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A note on degenerate r-Stirling numbers

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Abstract

The aim of this paper is to study the unsigned degenerate r-Stirling numbers of the first kind as degenerate versions of the r-Stirling numbers of the first kind and the degenerate r-Stirling numbers of the second kind as those of the r-Stirling numbers of the second kind. For the degenerate r-Stirling numbers of both kinds, we derive recurrence relations, generating functions, explicit expressions, and some identities involving them.

Keywords: Unsigned degenerate r-Stirling numbers of the first kind; Degenerate r-Stirling numbers of first kind; Degenerate r-Stirling numbers of second kind

1 Introduction

Let $[n] = \{1, 2, 3, \ldots, n\}$. The unsigned Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of permutations of the set $[n]$ with exactly k disjoint cycles, while the Stirling number of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}$ counts the number of partitions of the set $[n]$ into k nonempty disjoint subsets. Let r be a positive integer. The unsigned r-Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_r$ is the number of permutations of the set $[n]$ with exactly k disjoint cycles in such a way that the numbers $1, 2, \ldots, r$ are in distinct cycles, while the r-Stirling number of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_r$ counts the number of partitions of the set $[n]$ into k nonempty disjoint subsets in such a way that the numbers $1, 2, \ldots, r$ are in distinct subsets. We remark that Border [1] studied the combinatorial and algebraic properties of the r-Stirling numbers.

Carlitz initiated a study of the degenerate Bernoulli and Euler polynomials and numbers. In recent years, some mathematicians have explored various degenerate versions of many special polynomials and numbers by employing various tools like combinatorial methods, generating functions, differential equations, umbral calculus techniques, p-adic analysis, and probability theory. These degenerate versions include the degenerate Stirling numbers of the first and second kinds, degenerate Bernoulli numbers of the second kind, and degenerate Bell numbers and polynomials of which interesting arithmetical and combinatorial results were obtained (see [7, 9–11, 13] and references therein). Especially, it turns out that the degenerate Stirling numbers of the first and second kind (see (17), (18)) appear very frequently when we study degenerate versions of some special numbers and polynomials [7, 9, 13, 16].

The aim of this paper is to study the unsigned degenerate r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{r,\lambda}$ as degenerate versions of the r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_r$ and the

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degenerate r-Stirling numbers of the second kind $\left\{{n \atop k}\right\}_{r,\lambda}$ as degenerate versions of the r-Stirling numbers of the second kind $\left\{{n \atop k}\right\}_r$. They can be viewed also as natural extensions of the degenerate Stirling numbers of the first kind $\left[{n \atop k}\right]_\lambda$ and the degenerate Stirling numbers of the second kind $\left\{{n \atop k}\right\}_\lambda$ which were introduced earlier [7–9, 12]. For the degenerate r-Stirling numbers of both kinds, we derive recurrence relations, generating functions, explicit expressions, and some identities involving them.

The outline of this paper is as follows. In Sect. 1, we recall the degenerate exponential functions, the rising and falling λ-factorial sequences, recurrence relations and generating functions of r-Stirling numbers, Lah numbers, the degenerate Stirling numbers and the degenerate logarithm functions. In Sect. 2, we introduce the unsigned degenerate r-Stirling numbers of the first kind and the degenerate r-Stirling numbers of the second kind and derive recurrence relations, generating functions, expressions of degenerate r-Stirling numbers in terms of degenerate Stirling numbers, and a representation of the degenerate r-Bell polynomials in terms of the degenerate r-Stirling numbers of the second kind.

For any $0 \neq \lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \quad \text{(see [7, 10, 11]),} \tag{1}$$

where the falling λ-factorial sequence is given by

$$(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots \big(x - (n-1)\lambda\big) \quad (n \geq 1). \tag{2}$$

When $x = 1$, we have

$$e_\lambda(t) = e_\lambda^1(t) = \sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!}.$$

The rising λ-factorial sequence is defined by

$$\langle x \rangle_{0,\lambda} = 1, \qquad \langle x \rangle_{n,\lambda} = x(x + \lambda) \cdots \big(x + (n-1)\lambda\big) \quad (n \geq 1). \tag{3}$$

When $\lambda = 1$, we have

$$(x)_0 = 1, \qquad (x)_n = x(x - 1) \cdots (x - n + 1) \quad (n \geq 1), \tag{4}$$

$$\langle x \rangle_0 = 1, \qquad \langle x \rangle_n = x(x + 1) \cdots (x + n - 1) \quad (n \geq 1). \tag{5}$$

Throughout this paper, we assume that r is any positive integer.

It is known that the r-Stirling numbers satisfy the following recurrence relations:

$$\left[{n \atop m}\right]_r = (n - 1)\left[{n-1 \atop m}\right]_r + \left[{n-1 \atop m-1}\right]_r \quad (n > r), \tag{6}$$

$$\left\{{n \atop m}\right\}_r = m\left\{{n-1 \atop m}\right\}_r + \left\{{n-1 \atop m-1}\right\}_r \quad (n > r), \tag{7}$$

$$\left[{n \atop m}\right]_r = 0, \qquad \left\{{n \atop m}\right\}_r = 0, \quad \text{if } n < r, \tag{8}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \delta_{mr}, \qquad \begin{Bmatrix} n \\ m \end{Bmatrix}_r = \delta_{mr}, \quad \text{if } n = r \text{ (see [1–5, 14])}. \tag{9}$$

The r-Stirling numbers of the first kind have the "horizontal" generating function:

$$\sum_{k=r}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r z^k = z^r (z+r) \cdots (z+n-1) = z^r \langle z+r \rangle_{n-r}, \tag{10}$$

where $n \geq r$ (see [1–7, 9–11, 13, 15]).

Thus, by (10), we get

$$\langle z+r \rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r z^k \quad (n \geq 0). \tag{11}$$

The r-Stirling numbers of the second kind have the "vertical" generating function:

$$\sum_{n=m}^{\infty} \begin{Bmatrix} n \\ m \end{Bmatrix}_r z^n = \frac{z^m}{(1-rz)(1-(r+1)z) \cdots (1-mz)} \quad (m \geq r), \tag{12}$$

which is equivalent to

$$\begin{Bmatrix} n+r \\ m+r \end{Bmatrix}_r = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (r+k)^n \quad (n \geq m \geq 0), \tag{13}$$

and is also equivalent to

$$\sum_{n=m}^{\infty} \begin{Bmatrix} n+r \\ m+r \end{Bmatrix}_r \frac{t^n}{n!} = \frac{1}{m!} (e^t - 1)^m e^{rt}. \tag{14}$$

From (14), we note that

$$(z+r)^n = \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r (z)_k \quad (n \geq 0). \tag{15}$$

As is well known, the Lah numbers are defined by

$$L(n,k) = \binom{n-1}{k-1} \frac{n!}{k!} \quad (n \geq k \geq 0). \tag{16}$$

From (16), we note that

$$\frac{1}{k!} \left(\frac{t}{1-t} \right)^k = \sum_{n=k}^{\infty} L(n,k) \frac{t^n}{n!} \quad (k \geq 0) \text{ (see [2, 4, 5])}.$$

Recently, the degenerate Stirling numbers of the first kind were defined by

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda} \quad (n \geq 0) \text{ (see [1–7, 9–11, 13, 15])}, \tag{17}$$

and the degenerate Stirling numbers of the second kind were defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\lambda} (x)_k \quad (n \geq 0) \text{ (see [7, 9])}. \tag{18}$$

Note that

$$\lim_{\lambda \to 0} \left\{ {n \atop k} \right\}_{\lambda} = \left\{ {n \atop k} \right\} \quad (n \geq k \geq 0).$$

Let us define the degenerate "unsigned" Stirling numbers of the first kind by

$$\left[{n \atop k} \right]_{\lambda} = (-1)^{n-k} S_{1,\lambda}(n,k) \quad (n \geq k \geq 0). \tag{19}$$

Then, replacing x by $-x$ in (17), we get

$$\langle x \rangle_n = \sum_{k=0}^{n} \left[{n \atop k} \right]_{\lambda} \langle x \rangle_{k,\lambda} \quad (n \geq 0). \tag{20}$$

Note that

$$\lim_{\lambda \to 0} \left[{n \atop k} \right]_{\lambda} = \left[{n \atop k} \right].$$

Let $\log_{\lambda} t$ be the compositional inverse of $e_{\lambda}(t)$, called the degenerate logarithm function. Then we have

$$\log_{\lambda}(1 + t) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,\frac{1}{\lambda}} \frac{t^n}{n!} = \frac{1}{\lambda}\left((1 + t)^{\lambda} - 1\right) \quad \text{(see [7])}.$$

Note that

$$e_{\lambda}\left(\log_{\lambda}(1 + t)\right) = \log_{\lambda}\left(e_{\lambda}(1 + t)\right) = 1 + t. \tag{21}$$

2 Degenerate r-Stirling numbers

In view of (11) and (17), we define the unsigned degenerate r-Stirling numbers of the first kind by

$$\langle x + r \rangle_n = \sum_{k=0}^{n} \left[{n + r \atop k + r} \right]_{r,\lambda} \langle x \rangle_{k,\lambda} \quad (n \geq 0). \tag{22}$$

Note from (11) and (22) that

$$\lim_{\lambda \to 0} \left[{n + r \atop k + r} \right]_{r,\lambda} = \left[{n + r \atop k + r} \right]_{r}.$$

By (22), we get

$$\sum_{k=0}^{n+1} \begin{bmatrix} n+1+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x \rangle_{k,\lambda} = \langle x+r \rangle_{n+1} \tag{23}$$

$$= \langle x+r \rangle_n (x+r+n)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x \rangle_{k,\lambda} (x+k\lambda+r+n-k\lambda)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x \rangle_{k+1,\lambda} + \sum_{k=0}^{n} (n+r-k\lambda) \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x \rangle_{k,\lambda}$$

$$= \sum_{k=0}^{n+1} \begin{bmatrix} n+r \\ k-1+r \end{bmatrix}_{r,\lambda} \langle x \rangle_{k,\lambda} + \sum_{k=0}^{n} (n+r-k\lambda) \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x \rangle_{k,\lambda}$$

$$= \sum_{k=0}^{n+1} \left(\begin{bmatrix} n+r \\ k-1+r \end{bmatrix}_{r,\lambda} + (n+r-k\lambda) \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \right) \langle x \rangle_{k,\lambda},$$

as

$$\begin{bmatrix} n+r \\ r-1 \end{bmatrix}_{r,\lambda} = 0 \quad \text{and} \quad \begin{bmatrix} n+r \\ n+1+r \end{bmatrix}_{r,\lambda} = 0.$$

Therefore, we obtain the following theorem.

Theorem 2.1 *Let n, k be nonnegative integers. Then we have*

$$\begin{bmatrix} n+1+r \\ k+r \end{bmatrix}_{r,\lambda} = \begin{bmatrix} n+r \\ k-1+r \end{bmatrix}_{r,\lambda} + (n+r-k\lambda) \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \quad (n \ge k).$$

In particular,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{r,\lambda} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{r,\lambda} + \left(n-1-(k-r)\lambda \right) \begin{bmatrix} n-1 \\ k \end{bmatrix}_{r,\lambda} \quad (n > r, n > k).$$

From (22), we note that

$$\left(\frac{1}{1-t} \right)^x \left(\frac{1}{1-t} \right)^r = \sum_{n=0}^{\infty} \langle x+r \rangle_n \frac{t^n}{n!} \tag{24}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x \rangle_{k,\lambda} \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \frac{t^n}{n!} \right) \langle x \rangle_{k,\lambda}.$$

On the other hand, by (21), we get

$$\left(\frac{1}{1-t}\right)^x \left(\frac{1}{1-t}\right)^r = e_\lambda^{-x}\left(\log_\lambda(1-t)\right)\left(\frac{1}{1-t}\right)^r \tag{25}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k \left(\frac{1}{1-t}\right)^r \langle x \rangle_{k,\lambda}.$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 2.2 *For any integer $k \geq 0$, we have*

$$\frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k \left(\frac{1}{1-t}\right)^r = \sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \frac{t^n}{n!}.$$

From (19) and (20), we have

$$\frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \frac{t^n}{n!}. \tag{26}$$

Thus, we get

$$\frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k \left(\frac{1}{1-t}\right)^r = \left(\sum_{m=k}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_\lambda \frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} \langle r \rangle_l \frac{t^l}{l!}\right) \tag{27}$$

$$= \sum_{n=k}^{\infty} \left(\sum_{m=k}^{n} \binom{n}{m} \begin{bmatrix} m \\ k \end{bmatrix}_\lambda \langle r \rangle_{n-m}\right) \frac{t^n}{n!}.$$

Therefore, by Theorem 2.2 and (27), we obtain the following theorem.

Theorem 2.3 *For any nonnegative integers n, k, with $n \geq k$, we have*

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \sum_{m=k}^{n} \binom{n}{m} \begin{bmatrix} m \\ k \end{bmatrix}_\lambda \langle r \rangle_{n-m}.$$

Now, we observe that

$$\left(\frac{1}{1-t}\right)^r = \left(1 + \frac{t}{1-t}\right)^r = \sum_{l=0}^{r} \binom{r}{l}\left(\frac{t}{1-t}\right)^l = \sum_{l=0}^{r} (r)_l \frac{1}{l!}\left(\frac{t}{1-t}\right)^l \tag{28}$$

$$= \sum_{l=0}^{r} (r)_l \sum_{j=l}^{\infty} L(j,l)\frac{t^j}{j!} = \sum_{j=0}^{\infty} \left(\sum_{l=0}^{\min\{r,j\}} L(j,l)(r)_l\right)\frac{t^j}{j!}.$$

By (28), we get

$$\frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k\left(\frac{1}{1-t}\right)^r \tag{29}$$

$$= \sum_{m=k}^\infty \begin{bmatrix} m \\ k \end{bmatrix}_\lambda \frac{t^m}{m!} \sum_{j=0}^\infty \left(\sum_{l=0}^{\min\{r,j\}} L(j,l)(r)_l\right)\frac{t^j}{j!}$$

$$= \sum_{n=k}^\infty \left\{ \sum_{j=0}^{n-k} \sum_{l=0}^{\min\{r,j\}} L(j,l)(r)_l \binom{n}{j}\begin{bmatrix} n-j \\ k \end{bmatrix}_\lambda \right\}\frac{t^n}{n!}.$$

Therefore, by Theorem 2.2 and (29), we obtain the following theorem.

Theorem 2.4 *For any nonnegative integers n, k, with $n \geq k$, we have*

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \sum_{j=0}^{n-k} \sum_{l=0}^{\min\{r,j\}} L(j,l)(r)_l \binom{n}{j}\begin{bmatrix} n-j \\ k \end{bmatrix}_\lambda.$$

By (21), we get

$$\left(\frac{1}{1-t}\right)^{r+k} = e_\lambda^{-r-k}\left(\log_\lambda(1-t)\right) \tag{30}$$

$$= \sum_{l=0}^\infty \langle r+k\rangle_{l,\lambda} \sum_{n=l}^\infty \begin{bmatrix} n \\ l \end{bmatrix}_\lambda \frac{t^n}{n!}$$

$$= \sum_{n=0}^\infty \left(\sum_{l=0}^n \langle r+k\rangle_{l,\lambda} \begin{bmatrix} n \\ l \end{bmatrix}_\lambda\right)\frac{t^n}{n!}.$$

It is not difficult to show that

$$\left(\frac{1}{1-t}\right)^{r+k} = \sum_{n=0}^\infty \left(\sum_{l=0}^n \begin{bmatrix} n+r \\ l+r \end{bmatrix}_{r,\lambda} \langle k\rangle_{l,\lambda}\right)\frac{t^n}{n!}. \tag{31}$$

Therefore, by (28), (30), and (31), we obtain the following theorem.

Theorem 2.5 *For any nonnegative integers n, k, we have*

$$\sum_{l=0}^{\min\{r+k,n\}} (r+k)_l L(n,l) = \sum_{l=0}^n \langle r+k\rangle_{l,\lambda}\begin{bmatrix} n \\ l \end{bmatrix}_\lambda = \sum_{l=0}^n \begin{bmatrix} n+r \\ l+r \end{bmatrix}_{r,\lambda} \langle k\rangle_{l,\lambda}.$$

As an inversion formula of (22), we consider the degenerate r-Stirling numbers of second kind given by

$$(x+r)_{n,\lambda} = \sum_{k=0}^n \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} (x)_k, \tag{32}$$

where $n \geq 0$ and $r \in \mathbb{N}$.

Note from (15) and (32) that

$$\lim_{\lambda \to 0} \left\{ {n + r \atop k + r} \right\}_{r,\lambda} = \left\{ {n + r \atop k + r} \right\}.$$

From (32), we have

$$(x + r)_{n+1,\lambda} = (x + r)_{n,\lambda}(x + r - n\lambda) \tag{33}$$

$$= \sum_{k=0}^{n} \left\{ {n + r \atop k + r} \right\}_{r,\lambda} (x)_k (x - k + k + r - n\lambda)$$

$$= \sum_{k=0}^{n} \left\{ {n + r \atop k + r} \right\}_{r,\lambda} (x)_{k+1} + \sum_{k=0}^{n} \left\{ {n + r \atop k + r} \right\}_{r,\lambda} (k + r - n\lambda)(x)_k$$

$$= \sum_{k=0}^{n+1} \left\{ {n + r \atop k - 1 + r} \right\}_{r,\lambda} (x)_k + \sum_{k=0}^{n} \left\{ {n + r \atop k + r} \right\}_{r,\lambda} (k + r - n\lambda)(x)_k$$

$$= \sum_{k=0}^{n+1} \left(\left\{ {n + r \atop k - 1 + r} \right\}_{r,\lambda} + (k + r - n\lambda) \left\{ {n + r \atop k + r} \right\}_{r,\lambda} \right) (x)_k,$$

since

$$\left\{ {n + r \atop r - 1} \right\}_{r,\lambda} = 0 \quad \text{and} \quad \left\{ {n + r \atop n + 1 + r} \right\}_{r,\lambda} = 0.$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 2.6 *For any nonnegative integers n, k, with $n \geq k$, we have*

$$\left\{ {n + 1 + r \atop k + r} \right\}_{r,\lambda} = \left\{ {n + r \atop k - 1 + r} \right\}_{r,\lambda} + (k + r - n\lambda) \left\{ {n + r \atop k + r} \right\}_{r,\lambda}.$$

In particular,

$$\left\{ {n \atop k} \right\}_{r,\lambda} = \left\{ {n - 1 \atop k - 1} \right\}_{r,\lambda} + \left(k - (n - 1 - r)\lambda \right) \left\{ {n - 1 \atop k} \right\}_{r,\lambda},$$

where $n > r$.

By (32), we get

$$e_\lambda^{x+r}(t) = \sum_{n=0}^{\infty} (x + r)_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left\{ {n + r \atop k + r} \right\}_{r,\lambda} (x)_k \right) \frac{t^n}{n!} \tag{34}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \left\{ {n + r \atop k + r} \right\}_{r,\lambda} \frac{t^n}{n!} \right) (x)_k.$$

On the other hand,

$$e_\lambda^{x+r}(t) = \left(e_\lambda(t) - 1 + 1 \right)^x e_\lambda^r(t) = \sum_{k=0}^{\infty} \left(\frac{1}{k!} \left(e_\lambda(t) - 1 \right)^k e_\lambda^r(t) \right) (x)_k. \tag{35}$$

Therefore, by (34) and (35), we obtain the following theorem.

Theorem 2.7 *For any nonnegative integer k, we have*

$$\frac{1}{k!}\big(e_\lambda(t) - 1\big)^k e_\lambda^r(t) = \sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_{r,\lambda} \frac{t^n}{n!}.$$

Now, we observe that

$$
\begin{aligned}
\frac{1}{k!}\big(e_\lambda(t) - 1\big)^k e_\lambda^r(t) &= \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j} e_\lambda^{j+r}(t)\\
&= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{n=0}^{\infty}(j+r)_{n,\lambda}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left\{ \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j+r)_{n,\lambda}\right\}\frac{t^n}{n!}.
\end{aligned}
\tag{36}
$$

Therefore, by Theorem 2.7 and (36), we obtain the following theorem.

Theorem 2.8 *For any nonnegative integers n, k, we have*

$$\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j+r)_{n,\lambda} = \begin{cases} \left\{ {n+r \atop k+r} \right\}_{r,\lambda}, & \text{if } n \geq k,\\ 0, & \text{otherwise.}\end{cases}$$

In particular,

$$\frac{1}{(k-r)!}\sum_{j=0}^{k-r}\binom{k-r}{j}(-1)^{k-r-j}(j+r)_{n-r,\lambda} = \left\{ {n \atop k} \right\}_{r,\lambda} \quad (n \geq k \geq r).$$

The double generating function is given by

$$
\begin{aligned}
e_\lambda\big(t\big(e_\lambda(x) - 1\big)\big)e_\lambda^r(x) &= \sum_{m=0}^{\infty} t^m (1)_{m,\lambda}\frac{1}{m!}\big(e_\lambda(x) - 1\big)^m e_\lambda^r(x)\\
&= \sum_{m=0}^{\infty} t^m (1)_{m,\lambda}\sum_{k=m}^{\infty}\left\{ {k+r \atop m+r} \right\}_{r,\lambda}\frac{x^k}{k!}\\
&= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} t^m (1)_{m,\lambda}\left\{ {k+r \atop m+r} \right\}_{r,\lambda}\right)\frac{x^k}{k!}.
\end{aligned}
\tag{37}
$$

Let us define the degenerate r-Bell polynomials as follows:

$$e_\lambda\big(t\big(e_\lambda(x) - 1\big)\big)e_\lambda^r(x) = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}^{(r)}(t)\frac{x^n}{n!}. \tag{38}$$

From (37) and (38), we have

$$\mathrm{Bel}_{n,\lambda}^{(r)}(t) = \sum_{m=0}^{n} t^m (1)_{m,\lambda}\left\{ {n+r \atop m+r} \right\}_{r,\lambda} \quad (n \geq 0).$$

Theorem 2.9 *For any nonnegative integer n, we have*

$$\mathrm{Bel}_{n,\lambda}^{(r)}(t) = \sum_{m=0}^{n} \begin{Bmatrix} n+r \\ m+r \end{Bmatrix}_{r,\lambda} (1)_{m,\lambda} t^m.$$

By Theorem 2.7, we get

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k e_\lambda^r(t) = \frac{1}{k!}\left(e_\lambda(t)-1\right)^k \left(e_\lambda(t)-1+1\right)^r \qquad (39)$$

$$= \sum_{j=0}^{r} \binom{r}{j}\frac{1}{k!}\left(e_\lambda(t)-1\right)^{k+j}$$

$$= \sum_{j=0}^{r}(r)_j \binom{k+j}{k}\sum_{n=j+k}^{\infty}\begin{Bmatrix} n \\ j+k \end{Bmatrix}_\lambda \frac{t^n}{n!}$$

$$= \sum_{n=k}^{\infty}\left\{\sum_{j=0}^{\min\{r,n-k\}}(r)_j \binom{k+j}{k}\begin{Bmatrix} n \\ j+k \end{Bmatrix}_\lambda\right\}\frac{t^n}{n!}.$$

From (39), we have

$$\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} = \sum_{j=0}^{\min\{r,n-k\}}(r)_j \binom{k+j}{k}\begin{Bmatrix} n \\ j+k \end{Bmatrix}_\lambda. \qquad (40)$$

Therefore, by (40), we obtain the following theorem.

Theorem 2.10 *For any nonnegative integers n, k, with $n \geq k$, we have*

$$\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} = \sum_{j=0}^{\min\{r,n-k\}}(r)_j \binom{k+j}{k}\begin{Bmatrix} n \\ j+k \end{Bmatrix}_\lambda.$$

Now, we observe that

$$\frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k\left(\frac{1}{1-t}\right)^r = \frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k e_\lambda^{-r}\left(\log_\lambda(1-t)\right) \qquad (41)$$

$$= \frac{1}{k!}\left(-\log_\lambda(1-t)\right)^k \sum_{l=0}^{\infty}\langle r\rangle_{l,\lambda}\frac{1}{l!}\left(-\log_\lambda(1-t)\right)^l$$

$$= \sum_{l=0}^{\infty}\frac{(k+l)!}{k!l!}\langle r\rangle_{l,\lambda}\frac{1}{(k+l)!}\left(-\log_\lambda(1-t)\right)^{k+l}$$

$$= \sum_{l=0}^{\infty}\binom{k+l}{l}\langle r\rangle_{l,\lambda}\sum_{n=k+l}^{\infty}\begin{bmatrix} n \\ k+l \end{bmatrix}_\lambda \frac{t^n}{n!}$$

$$= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{n-k}\binom{k+l}{l}\langle r\rangle_{l,\lambda}\begin{bmatrix} n \\ k+l \end{bmatrix}_\lambda\right)\frac{t^n}{n!}.$$

By Theorem 2.2 and (41), we get

$$\begin{bmatrix} n + r \\ k + r \end{bmatrix}_{r,\lambda} = \sum_{l=0}^{n-k} \binom{k+l}{l} \begin{bmatrix} n \\ k + l \end{bmatrix}_{\lambda} \langle r \rangle_{l,\lambda}, \tag{42}$$

where n, k are nonnegative integers, with $n \geq k$.

Therefore, by (42), we obtain the following theorem.

Theorem 2.11 *For any nonnegative integers n, k, with $n \geq k$, we have*

$$\begin{bmatrix} n + r \\ k + r \end{bmatrix}_{r,\lambda} = \sum_{l=0}^{n-k} \binom{k+l}{l} \begin{bmatrix} n \\ k + l \end{bmatrix}_{\lambda} \langle r \rangle_{l,\lambda}.$$

3 Conclusion

In this paper, we studied the unsigned degenerate r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{r,\lambda}$ as degenerate versions of the r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{r}$ and the degenerate r-Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{r,\lambda}$ as degenerate versions of the r-Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{r}$. They can be viewed also as natural extensions of the degenerate Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{\lambda}$ and the degenerate Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda}$, which were introduced earlier. For the degenerate r-Stirling numbers of both kinds, we derived recurrence relations, generating functions, expressions of degenerate r-Stirling numbers in terms of degenerate Stirling numbers, and a representation of the degenerate r-Bell polynomials in terms of the degenerate r-Stirling numbers of the second kind.

As it turns out, the degenerate Stirling numbers appear very frequently when we study degenerate versions of many special numbers and polynomials [7, 9, 13]. It would be very interesting to discover many appearances of the degenerate r-Stirling numbers in such studies. It is one of our future projects to continue to explore degenerate versions of some special numbers and polynomials and their applications not only in mathematics but also in other disciplines like statistics, physics, engineering, and social sciences.

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