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Some Padé approximations and inequalities for the complete elliptic integrals of the first kind

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Abstract

In this paper, we present Padé approximations of some functions involving complete elliptic integrals of the first kind $K(x)$, and motivated by these approximations we also present the following double inequality:

$$\frac{1-x^2}{1-x^2+\frac{x^4}{62}} < \frac{2e^{\frac{2}{\pi}K(x)-1}}{(1+\frac{1}{\sqrt{1-x^2}})} < \frac{1-\frac{96}{100}x^2}{1-\frac{96}{100}x^2+\frac{x^4}{64}}, \quad x \in (0,1).$$

Our results have superiority over some new recent results.

MSC: 33E05; 26D15; 41A21

Keywords: Complete elliptic integrals; Hypergeometric function; Inequality; Padé approximant; Best possible constant; Error

1 Introduction

It is well known that the complete elliptic integrals of the first kind and of the second kind are classical integrals, and apart from their theoretical importance in the theory of theta functions, they have important applications in mechanics, statistical mechanics, electrodynamics, magnetic field calculations, astronomy, geodesy, quasiconformal mappings, and other fields of mathematics and mathematical physics. In most applications, we encounter complicated expressions involving the complete elliptic integrals (which are not always in a form that is immediately recognizable), and it is difficult to find numerical values of such expressions to a sufficient number of significant digits. The complete elliptic integrals cannot be expressed in terms of elementary functions and have representations as infinite series that slowly converge, so these series are not the most computationally efficient approach for most scientists and engineers. Therefore, there is a need for appropriate approximations and bounds for these integrals.

The complete elliptic integrals of the first and second kinds $K(x)$ and $E(x)$, respectively, are defined as [9, 14]

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-x^2\sin^2 t}}, \quad 0 < x < 1 \tag{1}$$

and

$$E(x) = \int_0^{\frac{\pi}{2}} \sqrt{1-x^2\sin^2 t}\,dt, \quad 0 < x < 1, \tag{2}$$

which satisfy

$$\begin{aligned} &\lim_{x\to 0^+} K(x) = \lim_{x\to 0^+} E(x) = \frac{\pi}{2}, \qquad \lim_{x\to 1^-} K(x) = \infty, \qquad \lim_{x\to 1^-} E(x) = 1, \\ &K'(x) = K\bigl(\sqrt{1-x^2}\bigr) \quad \text{and} \quad E'(x) = E\bigl(\sqrt{1-x^2}\bigr). \end{aligned} \tag{3}$$

The functions $K(x)$ and $E(x)$ have the following representation [23]:

$$K(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, x^2\right) \tag{4}$$

and

$$E(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{-1}{2}, 1, x^2\right),$$

where the hypergeometric function $F(a,b,c,x)$ is defined by [5]

$$F(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}, \quad -1 < x < 1, \tag{5}$$

with $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ and the Euler gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(x) = \int_0^{\infty} e^{-v} v^{x-1}\,dv, \quad x > 0.$$

The hypergeometric function $F(a,b,c,x)$ has the differentiation formula [5]

$$\frac{d^r}{dx^r} F(a,b,c,x) = \frac{(a)_r(b)_r}{(c)_r} F(a+r, b+r, c+r, x) \tag{6}$$

and the transformation

$$(1-x)^{a+b-c} F(a,b,c,x) = F(c-a, c-b, c, x), \quad a,b,c > 0; a+b > c. \tag{7}$$

Wallis's ratio W_n is defined as [10, 12]

$$W_n = \frac{\Gamma(n+1/2)}{\Gamma(\frac{1}{2})\Gamma(n+1)}, \quad n \in \mathbb{N}, \tag{8}$$

and satisfies the recurrence relation

$$W_{n+1} = \frac{n+1/2}{n+1} W_n.$$

In [31], Yang et al. show that

$$U_n = \pi \sum_{k=0}^{n} \frac{W_k^2 W_{n-k}^2}{(k+1)(n-k+1)} - \frac{6(2n+1)W_n^2}{(n+1)(n+2)} < 0, \quad n \geq 8. \tag{9}$$

$K(x)$ can be written using the notation W_n as follows:

$$K(x) = \frac{\pi}{2} \sum_{n=0}^{\infty} W_n^2 x^{2n}, \quad 0 < x < 1. \tag{10}$$

The importance of elliptic integrals led to deduction of many of their inequalities. In [11], Carlson and Gustafson presented the inequality

$$\log \frac{4}{\sqrt{1-x^2}} < K(x) < \frac{4}{3+x^2} \log \frac{4}{\sqrt{1-x^2}}, \quad 0 < x < 1. \tag{11}$$

In [16], Kühnau deduced the lower bound

$$K(x) > \frac{9}{8+x^2} \log \frac{4}{\sqrt{1-x^2}}, \quad 0 < x < 1, \tag{12}$$

which is an improvement of the left-hand side of inequality (11). In [4], Anderson et al. deduced the inequality

$$\frac{\pi}{2} \sqrt{\frac{\tanh^{-1}(x)}{x}} < K(x) < \frac{\pi}{2} \frac{\tanh^{-1}(x)}{x}, \quad 0 < x < 1. \tag{13}$$

Alzer and Qiu [1] presented the inequality

$$\frac{\pi}{2} \left(\frac{\tanh^{-1}(x)}{x} \right)^{\mu} < K(x) < \frac{\pi}{2} \left(\frac{\tanh^{-1}(x)}{x} \right)^{\nu}, \quad 0 < x < 1, \tag{14}$$

with the best possible constants $\mu = 3/4$ and $\nu = 1$, which improved the lower bound of (13). In [31], Yang et al. proved the inequality

$$\log \frac{4}{\sqrt{1-x^2}} < K(x) < \log \left(e^{\frac{\pi}{2}} - 4 + \frac{4}{\sqrt{1-x^2}} \right), \quad 0 < x < 1. \tag{15}$$

In 2019, Yang and Tian [32] deduced the inequality

$$\rho \log \left(1 + \frac{4}{\sqrt{1-x^2}} \right) < K(x) < \sigma \log \left(1 + \frac{4}{\sqrt{1-x^2}} \right), \tag{16}$$

with the best possible constants $\rho = \frac{\pi}{2 \ln 5}$ and $\sigma = 1$. Recently, Wang et al. [27] presented the inequality

$$K(x) < \log \left(1 + \frac{4}{\sqrt{1-x^2}} \right) \left[\frac{\pi}{2 \log 5} + \left(1 - \frac{\pi}{2 \log 5} \right) x^2 \right], \quad 0 < x < 1. \tag{17}$$

For more details about inequalities, applications, and other related special functions to $K(x)$ and $E(x)$, we refer to [2, 3, 13, 15, 17–22, 24–26, 28–30] and the references therein.

Padé approximant [6–8] of order (r,s) of a function $f(x)$ is a rational function

$$[r/s]_f(x) = \frac{\sum_{i=0}^{r} \alpha_i x^i}{1 + \sum_{i=1}^{s} \beta_i x^i}, \quad r;s \geq 0,$$

where singularities of $f(x)$ are only poles. There are many different ways to determine the other coefficients $\alpha_j s$ for $0 \leq j \leq r$ and $\beta_k s$ for $1 \leq k \leq s$. Among them is the matching between the first $r+s+1$ coefficients in Maclaurin series $f(x) = \sum_{k=0}^{\infty} c_k x^k$ and the first $r+s+1$ coefficients of Padé approximant by the relation

$$\sum_{k=0}^{r+s+1} c_k x^k = \frac{\sum_{i=0}^{r} \alpha_i x^i}{\sum_{i=0}^{s} \beta_i x^i} \quad \text{or} \quad \left(\sum_{k=0}^{r+s+1} c_k x^k\right)\left(\sum_{i=0}^{s} \beta_i x^i\right) = \left(\sum_{i=0}^{r} \alpha_i x^i\right).$$

Hence, we solve the following equations for $\alpha_i s$ and $\beta_i s$:

$$c_{r+1} + \sum_{k=1}^{s} c_k \beta_{r+1-k} = 0 \quad \text{and} \quad \alpha_r = \sum_{k=0}^{r} \beta_{r-k} c_k,$$

and we have

$$[r/s]_f(x) - f(x) = O\left(x^{r+s+1}\right).$$

2 Main results

Theorem 1 *The following inequality*

$$K(x) < \frac{\pi}{2} \log\left[\frac{e}{p+1}\left(p + \frac{1}{\sqrt{1-x^2}}\right)\right], \quad 0 < x < 1 \tag{18}$$

holds for the best possible constant $p = 1$.

Proof Consider the function

$$F_p(x) = \frac{e^{\frac{2K(x)}{\pi}}}{p + \frac{1}{\sqrt{1-x^2}}}.$$

Using (4) and (6), we have

$$F_p'(\sqrt{x}) = \frac{e^{F(\frac{1}{2},\frac{1}{2},1,x)}}{4(p + \frac{1}{\sqrt{1-x}})^2}\left[\left(p + \frac{1}{\sqrt{1-x}}\right) F\left(\frac{3}{2}, \frac{3}{2}, 2, x\right) - \frac{2}{(1-x)^{\frac{3}{2}}}\right],$$

and then the function $F_p(\sqrt{x})$ is strictly decreasing on $x \in (0,1)$ if and only if

$$p \leq \frac{2}{F(\frac{3}{2}, \frac{3}{2}, 2, x)(1-x)^{\frac{3}{2}}} - \frac{1}{\sqrt{1-x}} \doteqdot f(x).$$

Using relation (7), we have

$$f(x)=\frac{2-F(\frac{1}{2},\frac{1}{2},2,x)}{\sqrt{1-x}F(\frac{1}{2},\frac{1}{2},2,x)},$$

and hence

$$f'(x)=\frac{1}{4F(\frac{1}{2},\frac{1}{2},2,x)^2(1-x)^{\frac{3}{2}}}f_1(x),$$

where

$$f_1(x)=4F\left(\frac{1}{2},\frac{1}{2},2,x\right)-2F\left(\frac{1}{2},\frac{1}{2},2,x\right)^2-(1-x)F\left(\frac{3}{2},\frac{3}{2},3,x\right).$$

From (8), we have

$$\begin{aligned}f_1(x)&=4\sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n-2\left(\sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n\right)^2+\sum_{n=0}^{\infty}\frac{2(4n-1)W_n^2}{(n+1)(n+2)}x^n\\&=-2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{W_k^2W_{n-k}^2}{(k+1)(n-k+1)}x^n+\sum_{n=0}^{\infty}\frac{6(2n+1)W_n^2}{(n+1)(n+2)}x^n\\&=-\sum_{n=0}^{\infty}\mu_nx^n,\end{aligned}$$

where

$$\mu_n=2\sum_{k=0}^{n}\frac{W_k^2W_{n-k}^2}{(k+1)(n-k+1)}-\frac{6(2n+1)W_n^2}{(n+1)(n+2)}.$$

Using (9), we obtain

$$\mu_n<U_n<0,\quad n\geq 8,$$

and $\mu_0=-1$, $\mu_1=\frac{-1}{4}$, $\mu_2=\frac{-17}{128}$, $\mu_3=\frac{-43}{512}$, $\mu_4=\frac{-953}{16{,}384}$, $\mu_5=\frac{-2801}{65{,}536}$, $\mu_6=\frac{-137{,}401}{4{,}194{,}304}$, $\mu_7=\frac{-485{,}318}{16{,}777{,}216}$. Hence $\mu_n<0$ for $n\geq 0$, $f_1(x)>0$ and therefore the function $f(x)$ is increasing on $x\in(0,1)$ with

$$\lim_{x\to 0^+}f(x)=1,$$

which implies that $p\leq 1$. Therefore, the function $F_p(x)$ is strictly decreasing on $x\in(0,1)$ if and only if $p\leq 1$, and using the first limit in (3), we obtain inequality (18). □

Theorem 2 *The following inequality*

$$K(x)>\frac{\pi}{2}\log\left[\frac{e}{q+1}\left(q+\frac{1}{\sqrt{1-\frac{11}{12}x^2}}\right)\right],\quad 0<x<1 \tag{19}$$

holds for the best possible constant $q=\frac{5}{6}$.

Proof Consider the function

$$H_q(x) = \frac{e^{\frac{2K(x)}{\pi}}}{q + \frac{1}{\sqrt{1-\frac{11}{12}x^2}}}.$$

Using (4) and (6), we have

$$H_q'(x) = \frac{e^{F(\frac{1}{2},\frac{1}{2},1,x)}}{4(q + \frac{1}{\sqrt{1-\frac{11}{12}x}})^2}\left[\left(q + \frac{1}{\sqrt{1-\frac{11}{12}x}}\right)F\left(\frac{3}{2},\frac{3}{2},2,x\right) - \frac{11}{6(1-\frac{11}{12}x)^{\frac{3}{2}}}\right],$$

and then the function $H_q(\sqrt{x})$ is strictly increasing on $(0,1)$ if and only if

$$q \geq \frac{11}{6F(\frac{3}{2},\frac{3}{2},2,x)(1-\frac{11}{12}x)^{\frac{3}{2}}} - \frac{1}{\sqrt{1-\frac{11}{12}x}} \doteq h(x).$$

Then

$$h'(x) = \frac{11\sqrt{3}}{2F(\frac{3}{2},\frac{3}{2},2,x)^2(12-11x)^{\frac{5}{2}}}h_1(x),$$

where

$$h_1(x) = 132F\left(\frac{3}{2},\frac{3}{2},2,x\right) + (-24+22x)F\left(\frac{3}{2},\frac{3}{2},2,x\right)^2 + 9(-12+11x)F\left(\frac{5}{2},\frac{5}{2},3,x\right).$$

From (8), we have

$$\begin{aligned}
h_1(x) &= 132\sum_{n=0}^{\infty}\frac{(2n+1)^2W_n^2}{n+1}x^n + (-24+22x)\left(\sum_{n=0}^{\infty}\frac{(2n+1)^2W_n^2}{n+1}x^n\right)^2 \\
&\quad + 9(-12+11x)\sum_{n=0}^{\infty}\frac{2(2n+1)^2(2n+3)^2W_n^2}{9(n+1)(n+2)}x^n \\
&= 4\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{-2(2k+1)^2((n-k+\frac{1}{2})^2+\frac{11}{4})W_k^2W_{n-k}^2}{(k+1)(n-k+1)}x^n \\
&\quad - 4\sum_{n=1}^{\infty}\frac{(2n+1)^2(2n^2-5n-12)W_n^2}{(n+1)(n+2)}x^n \\
&= 4\sum_{n=1}^{\infty}V_nx^n
\end{aligned}$$

and

$$V_n = \sum_{k=0}^{n}\frac{-2(2k+1)^2((n-k+\frac{1}{2})^2+\frac{11}{4})W_k^2W_{n-k}^2}{(k+1)(n-k+1)} - \frac{(2n+1)^2(2n+3)(n-4)W_n^2}{(n+1)(n+2)}.$$

The sequence $V_n < 0$ for $n = 4, 5, 6, \ldots$ and

$$V_0 = 0, \qquad V_1 = \frac{-19}{8}, \qquad V_2 = \frac{-663}{128}, \qquad V_3 = \frac{-8367}{1024}.$$

Then $V_n < 0$ for $n \geq 0$, $h_1(x) < 0$ and therefore the function $h(x)$ is decreasing with

$$\lim_{x\to 0^+} h(x) = \frac{5}{6},$$

which implies that $q \geq \frac{5}{6}$. Therefore, the function $H_q(x)$ is strictly increasing on $x \in (0,1)$ if and only if $q \geq \frac{5}{6}$, and using the limits in (3), we obtain inequality (19). □

Based on the Padé approximation method, we can conclude the following approximations.

Proposition 3 *The Padé approximations of orders* $(3,4)$ *and* $(3,7)$ *of the function*

$$f(x) = \frac{2e^{\frac{2}{\pi}K(x)-1}}{(1+\frac{1}{\sqrt{1-x^2}})} = 1 - \frac{x^4}{64} - \frac{13x^6}{768} - \frac{261x^8}{16{,}384} - \frac{14{,}317x^{10}}{983{,}040} + \cdots, \quad x \to 0$$

are the following rational functions:

$$[3/4]_f(x) = \frac{1-\frac{13x^2}{12}}{1-\frac{13x^2}{12}+\frac{x^4}{64}} + O(x^8) \tag{20}$$

and

$$[3/7]_f(x) = \frac{1-\frac{795x^2}{832}}{1-\frac{795x^2}{832}+\frac{x^4}{64}+\frac{319x^6}{159{,}744}} + O(x^{11}). \tag{21}$$

Proposition 4 *The Padé approximations of orders* $(3,7)$ *and* $(3,9)$ *of the function*

$$g(x) = \frac{11e^{\frac{2}{\pi}K(x)-1}}{12(\frac{1}{\frac{5}{6}+\sqrt{1-\frac{11x^2}{12}}})} = \frac{1}{2} + \frac{19x^6}{9216} + \frac{403x^8}{110{,}592} + \frac{167{,}659x^{10}}{35{,}389{,}440} + \frac{1{,}862{,}857x^{12}}{339{,}738{,}624} + \cdots, \quad x \to 0$$

are the following rational functions:

$$[3/7]_g(x) = \frac{\frac{1}{2}-\frac{403x^2}{456}}{-\frac{19x^6}{4608}-\frac{403x^2}{228}+1} + O(x^{11}) \tag{22}$$

and

$$[3/9]_g(x) = \frac{\frac{1}{2}-\frac{167{,}659x^2}{257{,}920}}{1-\frac{167{,}659x^2}{128{,}960}-\frac{19x^6}{4608}-\frac{3{,}436{,}157x^8}{1{,}782{,}743{,}040}} + O(x^{13}). \tag{23}$$

Unfortunately, formulas (20),(21), (22), and (23) did not give bounds of the function $K(x)$ for all x in the domain $(0,1)$. But formula (20) motivates us to establish the following inequalities.

Theorem 5 *The following inequality*

$$K(x) < \frac{\pi}{2}\log\left(\frac{e}{2}\left(1+\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{1-\frac{96}{100}x^2}{1-\frac{96}{100}x^2+\frac{x^4}{64}}\right)\right) \tag{24}$$

holds for $0 < x < 1$.

Proof Consider the function

$$T(x) = \frac{e^{\frac{2K(x)}{\pi}}}{\frac{e}{2}(1+\frac{1}{\sqrt{1-x^2}})(\frac{1-\frac{96}{100}x^2}{1-\frac{96}{100}x^2+\frac{x^4}{64}})},$$

and hence

$$T'(\sqrt{x}) = \frac{(-1600+(1536-25x)x)e^{F(\frac{1}{2},\frac{1}{2},1,x)}}{128e\sqrt{1-x}(-25+24x)(1+\sqrt{1-x})}\frac{t_1(x)}{(x^2-1)},$$

where

$$t_1(x) = w_1(x) - w_2(x)$$

with

$$w_1(x) = (x^2-1)F\left(\frac{1}{2},\frac{1}{2},2,x\right)$$

and

$$\begin{aligned} w_2(x) = {} & \frac{(x^2-1)}{(1+\sqrt{1-x})(25-24x)(1600-(1536-25x)x)}\bigl[80{,}000 \\ & - 200(793+25\sqrt{1-x})x + (82{,}378+7400\sqrt{1-x})x^2 \\ & - 1200(3+2\sqrt{1-x})x^3\bigr]. \end{aligned}$$

Now

$$\begin{aligned} w_1(x) &= \sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^{n+2} - \sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n \\ &= -1-\frac{x}{8}+\sum_{n=2}^{\infty}\frac{(64n^3-104n^2+48n-9)W_n^2}{(n+1)(2n-1)^2(2n-3)^2}x^n \end{aligned}$$

and

$$w_1''(x) = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)(183+400n+280n^2+64n^3)W_{n+2}^2}{(n+3)(3+4n(n+2))^2}x^n.$$

Then $w_1(x)$ is a convex function between the points $(0,-1)$ and $(1,0)$. Also,

$$w_2''(1-u^2) = \frac{w_3(u)}{w_4(u)}, \quad 0 < u < 1,$$

where

$$w_4(u) = -2u\bigl(89 + 3622u^2 + 35{,}689u^4 + 600u^6\bigr)^3(1+u)^3 < 0$$

and

$$\begin{aligned} w_3(u) = {} & -422{,}9814 - 171{,}806{,}490u - 1{,}017{,}032{,}637u^2 - 107{,}1925{,}167u^3 \\ & - 26{,}188{,}833{,}484u^4 + 167{,}770{,}858{,}692u^5 - 106{,}178{,}812{,}935u^6 \\ & + 2{,}315{,}906{,}378{,}995u^7 - 2{,}725{,}881{,}138{,}990u^8 - 8{,}235{,}729{,}120{,}210u^9 \\ & - 93{,}146{,}282{,}277{,}931u^{10} - 135{,}010{,}050{,}058{,}665u^{11} - 381{,}354{,}896{,}855{,}688u^{12} \\ & - 469{,}576{,}678{,}642{,}848u^{13} - 381{,}450{,}253{,}571{,}121u^{14} \\ & - 157{,}869{,}853{,}296{,}507u^{15} - 117{,}933{,}228{,}362{,}600u^{16} \\ & - 118{,}127{,}740{,}267{,}800u^{17} - 38{,}943{,}594{,}964{,}800u^{18} - 1{,}971{,}366{,}120{,}000u^{19} \\ & - 629{,}017{,}920{,}000u^{20} - 11{,}016{,}000{,}000u^{21} - 3{,}456{,}000{,}000u^{22} \\ = {} & (1-u)^{24}\int_0^\infty e^{(1-u)t} w_5(t)\,dt < 0, \end{aligned}$$

where

$$\begin{aligned} w_5(t) = {} & -\frac{1{,}937{,}500t^{23}}{26{,}298{,}031{,}350{,}591} - \frac{415{,}555{,}625t^{22}}{18{,}294{,}282{,}678{,}672} - \frac{154{,}777{,}075t^{21}}{48{,}915{,}194{,}328} \\ & - \frac{167{,}679{,}321{,}125t^{20}}{633{,}568{,}231{,}296} - \frac{90{,}786{,}961{,}225t^{19}}{6{,}092{,}002{,}224} - \frac{100{,}101{,}056{,}444{,}021t^{18}}{166{,}728{,}481{,}920} \\ & - \frac{3{,}562{,}788{,}612{,}574{,}819t^{17}}{198{,}486{,}288{,}000} - \frac{133{,}198{,}651{,}249{,}637{,}299t^{16}}{326{,}918{,}592{,}000} \\ & - \frac{2{,}692{,}769{,}780{,}390{,}699t^{15}}{378{,}378{,}000} - \frac{174{,}766{,}864{,}343{,}435{,}699t^{14}}{1{,}816{,}214{,}400} \\ & - \frac{43{,}522{,}631{,}386{,}179{,}371t^{13}}{43{,}243{,}200} - \frac{5{,}123{,}113{,}557{,}369{,}119t^{12}}{633{,}600} \\ & - \frac{122{,}852{,}110{,}141{,}292{,}563t^{11}}{2{,}494{,}800} - \frac{809{,}450{,}870{,}289{,}838{,}177t^{10}}{3{,}628{,}800} \\ & - \frac{2{,}111{,}790{,}435{,}345{,}101t^9}{2880} - \frac{68{,}468{,}628{,}875{,}595{,}707t^8}{40{,}320} \\ & - \frac{169{,}054{,}344{,}340{,}120t^7}{63} - \frac{5{,}639{,}548{,}747{,}095t^6}{2} - 1{,}965{,}373{,}967{,}040t^5 \\ & - 924{,}471{,}855{,}000t^4 - 276{,}448{,}320{,}000t^3 - 43{,}524{,}000{,}000t^2 \\ & - 3{,}456{,}000{,}000t < 0. \end{aligned}$$

Then $w_2(x)$ is a convex function between the same two points $(0,-1)$ and $(1,0)$. Also,

$$\lim_{x\to 0^+} w_2'(x) = \frac{-1}{8}, \qquad \lim_{x\to 0^+} w_1'(x) = \frac{-1}{8}$$

and

$$\lim_{x\to 1^-} w_2'(x) = 4, \qquad \lim_{x\to 1^-} w_1'(x) = \frac{8}{\pi}.$$

Then $w_1(x) > w_2(x)$, $t_1(x) > 0$ and $T(x)$ is decreasing on $x \in (0,1)$. Hence, using the limits in (3), we obtain inequality (24). □

Theorem 6 *The inequality*

$$K(x) > \frac{\pi}{2}\log\left(\frac{e}{2}\left(1+\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{1-x^2}{1-x^2+\frac{x^4}{62}}\right)\right) \tag{25}$$

holds for $x \in (0,1)$.

Proof Consider the function

$$G(x) = \frac{e^{\frac{2K(x)}{\pi}}}{\frac{e}{2}(1+\frac{1}{\sqrt{1-x^2}})(\frac{1-x^2}{1-x^2+\frac{x^4}{62}})}$$

and hence

$$G'(\sqrt{x}) = \frac{(62-62x+x^2)e^{F(\frac{1}{2},\frac{1}{2},1,x)}}{124e(1-x)^{\frac{3}{2}}(1+\sqrt{1-x})}g_1(x),$$

where

$$g_1(x) = F\left(\frac{1}{2},\frac{1}{2},2,x\right) - \frac{2(62-2(33+2\sqrt{1-x})x+(3+2\sqrt{1-x})x^2}{(1+\sqrt{1-x})(62-62x+x^2)}.$$

Using (5), we get

$$F\left(\frac{1}{2},\frac{1}{2},2,x\right) = \sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n > 1+\frac{x}{8}+\frac{3x^2}{64}+\frac{25x^3}{1024}. \tag{26}$$

Now let

$$\begin{aligned} g_2(x) &= (1+\sqrt{1-x})^2\Big(2\big(62-2(33+2\sqrt{1-x})x+(3+2\sqrt{1-x})x^2\big) \\ &\quad - \left(1+\frac{x}{8}+\frac{3x^2}{64}+\frac{25x^3}{1024}\right)(1+\sqrt{1-x})\big(62-62x+x^2\big)\Big). \\ g_2'\big(1-u^2\big) &= \frac{g_3(u)}{2048u}, \quad 0<u<1, \end{aligned}$$

where

$$\begin{aligned} g_3(u) &= -1225 - 114{,}096u + 533{,}640u^2 - 875{,}268u^3 + 489{,}925u^4 + 154{,}176u^5 \\ &\quad - 194{,}208u^6 - 67{,}464u^7 + 75{,}897u^8 + 13{,}520u^9 - 14{,}872u^{10} + 300u^{11} \end{aligned}$$

$$\begin{aligned}
&- 325u^{12} - 194{,}208u^6 - 67{,}464u^7 + 75{,}897u^8 + 13{,}520u^9 - 14{,}872u^{10} \\
&+ 300u^{11} - 325u^{12} \\
&= (1-u)^{14} \int_0^\infty e^{(1-u)t} g_4(t)\,dt \\
&< 0,
\end{aligned}$$

where

$$\begin{aligned}
g_4(t) = -\frac{1}{623{,}700}\big[&48t^{11} + 6864t^{10} + 313{,}225t^9 + 5{,}268{,}780t^8 \\
&+ 85{,}446{,}900t^7 + 794{,}011{,}680t^6 + 3{,}030{,}340{,}005t^5 + 4{,}942{,}822{,}500t^4 \\
&+ 3{,}432{,}636{,}900t^3 + 1{,}122{,}660{,}000t^2 + 202{,}702{,}500t\big].
\end{aligned}$$

Then $g_2(x)$ is deceasing with

$$\lim_{x\to 0^+} g_2(x) = 0.$$

Hence $g_2(x) < 0$, and we have

$$\frac{2(62 - 2(33 + 2\sqrt{1-x})x + (3 + 2\sqrt{1-x})x^2}{(1+\sqrt{1-x})(62 - 62x + x^2)} < 1 + \frac{x}{8} + \frac{3x^2}{64} + \frac{25x^3}{1024}. \tag{27}$$

From inequalities (26) and (27), we get $g_1(x) > 0$ and the function $G(x)$ is increasing. Hence, using the limits in (3), we obtain inequality (25) □

3 Remarks

Comparing our new bounds of the function $K(x)$ with its previous ones presents the following remarks.

Remark 7 Our upper bound in (24) is better than our upper bound in (18) for $x \in (0, 1)$.

Remark 8 The upper bound in (24) is better than the upper bound in (11) for $x \in (0, 0.97)$.

Remark 9 The upper bound in (24) is better than the upper bound in (14) for $x \in (0, 1)$.

Remark 10 The upper bound in (24) is better than the upper bound in each of (15), (16), and (17) for $x \in (0, 0.98)$.

Remark 11 Our lower bounds in (19) and (25) are not contained in each other for $x \in (0, 1)$.

Remark 12 Our lower bound in (25) is better than the lower bound in (12) for $x \in (0, 9)$.

Remark 13 The lower bound in (25) is better than the lower bound in (14) for $x \in (0, 87)$.

Remark 14 The lower bound in (25) is better than the lower bound in (15) for $x \in (0, 94)$.

Remark 15 The lower bound in (25) is better than the lower bound in (16) for $x \in (0, 91)$.

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