}^{\lambda_1}/\|y\|_{n,\rho}^{\lambda_2})$, $\alpha,\beta \in \mathbb{R}$. Then*

$$\begin{aligned}\omega_1(m,\alpha,p,y) &= \int_{\mathbb{R}_+^m}K\left(\|x\|_{m,\rho},\|y\|_{n,\rho}\right)\|x\|_{m,\rho}^{-\frac{\alpha+m}{p}}\,\mathrm{d}x\\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\|y\|_{n,\rho}^{\frac{\lambda_2}{\lambda_1}(-\frac{\alpha+m}{p}+m)}\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t\\ &:= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\|y\|_{n,\rho}^{\frac{\lambda_2}{\lambda_1}(-\frac{\alpha+m}{p}+m)}W_1,\end{aligned} \tag{3}$$

$$\begin{aligned}\omega_2(n,\beta,q,x) &= \int_{\mathbb{R}_+^n}K\left(\|x\|_{m,\rho},\|y\|_{n,\rho}\right)\|y\|_{n,\rho}^{-\frac{\beta+n}{q}}\,\mathrm{d}y\\ &= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\|x\|_{m,\rho}^{\frac{\lambda_1}{\lambda_2}(-\frac{\beta+n}{q}+n)}\int_0^{+\infty}K(1,t)t^{-\frac{\beta+n}{q}+n-1}\,\mathrm{d}t\\ &:= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\|x\|_{m,\rho}^{\frac{\lambda_1}{\lambda_2}(-\frac{\beta+n}{q}+n)}W_2.\end{aligned} \tag{4}$$

Moreover, if $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q} = 0$, then $\lambda_1W_1 = \lambda_2W_2$.

Proof It follows from (2) that

$$\begin{aligned}\omega_1(m,\alpha,p,y) &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^{+\infty}K\big(u,\|y\|_{n,\rho}\big)u^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}u \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^{+\infty}K\big(\|y\|_{n,\rho}^{-\lambda_2/\lambda_1}u,1\big)u^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}u \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\|y\|_{n,\rho}^{\frac{\lambda_2}{\lambda_1}(-\frac{\alpha+m}{p}+m-1)+\frac{\lambda_2}{\lambda_1}} \\ &\quad\times\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\|y\|_{n,\rho}^{\frac{\lambda_2}{\lambda_1}(-\frac{\alpha+m}{p}+m)}W_1.\end{aligned}$$

(4) can be proved at the same time.

When $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=0$, notice that $\lambda_1\lambda_2>0$, we have

$$\begin{aligned}W_1 &= \int_0^{+\infty}K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t \\ &= \int_0^{+\infty}K\big(1,t^{-\lambda_1/\lambda_2}\big)t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t \\ &= \frac{\lambda_2}{\lambda_1}\int_0^{+\infty}K(1,u)u^{-\frac{\lambda_2}{\lambda_1}(-\frac{\alpha+m}{p}+m-1)-\frac{\lambda_2}{\lambda_1}-1}\,\mathrm{d}u \\ &= \frac{\lambda_2}{\lambda_1}\int_0^{+\infty}K(1,u)u^{-\frac{\beta+n}{q}+n-1}\,\mathrm{d}u \\ &= \frac{\lambda_2}{\lambda_1}W_2.\end{aligned}$$

Thus $\lambda_1W_1=\lambda_2W_2$. □

2 Main results

Theorem 2.1 *Let* $\frac{1}{p}+\frac{1}{q}=1(p>1)$, $\rho>0$, $m,n\in\mathbb{N}$, $\lambda_1\lambda_2>0$, $\alpha,\beta\in\mathbb{R}$, $K(\|x\|_{m,\rho},\|y\|_{n,\rho})=G(\|x\|_{m,\rho}^{\lambda_1}/\|y\|_{n,\rho}^{\lambda_2})(\lambda_1\lambda_2>0)$ *be nonnegative measurable and*

$$W_0=|\lambda_1|\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t$$

be convergent. Then

(*i*) *If and only if* $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=0$, *there exists a constant* $M>0$ *such that*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m}K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\le M\|f\|_{p,\alpha}\|g\|_{q,\beta}, \tag{5}$$

where $f(x)\in L_p^\alpha(\mathbb{R}_+^m)$, $g(y)\in L_q^\beta(\mathbb{R}_+^n)$.

(*ii*) *When* (5) *holds, the best constant factor is*

$$\inf M=\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}.$$

Proof Let $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=c$.

(i) Suppose that (5) holds. We prove that $c=0$. Consider the case of $\lambda_1>0$, $\lambda_2>0$. If $c>0$, take $0<\varepsilon<\frac{c}{\lambda_1\lambda_2}$ and

$$f(x)=\begin{cases}\|x\|_{m,\rho}^{(-\alpha-m-\lambda_1\varepsilon)/p}, & \|x\|_{m,\rho}\geq 1,\\ 0, & 0<\|x\|_{m,\rho}<1,\end{cases}$$

$$g(y)=\begin{cases}\|y\|_{n,\rho}^{(-\beta-n-\lambda_2\varepsilon)/q}, & \|y\|_{n,\rho}\geq 1,\\ 0, & 0<\|y\|_{n,\rho}<1.\end{cases}$$

Then

$$\begin{aligned}M\|f\|_{p,\alpha}\|g\|_{q,\beta}&=M\left(\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-m-\lambda_1\varepsilon}\,\mathrm{d}x\right)^{1/p}\left(\int_{\|y\|_{n,\rho}\geq 1}\|y\|_{n,\rho}^{-n-\lambda_2\varepsilon}\,\mathrm{d}y\right)^{1/q}\\&=M\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q}\\&\quad\times\left(\int_1^{+\infty}u^{-1-\lambda_1\varepsilon}\,\mathrm{d}u\right)^{1/p}\left(\int_1^{+\infty}u^{-1-\lambda_2\varepsilon}\,\mathrm{d}u\right)^{1/q}\\&=\frac{M}{\varepsilon\lambda_1^{1/p}\lambda_2^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q},\end{aligned}$$

$$\begin{aligned}&\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m}K\left(\|x\|_{m,\rho},\|y\|_{n,\rho}\right)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{(-\alpha-m-\lambda_1\varepsilon)/p}\left(\int_{\|y\|_{m,\rho}\geq 1}K\left(\|x\|_{m,\rho},\|y\|_{n,\rho}\right)\|y\|_{n,\rho}^{(-\beta-n-\lambda_2\varepsilon)/q}\,\mathrm{d}y\right)\mathrm{d}x\\&=\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-\frac{\alpha+m+\lambda_1\varepsilon}{p}}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_1^{+\infty}K\left(\|x\|_{m,\rho},u\right)u^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}u\right)\mathrm{d}x\\&=\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-\frac{\alpha+m+\lambda_1\varepsilon}{p}}\left(\int_1^{+\infty}K\left(1,u\|x\|_{m,\rho}^{-\lambda_1/\lambda_2}\right)u^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}u\right)\mathrm{d}x\\&=\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-\frac{\alpha+m+\lambda_1\varepsilon}{p}+\frac{\lambda_1}{\lambda_2}(-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1)+\frac{\lambda_1}{\lambda_2}}\\&\quad\times\left(\int_{\|x\|_{m,\rho}^{-\lambda_1/\lambda_2}}^{+\infty}K(1,t)t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}t\right)\mathrm{d}x\\&=\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-m+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\left(\int_{\|x\|_{m,\rho}^{-\lambda_1/\lambda_2}}^{+\infty}K(1,t)t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}t\right)\mathrm{d}x\\&\geq\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-m+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\left(\int_1^{+\infty}K(1,t)t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}t\right)\mathrm{d}x\\&=\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_1^{+\infty}K(1,t)t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}t\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-m+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,\mathrm{d}x\\&=\frac{\Gamma^{m+n}(1/\rho)}{\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\int_1^{+\infty}K(1,t)t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}t\int_1^{+\infty}u^{-1+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,\mathrm{d}u.\end{aligned}$$

It follows that

$$\frac{\Gamma^{m+n}(1/\rho)}{\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\int_1^{+\infty}K(1,t)t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}t\int_1^{+\infty}u^{-1+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,\mathrm{d}u \le \frac{M}{\varepsilon\lambda_1^{1/p}\lambda_2^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q} < +\infty.$$

But since $0 < \varepsilon < \frac{c}{\lambda_1\lambda_2}$, we have $\frac{c}{\lambda_2} - \lambda_1\varepsilon > 0$ and $\int_1^{+\infty} u^{-1+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,\mathrm{d}u = +\infty$, which is contradictory, hence $c > 0$ is not valid.

If $c < 0$, take $0 < \varepsilon < \frac{-c}{\lambda_1\lambda_2}$ and

$$f(x) = \begin{cases} \|x\|_{m,\rho}^{(-\alpha-m+\lambda_1\varepsilon)/p}, & 0 < \|x\|_{m,\rho} \le 1, \\ 0, & \|x\|_{m,\rho} > 1. \end{cases}$$

$$g(y) = \begin{cases} \|y\|_{n,\rho}^{(-\beta-n+\lambda_2\varepsilon)/q}, & 0 < \|y\|_{n,\rho} \le 1, \\ 0, & \|y\|_{n,\rho} > 1. \end{cases}$$

Similarly, we can get

$$\frac{\Gamma^{m+n}(1/\rho)}{\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\int_0^{1}K(1,t)t^{-\frac{\beta+n-\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}t\int_0^{1}u^{-1+\frac{c}{\lambda_2}+\lambda_1\varepsilon}\,\mathrm{d}u \le \frac{M}{\varepsilon\lambda_1^{1/p}\lambda_2^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q} < +\infty.$$

Since $0 < \varepsilon < \frac{-c}{\lambda_1\lambda_2}$, we obtain $\frac{c}{\lambda_2} + \lambda_1\varepsilon < 0$ and $\int_0^1 u^{-1+\frac{c}{\lambda_2}+\lambda_1\varepsilon}\,\mathrm{d}u = +\infty$, this is still a contradiction, hence $c < 0$ cannot hold.

To sum up, when $\lambda_1 > 0$, $\lambda_2 > 0$, we have $c = 0$, that is, $\frac{n\lambda_1-\alpha\lambda_2}{p} + \frac{m\lambda_2-\beta\lambda_1}{q} = 0$.

Moreover, consider the case of $\lambda_1 < 0$, $\lambda_2 < 0$. If $c > 0$, take $0 < \varepsilon < \frac{c}{\lambda_1\lambda_2}$ and

$$f(x) = \begin{cases} \|x\|_{m,\rho}^{(-\alpha-m-\lambda_1\varepsilon)/p}, & 0 < \|x\|_{m,\rho} \le 1, \\ 0, & \|x\|_{m,\rho} > 1, \end{cases}$$

$$g(y) = \begin{cases} \|y\|_{n,\rho}^{(-\beta-n-\lambda_2\varepsilon)/q}, & 0 < \|y\|_{n,\rho} \le 1, \\ 0, & \|y\|_{n,\rho} > 1. \end{cases}$$

Then, by calculation,

$$M\|f\|_{p,\alpha}\|g\|_{q,\beta} = \frac{M}{\varepsilon(-\lambda_1)^{1/p}(-\lambda_2)^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q},$$

$$\begin{aligned}
&\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m}K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{0<\|x\|_{m,\rho}\le 1}\|x\|_{m,\rho}^{(-\alpha-m-\lambda_1\varepsilon)/p}\left(\int_{0<\|y\|_{n,\rho}\le 1}K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)\|y\|_{n,\rho}^{(-\beta-n-\lambda_2\varepsilon)/q}\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{0<\|x\|_{m,\rho}\le 1}\|x\|_{m,\rho}^{-\frac{\alpha+m+\lambda_1\varepsilon}{p}}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\int_0^1 K\big(\|x\|_{m,\rho},u\big)u^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1}\,\mathrm{d}u\right)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)} \int_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-\frac{\alpha+m+\lambda_1\varepsilon}{p}} \left(\int_0^1 K\big(1, u\|x\|_{m,\rho}^{-\lambda_1/\lambda_2}\big) u^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}u \right) \mathrm{d}x \\
&= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)} \int_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-\frac{\alpha+m+\lambda_1\varepsilon}{p}+\frac{\lambda_1}{\lambda_2}(-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1)+\frac{\lambda_1}{\lambda_2}} \\
&\quad \times \left(\int_0^{\|x\|_{m,\rho}^{-\lambda_1/\lambda_2}} K(1,t) t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}t \right) \mathrm{d}x \\
&= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)} \int_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-m+\frac{c}{\lambda_2}-\lambda_1\varepsilon} \left(\int_0^{\|x\|_{m,\rho}^{-\lambda_1/\lambda_2}} K(1,t) t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}t \right) \mathrm{d}x \\
&\ge \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)} \int_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-m+\frac{c}{\lambda_2}-\lambda_1\varepsilon} \left(\int_0^1 K(1,t) t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}t \right) \mathrm{d}x \\
&= \frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)} \int_0^1 K(1,t) t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}t \int_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-m+\frac{c}{\lambda_2}-\lambda_1\varepsilon} \,\mathrm{d}x \\
&= \frac{\Gamma^{m+n}(1/\rho)}{\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)} \int_0^1 K(1,t) t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}t \int_0^1 u^{-1+\frac{c}{\lambda_2}-\lambda_1\varepsilon} \,\mathrm{d}u.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{\Gamma^{m+n}(1/\rho)}{\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)} \int_0^1 K(1,t) t^{-\frac{\beta+n+\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}t \int_0^1 u^{-1+\frac{c}{\lambda_2}-\lambda_1\varepsilon} \,\mathrm{d}u \\
&\quad \le \frac{M}{\varepsilon(-\lambda_1)^{1/p}(-\lambda_2)^{1/q}} \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \right)^{1/p} \left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)} \right)^{1/q} < +\infty.
\end{aligned}
$$

Since $0 < \varepsilon < \frac{c}{\lambda_1\lambda_2}$ and $\lambda_1 < 0$, then $\frac{c}{\lambda_2} - \lambda_1\varepsilon < 0$ and $\int_0^1 u^{-1+\frac{c}{\lambda_2}-\lambda_1\varepsilon} \,\mathrm{d}u = +\infty$. This is a contradiction, therefore $c > 0$ cannot hold.

If $c < 0$, take $0 < \varepsilon < \frac{-c}{\lambda_1\lambda_2}$ and

$$
f(x) = \begin{cases} \|x\|_{m,\rho}^{(-\alpha-m+\lambda_1\varepsilon)/p}, & \|x\|_{m,\rho} \ge 1, \\ 0, & 0 < \|x\|_{m,\rho} < 1. \end{cases}
$$

$$
g(y) = \begin{cases} \|y\|_{n,\rho}^{(-\beta-n+\lambda_2\varepsilon)/q}, & \|y\|_{n,\rho} \ge 1, \\ 0, & 0 < \|y\|_{n,\rho} < 1. \end{cases}
$$

Similarly,

$$
\begin{aligned}
&\frac{\Gamma^{m+n}(1/\rho)}{\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)} \int_1^{+\infty} K(1,t) t^{-\frac{\beta+n-\lambda_2\varepsilon}{q}+n-1} \,\mathrm{d}t \int_1^{+\infty} u^{-1+\frac{c}{\lambda_2}+\lambda_1\varepsilon} \,\mathrm{d}u \\
&\quad \le \frac{M}{\varepsilon(-\lambda_1)^{1/p}(-\lambda_2)^{1/q}} \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \right)^{1/p} \left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)} \right)^{1/q} < +\infty.
\end{aligned}
$$

Since $0 < \varepsilon < \frac{-c}{\lambda_1\lambda_2}$ and $\lambda_1 < 0$, we have $\frac{c}{\lambda_2} + \lambda_1\varepsilon > 0$ and $\int_1^{+\infty} u^{-1+\frac{c}{\lambda_2}+\lambda_1\varepsilon} \,\mathrm{d}u = +\infty$. That is still a contradiction, so $c < 0$ does not hold either.

To sum up, when $\lambda_1 < 0$, $\lambda_2 < 0$, we still have $c = 0$, that is, $\frac{n\lambda_1-\alpha\lambda_2}{p} + \frac{m\lambda_2-\beta\lambda_1}{q} = 0$.

Conversely, if $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=0$, set $a=\frac{\alpha+m}{pq}$, $b=\frac{\beta+n}{pq}$, it follows from Hölder's inequality and Lemma 1.2 that

$$\begin{aligned}
&\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad=\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)\left(\frac{\|x\|_{m,\rho}^{a}}{\|y\|_{n,\rho}^{b}}f(x)\right)\left(\frac{\|y\|_{n,\rho}^{b}}{\|x\|_{m,\rho}^{a}}g(y)\right)\mathrm{d}x\,\mathrm{d}y \\
&\quad\le\left(\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)\frac{\|x\|_{m,\rho}^{ap}}{\|y\|_{n,\rho}^{bp}}f^p(x)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \\
&\qquad\times\left(\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)\frac{\|y\|_{n,\rho}^{bq}}{\|x\|_{m,\rho}^{aq}}g^q(y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/q} \\
&\quad=\left[\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\frac{\alpha+m}{q}}f^p(x)\left(\int_{\mathbb{R}_+^n}\|y\|_{n,\rho}^{-\frac{\beta+n}{q}}K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)\,\mathrm{d}y\right)\mathrm{d}x\right]^{1/p} \\
&\qquad\times\left[\int_{\mathbb{R}_+^n}\|y\|_{n,\rho}^{\frac{\beta+n}{p}}g^q(y)\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{-\frac{\alpha+m}{p}}K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)\,\mathrm{d}x\right)\mathrm{d}y\right]^{1/q} \\
&\quad=\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\frac{\alpha+m}{q}}f^p(x)\omega_2(n,\beta,q,x)\,\mathrm{d}x\right)^{1/p}\left(\int_{\mathbb{R}_+^n}\|y\|_{n,\rho}^{\frac{\beta+n}{p}}g^q(y)\omega_1(m,\alpha,p,y)\,\mathrm{d}y\right)^{1/q} \\
&\quad=\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} \\
&\qquad\times W_1^{1/q}W_2^{1/p}\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\frac{\alpha+m}{q}+\frac{\lambda_1}{\lambda_2}(-\frac{\beta+n}{q}+n)}f^p(x)\,\mathrm{d}x\right)^{1/p} \\
&\qquad\times\left(\int_{\mathbb{R}_+^n}\|y\|_{n,\rho}^{\frac{\beta+n}{p}+\frac{\lambda_2}{\lambda_1}(-\frac{\alpha+m}{p}+m)}g^q(y)\,\mathrm{d}y\right)^{1/q} \\
&\quad=\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}W_1^{1/q}W_2^{1/p} \\
&\qquad\times\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\alpha}f^p(x)\,\mathrm{d}x\right)^{1/p}\left(\int_{\mathbb{R}_+^n}\|y\|_{n,\rho}^{\beta}g^q(y)\,\mathrm{d}y\right)^{1/q} \\
&\quad=\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}W_1^{1/q}W_2^{1/p}\|f\|_{p,\alpha}\|g\|_{q,\beta}.
\end{aligned}$$

Arbitrarily take a constant M satisfying

$$M\ge\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}W_1^{1/q}W_2^{1/p},$$

then

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\le M\|f\|_{p,\alpha}\|g\|_{q,\beta}.$$

Thus (5) holds.

(ii) Assume that there is a constant M_0 satisfying

$$M_0 < \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p} \tag{6}$$

such that, for any $f(x) \in L_p^\alpha(\mathbb{R}_+^m)$, $g(y) \in L_q^\beta(\mathbb{R}_+^n)$, we have

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho}, \|y\|_{n,\rho}\big)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \le M_0\|f\|_{p,\alpha}\|g\|_{q,\beta}.$$

Take sufficiently small $\varepsilon > 0$, $\delta > 0$, and set

$$f(x) = \begin{cases} \|x\|_{m,\rho}^{(-\alpha-m-|\lambda_1|\varepsilon)/p}, & \|x\|_{m,\rho} \ge \delta, \\ 0, & 0 < \|x\|_{m,\rho} < \delta. \end{cases}$$

$$g(y) = \begin{cases} \|y\|_{n,\rho}^{(-\beta-n-|\lambda_2|\varepsilon)/q}, & \|y\|_{n,\rho} \ge 1, \\ 0, & 0 < \|y\|_{n,\rho} < 1. \end{cases}$$

It can be obtained by calculation that

$$\begin{aligned} M_0\|f\|_{p,\alpha}\|g\|_{q,\beta} &= M_0\left(\int_{\|x\|_{m,\rho}\ge\delta} \|x\|_{m,\rho}^{-m-|\lambda_1|\varepsilon}\,\mathrm{d}x\right)^{1/p}\left(\int_{\|y\|_{m,\rho}\ge 1} \|y\|_{n,\rho}^{-n-|\lambda_2|\varepsilon}\,\mathrm{d}y\right)^{1/q} \\ &= \frac{M_0\cdot\delta^{-|\lambda_1|\varepsilon/p}}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q}. \end{aligned}$$

Since $\frac{n\lambda_1-\alpha\lambda_2}{p} + \frac{m\lambda_2-\beta\lambda_1}{q} = 0$,

$$\begin{aligned} &\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K\big(\|x\|_{m,\rho}, \|y\|_{n,\rho}\big)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\|y\|_{n,\rho}\ge 1} \|y\|_{n,\rho}^{(-\beta-n-|\lambda_2|\varepsilon)/q}\left(\int_{\|x\|_{m,\rho}\ge\delta} \|x\|_{m,\rho}^{(-\alpha-m-|\lambda_1|\varepsilon)/p} K\big(\|x\|_{m,\rho}, \|y\|_{n,\rho}\big)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\|y\|_{n,\rho}\ge 1} \|y\|_{n,\rho}^{(-\beta-n-|\lambda_2|\varepsilon)/q}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_\delta^{+\infty} u^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K\big(u, \|y\|_{n,\rho}\big)\,\mathrm{d}u\right)\mathrm{d}y \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_{\|y\|_{n,\rho}\ge 1} \|y\|_{n,\rho}^{(-\beta-n-|\lambda_2|\varepsilon)/q} \\ &\quad\times\left(\int_\delta^{+\infty} u^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K\big(u\|y\|_{n,\rho}^{-\lambda_2/\lambda_1}, 1\big)\,\mathrm{d}u\right)\mathrm{d}y \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_{\|y\|_{n,\rho}\ge 1} \|y\|_{n,\rho}^{-\frac{\beta+n+|\lambda_2|\varepsilon}{q}+\frac{\lambda_2}{\lambda_1}(-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1)+\frac{\lambda_2}{\lambda_1}} \\ &\quad\times\left(\int_{\delta\|y\|_{n,\rho}^{-\lambda_2/\lambda_1}}^{+\infty} t^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K(t,1)\,\mathrm{d}t\right)\mathrm{d}y \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_{\|y\|_{n,\rho}\ge 1} \|y\|_{n,\rho}^{\frac{-1}{\lambda_1}(n\lambda_1+\frac{\lambda_1|\lambda_2|\varepsilon}{q}+\frac{|\lambda_1|\lambda_2\varepsilon}{p})} \\ &\quad\times\left(\int_{\delta\|y\|_{n,p}^{-\lambda_2/\lambda_1}}^{+\infty} t^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K(t,1)\,\mathrm{d}t\right)\mathrm{d}y \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_{\|y\|_{n,\rho}\geq 1}\|y\|_{n,\rho}^{-n-|\lambda_2|\varepsilon}\left(\int_\delta^{+\infty}t^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K(t,1)\,\mathrm{d}t\right)\mathrm{d}y\\
&=\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_\delta^{+\infty}t^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K(t,1)\,\mathrm{d}t\int_{\|y\|_{n,\rho}\geq 1}\|y\|_{n,\rho}^{-n-|\lambda_2|\varepsilon}\,\mathrm{d}y\\
&=\frac{\Gamma^{m+n}(1/\rho)}{\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\int_\delta^{+\infty}t^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K(t,1)\,\mathrm{d}t\int_1^{+\infty}u^{-1-|\lambda_2|\varepsilon}\,\mathrm{d}u\\
&=\frac{\Gamma^{m+n}(1/\rho)}{\varepsilon|\lambda_2|\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\int_\delta^{+\infty}t^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K(t,1)\,\mathrm{d}t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{\Gamma^{m+n}(1/\rho)}{|\lambda_2|\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\int_\delta^{+\infty}t^{-\frac{\alpha+m+|\lambda_1|\varepsilon}{p}+m-1}K(t,1)\,\mathrm{d}t\\
&\quad\leq\frac{M_0\delta^{-|\lambda_1|\varepsilon/p}}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q}.
\end{aligned}$$

Let $\varepsilon\to 0^+$, and by using the famous Fatou lemma, we obtain

$$\begin{aligned}
&\frac{\Gamma^{m+n}(1/\rho)}{|\lambda_2|\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\int_\delta^{+\infty}t^{-\frac{\alpha+m}{p}+m-1}K(t,1)\,\mathrm{d}t\\
&\quad\leq\frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q}.
\end{aligned}$$

Let again $\delta\to 0^+$, then

$$\frac{\Gamma^{m+n}(1/\rho)W_1}{|\lambda_2|\rho^{m+n-2}\Gamma(n/\rho)\Gamma(m/\rho)}\leq\frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/q}.$$

It follows that

$$\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\leq M_0.$$

This contradicts (6). Thus

$$\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}$$

is the best constant factor of (5). □

3 Applications in operator theory

Let $p>1$, $\rho>0$, $m,n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$, $K(u,v)$ be nonnegative measurable. Define

$$T(f)(y)=\int_{\mathbb{R}_+^m}K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)f(x)\,\mathrm{d}x,\quad f(x)\in L_p^\alpha\big(\mathbb{R}_+^m\big). \tag{7}$$

Then T is a singular integral operator defined on $L_p^\alpha(\mathbb{R}_+^m)$. Using this operator and according to Hilbert type integral operator theory, (5) is equivalent to

$$\big\|T(f)\big\|_{p,\beta(1-p)}\leq M\|f\|_{p,\alpha},$$

so we get the following.

Theorem 3.1 *Under the same conditions as in Theorem 2.1, let the singular integral operator T be defined as in (7). Then*

(*i*) *T is a bounded operator from $L_p^{\alpha}(\mathbb{R}_+^m)$ to $L_p^{\beta(1-p)}(\mathbb{R}_+^n)$ if and only if $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=0$.*

(*ii*) *When T is a bounded operator from $L_p^{\alpha}(\mathbb{R}_+^m)$ to $L_p^{\beta(1-p)}(\mathbb{R}_+^n)$, the operator norm of T is*

$$\|T\|=\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}.$$

Corollary 3.1 *Let $\frac{1}{p}+\frac{1}{q}=1(p>1)$, $\rho>0$, $\lambda>0$, $\lambda_1\lambda_2>0$, $m,n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$, $0<\frac{1}{\rho\lambda_1}(\frac{m}{q}-\frac{\alpha}{p})<\lambda$. Define a singular integral operator T by*

$$T(f)(y)=\int_{\mathbb{R}_+^m}\frac{f(x)}{[1+(\sum_{k=1}^m x_k^{\rho})^{\lambda_1}/(\sum_{k=1}^n y_k^{\rho})^{\lambda_2}]^{\lambda}}\,\mathrm{d}x.$$

Then $T:L_p^{\alpha}(\mathbb{R}_+^m)\to L_p^{\beta(1-p)}(\mathbb{R}_+^n)$ is a bounded operator if and only if $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=0$. And when T is bounded, its operator norm is

$$\begin{aligned}\|T\|&=\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}B\left(\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right),\lambda-\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right)\right)\\&\quad\times\left(\frac{\Gamma^m(1/\rho)}{\rho^{m}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n}\Gamma(n/\rho)}\right)^{1/p},\end{aligned}$$

where $B(u,v)$ represents the beta function.

Proof First, notice that

$$\begin{aligned}\frac{1}{[1+(\sum_{k=1}^m x_k^{\rho})^{\lambda_1}/(\sum_{k=1}^n y_k^{\rho})^{\lambda_2}]^{\lambda}}&=\frac{1}{(1+\|x\|_{m,\rho}^{\rho\lambda_1}/\|y\|_{n,\rho}^{\rho\lambda_2})^{\lambda}}\\&=G\big(\|x\|_{m,\rho}^{\rho\lambda_1}/\|y\|_{n,\rho}^{\rho\lambda_2}\big)=K\big(\|x\|_{m,\rho},\|y\|_{n,\rho}\big)\end{aligned}$$

and

$$\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=0$$

is equivalent to

$$\frac{n(\rho\lambda_1)-\alpha(\rho\lambda_2)}{p}+\frac{m(\rho\lambda_2)-\beta(\rho\lambda_1)}{q}=0.$$

Since

$$\begin{aligned}W_0&=|\rho\lambda_1|W_1=|\rho\lambda_1|\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t\\&=|\rho\lambda_1|\int_0^{+\infty}\frac{1}{(1+t^{\rho\lambda_1})^{\lambda}}t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t=\int_0^{+\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{1}{\rho\lambda_1}(\frac{m}{q}-\frac{\alpha}{p})-1}\,\mathrm{d}u\\&=B\left(\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right),\lambda-\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right)\right),\end{aligned}$$

we have

$$\begin{aligned}
&\frac{W_0}{|\rho\lambda_1|^{1/q}|\rho\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\\
&=\frac{1}{\rho|\lambda_1|^{1/q}|\lambda_2|^{1/p}}B\left(\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right),\lambda-\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right)\right)\\
&\quad\times\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\\
&=\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}B\left(\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right),\lambda-\frac{1}{\rho\lambda_1}\left(\frac{m}{q}-\frac{\alpha}{p}\right)\right)\\
&\quad\times\left(\frac{\Gamma^m(1/\rho)}{\rho^{m}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n}\Gamma(n/\rho)}\right)^{1/p}.
\end{aligned}$$

According to Theorem 3.1, Corollary 3.1 holds. □

Corollary 3.2 *Let* $\frac{1}{p}+\frac{1}{q}=1(p>1)$, $\rho>0$, $\lambda_1>0$, $\lambda_2>0$, $m,n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$, $-\lambda_1<\frac{m}{q}-\frac{\alpha}{p}<\lambda_1$. *Define a singular integral operator* T *by*

$$T(f)(y)=\int_{\mathbb{R}_+^m}\frac{\min\{1,\|x\|_{m,\rho}^{\lambda_1}/\|y\|_{n,\rho}^{\lambda_2}\}}{\max\{1,\|x\|_{m,\rho}^{\lambda_1}/\|y\|_{n,\rho}^{\lambda_2}\}}f(x)\,\mathrm{d}x,\quad f(x)\in L_p^\alpha(\mathbb{R}_+^m).$$

Then $T:L_p^\alpha(\mathbb{R}_+^m)\to L_p^{\beta(1-p)}(\mathbb{R}_+^n)$ *is a bounded operator if and only if* $\frac{n\lambda_1-\alpha\lambda_2}{p}+\frac{m\lambda_2-\beta\lambda_1}{q}=0$, *and when* T *is bounded, its operator norm is*

$$\|T\|=\frac{2\lambda_1^2}{\lambda_1^{1/q}\lambda_2^{1/p}(\lambda_1+\frac{m}{q}-\frac{\alpha}{p})(\lambda_1-\frac{m}{q}+\frac{\alpha}{p})}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}.$$

Proof Since $-\lambda_1<\frac{m}{q}-\frac{\alpha}{p}<\lambda_1$, then $\frac{m}{q}-\frac{\alpha}{p}+\lambda_1>0$ and $\frac{m}{q}-\frac{\alpha}{p}-\lambda_1<0$, therefore

$$\begin{aligned}
W_0&=\lambda_1\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\,\mathrm{d}t\\
&=\lambda_1\int_0^{+\infty}\frac{\min\{1,t^{\lambda_1}\}}{\max\{1,t^{\lambda_1}\}}t^{\frac{m}{q}-\frac{\alpha}{p}-1}\,\mathrm{d}t\\
&=\lambda_1\int_0^1 t^{\frac{m}{q}-\frac{\alpha}{p}+\lambda_1-1}\,\mathrm{d}t+\lambda_1\int_1^{+\infty}t^{\frac{m}{q}-\frac{\alpha}{p}-\lambda_1-1}\,\mathrm{d}t\\
&=\frac{\lambda_1}{\frac{m}{q}-\frac{\alpha}{p}+\lambda_1}-\frac{\lambda_1}{\frac{m}{q}-\frac{\alpha}{p}-\lambda_1}=\frac{-2\lambda_1^2}{(\frac{m}{q}-\frac{\alpha}{p}+\lambda_1)(\frac{m}{q}-\frac{\alpha}{p}-\lambda_1)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}\\
&=\frac{2\lambda_1^2}{\lambda_1^{1/q}\lambda_2^{1/p}(\lambda_1+\frac{m}{q}-\frac{\alpha}{p})(\lambda_1-\frac{m}{q}+\frac{\alpha}{p})}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\frac{\Gamma^n(1/\rho)}{\rho^{n-1}\Gamma(n/\rho)}\right)^{1/p}.
\end{aligned}$$

According to Theorem 3.1, Corollary 3.2 holds. □

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Authors' contributions
BH carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. ZL and YH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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