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Complete moment convergence of extended negatively dependent random variables

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Abstract

In this paper, some results on the complete moment convergence of extended negatively dependent (END) random variables are established. The results in the paper improve and extend the corresponding ones of Qiu et al. (Acta Math. Appl. Sin. 40(3):436–448, 2017) under some weaker conditions. Our results also improve and extend the related known works in the literature.

MSC: 60F15

Keywords: END random variables; Weighted sums; Complete convergence; Complete moment convergence

1 Introduction

In many statistical models, it is not reasonable to assume that random variables are independent, and so it is very meaningful to extend the concept of independence to dependence cases. One important dependence sequence of these dependences is extended negatively dependent (END) random variables, we recall the concept of END random variables as follows.

Definition 1.1 The random variables $\{X_n, n \geq 1\}$ are said to be extended negatively dependent (END) random variables if there exists a positive constant $M > 0$ such that both

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i)$$

hold for each $n \geq 1$ and all real $x_1, x_2, \ldots, x_n$.

The concept of END random variables was introduced by Liu [2]. Obviously, END random variables ($M = 1$) imply NOD (negatively orthant dependent) random variables (Joag-Dev and Proschan [3]). Liu [2] pointed out that the END random variables are more comprehensive, and they can reflect not only negative dependence random variables but also positive ones, to some extent. Joag-Dev and Proschan [3] once pointed out that NOD random variables imply NA (negatively associated) random variables, but NA random variables do not imply NOD random variables, so END random variables imply NA random variables. Thus, it is interesting to investigate convergence properties for END random variables.

After the appearance of Liu [2], many scholars have focused on the properties of END random variables, and a lot of results have been gained. For example, Liu [4] studied necessary and sufficient conditions for moderate deviations of dependent random variables with heavy tails; Chen et al. [5] established strong law of large numbers for END random variables; Wu and Guan [6] presented convergence properties of the partial sums of END random variables; Shen [7] presented probability inequalities for END sequence and their applications; Wang and Wang [8] investigated large deviations for random sums of END random variables; Wang et al. and Qiu et al. [9–13] studied complete convergence of END random variables, etc.

The complete convergence plays a very important role in the probability theory and mathematical statistics. The concept of complete convergence was introduced by Hsu and Robbins [14] as follows: A sequence $\{U_n, n \geq 1\}$ of random variables is said to converge completely to a constant θ if, for $\forall \varepsilon > 0$, $\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty$. In view of the Borel–Cantelli lemma, the complete convergence implies that $U_n \to \theta$ almost surely. Therefore, complete convergence is a very important tool in establishing almost sure convergence for partial of random variables as well as weighted sums of random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, $a_n > 0$, $b_n > 0$, $\gamma > 0$. If for $\forall \varepsilon > 0$, $\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^{\gamma} < \infty$, then $\{X_n, n \geq 1\}$ is called the complete moment convergence (Chow [15]). It is well known that complete moment convergence implies complete convergence, i.e., the complete moment convergence is more general than complete convergence. The following result is from Chow [15].

Theorem A *Let $r > 1$, $1 \leq p < 2$, $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables and $EX_1 = 0$, if $E\{|X_1|^{rp} + |X_1| \log(1 + |X_1|)\} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2-1/p} E\left\{\left|\sum_{i=1}^{n} X_i\right| - \varepsilon n^{1/p}\right\}_+ < \infty, \quad \forall \varepsilon > 0.$$

It should be noted that Theorem A has been extended and improved by many scholars (see [16–19]).

Recently, Chen and Sung [20] obtained complete and complete moment convergence of ρ-mixing random variables, and Qiu et al. [1] obtained the following complete moment convergence for weighted sums of END random variables.

Theorem B *Let $r > 1$, $1 \le p < 2$, $\lambda > 0$, $\alpha > 1$, $\beta > 1$ with $1/\alpha + 1/\beta = 1/p$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{a} \le Dn, \quad \forall n \ge 1, \tag{1.1}$$

where D is a positive constant. $\{X, X_n, 1 \le n\}$ is a sequence of identically distributed END random variables with $EX = 0$. If

$$\begin{cases} E|X|^{(r-1)\beta} < \infty & \text{if } \alpha < rp, \lambda < (r-1)\beta, \\ E|X|^{(r-1)\beta} \log(1+|X|) < \infty & \text{if } \alpha = rp, \lambda < (r-1)\beta, \\ E|X|^{(r-1)\beta} \log(1+|X|) < \infty & \text{if } \alpha < rp, \lambda = (r-1)\beta, \\ E|X|^{(r-1)\beta} \log^2(1+|X|) < \infty & \text{if } \alpha = rp, \lambda = (r-1)\beta, \\ E|X|^{rp} < \infty & \text{if } \alpha > rp, \lambda < rp, \\ E|X|^{rp} \log(1+|X|) < \infty & \text{if } \alpha > rp, \lambda = rp, \\ E|X|^{\lambda} < \infty & \text{if } \lambda > \max\{rp, (r-1)\beta\}, \\ & \text{when } \alpha > rp, \text{assume } \lambda < \alpha, \end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon n^{1/p} \right\}_{+}^{\lambda} < \infty, \quad \forall \varepsilon > 0.$$

In this article, our goal is to further study complete moment convergence for weighted sums of END random variables with suitable conditions. By using the truncated method, we obtain a novel result, which extends that in Qiu et al. [1] under some weaker conditions. Our result also improves and extends those in Chen and Sung [20], Sung [21], and Qiu and Xiao [22].

The layout of this paper is as follows. Main results and some lemmas are provided in Sect. 2. Proofs of the main results are given in Sect. 3. Throughout the paper, the symbol C denotes a positive constant, which may take different values in different places. $I(A)$ is the indicator function of an event A.

2 Main results and some lemmas

Theoremm 2.1 *Let $r > 1$, $1 \le p < 2$, $\lambda > 0$, $\alpha > 0$, $\beta > 0$ with $1/\alpha + 1/\beta = 1/p$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1). $\{X, X_n, n \ge 1\}$ is a sequence of identically distributed END random variables with $EX = 0$. Assume that one of the following conditions holds*:

(1) *If $\alpha < rp$, then*

$$\begin{cases} E|X|^{(r-1)\beta} < \infty & \text{if } \lambda < (r-1)\beta, \\ E|X|^{(r-1)\beta} \log(1+|X|) < \infty & \text{if } \lambda = (r-1)\beta, \\ E|X|^{\lambda} < \infty & \text{if } \lambda > (r-1)\beta. \end{cases} \tag{2.1}$$

(2) *If* $\alpha = rp$, *then*

$$\begin{cases} E|X|^{(r-1)\beta}\log(1+|X|) < \infty & \text{if } \lambda \le (r-1)\beta = rp, \\ E|X|^{\lambda} < \infty & \text{if } \lambda > (r-1)\beta = rp. \end{cases} \tag{2.2}$$

(3) *If* $\alpha > rp$, *then*

$$\begin{cases} E|X|^{rp} < \infty & \text{if } \lambda < rp, \\ E|X|^{rp}\log(1+|X|) < \infty & \text{if } \lambda = rp, \\ E|X|^{\lambda} < \infty & \text{if } \lambda > rp. \end{cases} \tag{2.3}$$

Then

$$\sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p}\right\}_+^{\lambda} < \infty, \quad \forall \varepsilon > 0. \tag{2.4}$$

Conversely, if (2.4) *holds for any array* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *satisfying* (1.1), *then* $EX = 0$, $E|X|^{(r-1)\beta} < \infty$, $E|X|^{rp} < \infty$.

Remark 2.1 The Rademacher–Menshov inequality is only used in the proof process of Theorem 2.1. The results in this paper still hold for random variable satisfying Rosenthal's inequality. Therefore, our results improve and extend the result of Chen and Sung [20].

Remark 2.2 In this paper, the conditions of Theorem 2.1 are weaker than those in Theorem 1.1 of Qiu et al. [1], and the condition of "if $\alpha > rp$, assume $\lambda < \alpha$ (Qiu et al. [1])" is not necessary for (2.4) in our paper. Therefore our results improve and extend the result of Qiu et al. [1]. It is worth pointing out that the method applied in this article is different from that in Qiu et al. [1].

To prove Theorem 2.1 of the paper, we need the following important lemmas.

Lemma 2.1 (Qiu [22]; Rademacher–Menshov inequality) *Let* $p > 1$, $\{X_n, n \ge 1\}$ *be a sequence of END random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$. *Then there exists a positive constant* C_p *only depending on* p *such that*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C_p \log^p n \sum_{i=1}^{n} E|X_i|^p, \quad 1 < p \le 2, \tag{2.5}$$

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C_p \log^p n \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right\}, \quad p > 2. \tag{2.6}$$

Lemma 2.2 (Qiu [22]) *Let* $p \ge 1$, $\{X_n, n \ge 1\}$ *be a sequence of END random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$. *Then there exists a positive constant* C_p *only depending on* p *such* *that*

$$E\left(\left|\sum_{i=1}^{n} X_i\right|^p\right) \le C_p \sum_{i=1}^{n} E|X_i|^p, \quad 1 \le p < 2, \tag{2.7}$$

$$E\left(\left|\sum_{i=1}^{n} X_i\right|^p\right) \le C_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E(X_i)^2\right)^{p/2}\right\}, \quad p \ge 2. \tag{2.8}$$

Lemma 2.3 (Liu [2]) *Let $\{X_n, n \ge 1\}$ be a sequence of END random variables. If $f_1, f_2, \dots, f_n$ are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \dots, f_n(X_n)$ are still END random variables.*

Lemma 2.4 (Wu [23]) *Let $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ be sequences of random variables, for any $q > r > 0$, $\varepsilon > 0$, $a > 0$, then*

$$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_i + Y_i)\right| - \varepsilon a\right)_+^r \le C_r\left(\frac{1}{\varepsilon^q} + \frac{r}{q-r}\right)\frac{1}{a^{q-r}} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^q\right) + C_r E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_i\right|^r\right),$$

where $C_r = 1$ if $0 < r \le 1$ or $C_r = 2^{r-1}$ if $r > 1$.

Chen and Sung [20] obtained the following theorems (see Lemmas 2.5–2.7).

Lemma 2.5 (Chen [20]) *Let $r > 1$, $1 \le p < 2$, $\alpha > 0$, $\beta > 0$ with $1/\alpha + 1/\beta = 1/p$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1). X is a random variable, then*

$$\sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{n} P\big(|a_{ni}X| > n^{1/p}\big) \le \begin{cases} CE|X|^{(r-1)\beta} & \text{if } \alpha < rp, \\ CE|X|^{(r-1)\beta}\log(1+|X|) & \text{if } \alpha = rp, \\ CE|X|^{rp} & \text{if } \alpha > rp. \end{cases}$$

Lemma 2.6 (Chen [20]) *Let $r > 1$, $1 \le p < 2$, $\alpha > 0$, $\beta > 0$ with $1/\alpha + 1/\beta = 1/p$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1). If X is a random variable, then for any $v > \max\{\alpha, (r-1)\beta\}$*

$$\sum_{n=1}^{\infty} n^{r-2-v/p} \sum_{i=1}^{n} E|a_{ni}X|^v I\big(|a_{ni}X| \le n^{1/p}\big) \le \begin{cases} CE|X|^{(r-1)\beta} & \text{if } \alpha < rp, \\ CE|X|^{(r-1)\beta}\log(1+|X|) & \text{if } \alpha = rp, \\ CE|X|^{rp} & \text{if } \alpha > rp. \end{cases}$$

Lemma 2.7 (Chen [20]) *Let $\lambda > 0$, $r > 1$, $1 \le p < 2$, $\alpha > 0$, $\beta > 0$ with $1/\alpha + 1/\beta = 1/p$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1) and X be a random variable. Then the following statements hold:*

(1) *If* $\alpha < rp$, *then*

$$\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E|a_{ni}X|^{\lambda} I\big(|a_{ni}X| > n^{1/p}\big) \le \begin{cases} CE|X|^{(r-1)\beta} & \text{if } \lambda < (r-1)\beta, \\ CE|X|^{(r-1)\beta}\log(1+|X|) & \text{if } \lambda = (r-1)\beta, \\ CE|X|^{\lambda} & \text{if } \lambda > (r-1)\beta. \end{cases}$$

(2) *If* $\alpha = rp$, *then*

$$\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E|a_{ni}X|^{\lambda} I\big(|a_{ni}X| > n^{1/p}\big) \le \begin{cases} CE|X|^{(r-1)\beta}\log(1+|X|) & \text{if } \lambda \le (r-1)\beta = rp, \\ CE|X|^{\lambda} & \text{if } \lambda > (r-1)\beta = rp. \end{cases}$$

(3) *If* $\alpha > rp$, *then*

$$\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E|a_{ni}X|^{\lambda} I\big(|a_{ni}X| > n^{1/p}\big) \le \begin{cases} CE|X|^{rp} & \text{if } \lambda < rp, \\ CE|X|^{rp}\log(1+|X|) & \text{if } \lambda = rp, \\ CE|X|^{\lambda} & \text{if } \lambda > rp. \end{cases}$$

3 Proofs of theorems

Proof of Theorem 2.1 Noting $\alpha > 0$, $\beta > 0$, $1/\alpha + 1/\beta = 1/p$, we have

$$\begin{cases} \alpha < rp & \Leftrightarrow & rp < (r-1)\beta, \\ \alpha = rp & \Leftrightarrow & rp = (r-1)\beta, \\ \alpha > rp & \Leftrightarrow & rp > (r-1)\beta. \end{cases}$$

For $\forall t : 0 < t \le \alpha$, by the *Hölder* inequality and (1.1), we have

$$\sum_{i=1}^{n} |a_{ni}|^{t} \le \left(\sum_{i=1}^{n} |a_{ni}|^{a}\right)^{t/a} \left(\sum_{i=1}^{n} 1\right)^{1-t/a} \le Cn. \tag{3.1}$$

For $\forall t : t > \alpha$, it follows from the C_r inequality and (1.1) that

$$\sum_{i=1}^{n} |a_{ni}|^{t} \le \left(\sum_{i=1}^{n} |a_{ni}|^{a}\right)^{t/a} \le Cn^{t/a}. \tag{3.2}$$

Noting that $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, without loss of generality, we can assume $a_{ni} > 0$.

Sufficiency. Set $\theta \in (\frac{p}{\alpha\wedge rp}, 1)$ for $1 \le i \le n$, $n \ge 1$, and let

$$
\begin{aligned}
X_{ni}^{(1)} &= -n^{\theta/p} I\big(a_{ni}X_i < -n^{\theta/p}\big) + a_{ni}X_i I\big(|a_{ni}X_i| \le n^{\theta/p}\big) + n^{\theta/p} I\big(a_{ni}X_i > n^{\theta/p}\big),\\
X_{ni}^{(2)} &= \big(a_{ni}X_i - n^{\theta/p}\big) I\big(n^{\theta/p} < a_{ni}X_i \le n^{\theta/p} + n^{1/p}\big) + n^{1/p} I\big(a_{ni}X_i > n^{\theta/p} + n^{1/p}\big),\\
X_{ni}^{(3)} &= \big(a_{ni}X_i + n^{\theta/p}\big) I\big(-n^{\theta/p} - n^{1/p} \le a_{ni}X_i < -n^{\theta/p}\big) - n^{1/p} I\big(a_{ni}X_i < -n^{\theta/p} - n^{1/p}\big),\\
X_{ni}^{(4)} &= \big(a_{ni}X_i - n^{\theta/p} - n^{1/p}\big) I\big(a_{ni}X_i > n^{\theta/p} + n^{1/p}\big),\\
X_{ni}^{(5)} &= \big(a_{ni}X_i + n^{\theta/p} + n^{1/p}\big) I\big(a_{ni}X_i < -n^{\theta/p} - n^{1/p}\big).
\end{aligned}
$$

Then $a_{ni}X_i = \sum_{l=1}^{5} X_{ni}^{(l)}$. It follows from the definition of $X_{ni}^{(2)}, \theta \in (\frac{p}{\alpha\wedge rp}, 1)$, (3.1), and (2.1)–(2.3) that

$$
\begin{aligned}
n^{-1/p} \max_{1\le k\le n} \left|\sum_{i=1}^{k} EX_{ni}^{(2)}\right| &= n^{-1/p} \sum_{i=1}^{n} EX_{ni}^{(2)}\\
&\le n^{-1/p} \sum_{i=1}^{n} E|a_{ni}X_i| I\big(|a_{ni}X_i| > n^{\theta/p}\big)\\
&\le n^{-1/p} \sum_{i=1}^{n} E|a_{ni}X_i| \left(\frac{|a_{ni}X_i|}{n^{\theta/p}}\right)^{(\alpha\wedge rp-1)} I\big(|a_{ni}X_i| > n^{\theta/p}\big)\\
&\le n^{1-1/p-(\alpha\wedge rp-1)\theta/p} E|X|^{\alpha\wedge rp} \to 0, \quad n \to \infty.
\end{aligned}
$$

By the definition $X_{ni}^{(4)}$ and (3.1), from the above proof process, we have

$$
\begin{aligned}
n^{-1/p} \max_{1\le k\le n} \left|\sum_{i=1}^{k} EX_{ni}^{(4)}\right| &= n^{-1/p} \sum_{i=1}^{n} EX_{ni}^{(4)}\\
&\le n^{-1/p} \sum_{i=1}^{n} E|a_{ni}X_i| I\big(|a_{ni}X_i| > n^{\theta/p} + n^{1/p}\big)\\
&\le n^{-1/p} \sum_{i=1}^{n} E|a_{ni}X_i| I\big(|a_{ni}X_i| > n^{\theta/p}\big) \to 0, \quad n \to \infty.
\end{aligned}
$$

Similarly, we can obtain

$$
\lim_{n\to\infty} n^{-1/p} \max_{1\le k\le n} \left|\sum_{i=1}^{k} EX_{ni}^{(3)}\right| = \lim_{n\to\infty} -n^{-1/p} \sum_{i=1}^{n} EX_{ni}^{(3)} = 0
$$

and

$$
\lim_{n\to\infty} n^{-1/p} \max_{1\le k\le n} \left|\sum_{i=1}^{k} EX_{ni}^{(5)}\right| = \lim_{n\to\infty} -n^{-1/p} \sum_{i=1}^{n} EX_{ni}^{(5)} = 0.
$$

Noting that $EX_i = 0$, it follows from Lemma 2.4 and the C_r inequality that, for $\nu > \lambda \ge 1$,

$$
\sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p}\right\}_{+}^{\lambda}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} \sum_{l=1}^{5} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right) \right| - \varepsilon n^{1/p} \right\}_{+}^{\lambda} \\
&\le \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{ \sum_{l=1}^{5} \max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right) \right| - \varepsilon n^{1/p} \right\}_{+}^{\lambda} \\
&\le \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right) \right| + \sum_{l=2}^{5} \left| \sum_{i=1}^{n} X_{ni}^{(l)} \right| - 3\varepsilon n^{1/p}/4 \right\}_{+}^{\lambda} \\
&\le \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right) \right| + \sum_{l=2}^{5} \left| \sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right) \right| - \varepsilon n^{1/p}/2 \right\}_{+}^{\lambda} \\
&\le C \sum_{n=1}^{\infty} n^{r-2-\nu/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right) \right|^{\nu} \right\} \\
&\quad + C \sum_{l=2}^{3} \sum_{n=1}^{\infty} n^{r-2-\nu/p} E\left| \sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right) \right|^{\nu} \\
&\quad + C \sum_{l=4}^{5} \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left| \sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right) \right|^{\lambda} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{3.3}
$$

Similarly, for $\nu > \lambda$, $0 < \lambda < 1$, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon n^{1/p} \right\}_{+}^{\lambda} \\
&\quad\le \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right) \right| + \sum_{l=2}^{3} \left| \sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right) \right| \right. \\
&\qquad \left. + \sum_{l=4}^{5} \left| \sum_{i=1}^{n} X_{ni}^{(l)} \right| - \varepsilon n^{1/p}/2 \right\}_{+}^{\lambda} \\
&\quad\le C \sum_{n=1}^{\infty} n^{r-2-\nu/p} E\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right) \right|^{\nu} \right\} \\
&\qquad + C \sum_{l=2}^{3} \sum_{n=1}^{\infty} n^{r-2-\nu/p} E\left| \sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right) \right|^{\nu} \\
&\qquad + C \sum_{l=4}^{5} \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left| \sum_{i=1}^{n} X_{ni}^{(l)} \right|^{\lambda} \\
&\quad =: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{3.4}
$$

In order to prove Theorem 2.1, we need to prove $I_i < \infty$, $i = 1, 2, \ldots, 5$.

Taking $\nu > \max\{2, 2rp/[(2-p)(1-\theta)], 2pr/(a-p), 2pr/(2-p), a, (r-1)\beta, \lambda\}$, it follows from Lemmas 2.1 and 2.3 that

$$
\begin{aligned}
I_1 &\le C \sum_{n=1}^{\infty} n^{r-2-\nu/p} \log^{\nu} n \sum_{i=1}^{n} \left\{ E\left|X_{ni}^{(1)}\right|^{\nu} + \left(\sum_{i=1}^{n} E\left|X_{ni}^{(1)}\right|^{2} \right)^{\nu/2} \right\} \\
&:= I_{11} + I_{12}.
\end{aligned}
$$

By the definition of $X_{ni}^{(1)}$ and $v > 2rp/[(2-p)(1-\theta)] > rp/(1-\theta)$, we have

$$\begin{aligned} I_{11} &\leq C\sum_{n=1}^{\infty} n^{r-2-v/p}\log^v n\left[\sum_{i=1}^{n} E|a_{ni}X_i|^v I\left(|a_{ni}X_i| \leq n^{\theta/p}\right) + \sum_{i=1}^{n} n^{v\theta/p} P\left(|a_{ni}X_i| > n^{\theta/p}\right)\right] \\ &\leq C\sum_{n=1}^{\infty} n^{r-2-v/p}\log^v n\left(\sum_{i=1}^{n} n^{v\theta/p}\right) \\ &\leq C\sum_{n=1}^{\infty} n^{r-1-(1-\theta)v/p}\log^v n < \infty. \end{aligned} \tag{3.5}$$

Since $r > 1$, $1 \leq p < 2$, $\alpha > 0$, $\beta > 0$ with $1/\alpha + 1/\beta = 1/p$, then $p < \alpha \wedge rp$. By (3.1) and (2.1)–(2.3), we obtain

$$\begin{aligned} I_{12} &\leq C\sum_{n=1}^{\infty} n^{r-2-v/p}\log^v n\left[\sum_{i=1}^{n} E|a_{ni}X_i|^2 I\left(|a_{ni}X_i| \leq n^{\theta/p}\right)\right. \\ &\quad \left. + \sum_{i=1}^{n} n^{2\theta/p} P\left(|a_{ni}X_i| > n^{\theta/p}\right)\right]^{v/2} \\ &\leq C\sum_{n=1}^{\infty} n^{r-2-v/p}\log^v n\left(\sum_{i=1}^{n} E|a_{ni}X_i|^p n^{(2-p)\theta/p}\right)^{v/2} \\ &\leq C\sum_{n=1}^{\infty} n^{r-2-(2-p)(1-\theta)v/2p}\log^v n\left(E|X|^p\right)^{v/2} < \infty. \end{aligned} \tag{3.6}$$

Then it follows from (3.5) and (3.6) that $I_1 < \infty$ holds.

By the definition of $X_{ni}^{(2)}$, Lemmas 2.2 and 2.3, we get

$$\begin{aligned} I_2 &\leq C\sum_{n=1}^{\infty} n^{r-2-v/p}\left[\sum_{i=1}^{n} E\left|X_{ni}^{(2)}\right|^v + \left(\sum_{i=1}^{n} E\left|X_{ni}^{(2)}\right|^2\right)^{v/2}\right] \\ &\leq C\sum_{n=1}^{\infty} n^{r-2-v/p}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^v I\left(|a_{ni}X_i| \leq 2n^{1/p}\right) + \sum_{i=1}^{n} n^{v/p} P\left(|a_{ni}X_i| > n^{1/p}\right)\right] \\ &\quad + C\sum_{n=1}^{\infty} n^{r-2-v/p}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^2 I\left(|a_{ni}X_i| \leq 2n^{1/p}\right) + \sum_{i=1}^{n} n^{2/p} P\left(|a_{ni}X_i| > n^{1/p}\right)\right]^{v/2} \\ &:= I_{21} + I_{22}. \end{aligned}$$

Combining Lemmas 2.5 and 2.6, we obtain $I_{21} < \infty$.

The proof of $I_{22} < \infty$ will mainly be conducted under the following four cases.

Case 1: $1 < \alpha < 2$, $\alpha \leq rp$. Noting that $p < \alpha$, by (2.1)–(2.2), we have $E|X|^\alpha < \infty$, then

$$\begin{aligned} I_{22} &\leq C\sum_{n=1}^{\infty} n^{r-2-v/p}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^2 I\left(|a_{ni}X_i| \leq 2n^{1/p}\right)\right]^{v/2} \\ &\quad + C\sum_{n=1}^{\infty} n^{r-2}\left[\sum_{i=1}^{n} P\left(|a_{ni}X_i| > n^{1/p}\right)\right]^{v/2} \end{aligned}$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{r-2-v/p}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^{\alpha}\left(2n^{1/p}\right)^{2-\alpha}\right]^{v/2} + C\sum_{n=1}^{\infty} n^{r-2}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^{\alpha}\left(n^{-\alpha/p}\right)\right]^{v/2}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-[(\alpha/p)-1]v/2}\left(E|X|^{\alpha}\right)^{v/2} < \infty. \qquad (3.7)
\end{aligned}$$

Case 2: $1 < \alpha < 2$, $\alpha > rp$. Noting that $rp < 2$, by (2.3), we obtain $E|X|^{rp} < \infty$, then

$$\begin{aligned}
I_{22} &\le C\sum_{n=1}^{\infty} n^{r-2-v/p}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^{rp}\left(2n^{1/p}\right)^{2-rp}\right]^{v/2}\\
&\quad + C\sum_{n=1}^{\infty} n^{r-2}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^{rp} n^{-rp/p})\right]^{v/2}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-(r-1)v/2}\left(E|X|^{rp}\right)^{v/2} < \infty. \qquad (3.8)
\end{aligned}$$

Case 3: $\alpha \ge 2$, $\alpha \le rp$. Noting that $rp \ge 2$, by (2.1)–(2.2), we get $E|X|^2 < \infty$, and then

$$\begin{aligned}
I_{22} &\le C\sum_{n=1}^{\infty} n^{r-2-v/p}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^{2}\right]^{v/2} + C\sum_{n=1}^{\infty} n^{r-2}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^{2} n^{-2/p}\right]^{v/2}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-[(2/p)-1]v/2}\left(E|X|^{2}\right)^{v/2} < \infty. \qquad (3.9)
\end{aligned}$$

Case 4: $\alpha \ge 2$, $\alpha > rp$, then $E|X|^{rp} < \infty$. If $rp < 2$, the proof is the same as that of Case 2. If $rp \ge 2$, the proof is the same as that of Case 3.

Then it follows from (3.7)–(3.9) that $I_2 < \infty$ holds.

The proof of $I_4 < \infty$ will mainly be conducted under the following three cases.

Case 1: $0 < \lambda < 1$. By (3.4), the C_r inequality, Lemma 2.7, and (2.1)–(2.3), we have

$$\begin{aligned}
I_4 &= \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left|\sum_{i=1}^{n} X_{ni}^{(4)}\right|^{\lambda}\\
&\le \sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E|X_{ni}^{(4)}|^{\lambda}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E|a_{ni}X_i|^{\lambda} I\left(|a_{ni}X_i| > n^{1/p}\right) < \infty. \qquad (3.10)
\end{aligned}$$

Case 2: $1 \le \lambda \le 2$. It follows from (3.3), the C_r inequality, Jensen's inequality, Lemmas 2.2–2.3, 2.7, and (2.1)–(2.3) that

$$I_4 = \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left|\sum_{i=1}^{n} \left(X_{ni}^{(4)} - EX_{ni}^{(4)}\right)\right|^{\lambda}$$

$$\begin{aligned}
&\le \sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E\big|X_{ni}^{(4)}\big|^{\lambda} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E|a_{ni}X_i|^{\lambda} I\big(|a_{ni}X_i| > n^{1/p}\big) < \infty. \qquad (3.11)
\end{aligned}$$

Case 3. $\lambda > 2$. By (3.3), the C_r inequality, Jensen's inequality, Lemmas 2.2, 2.7, and (2.1)–(2.3), we have

$$\begin{aligned}
I_4 &= \sum_{n=1}^{\infty} n^{r-2-\lambda/p} E\left|\sum_{i=1}^{n}\big(X_{ni}^{(4)} - EX_{ni}^{(4)}\big)\right|^{\lambda} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \left\{\sum_{i=1}^{n} E\big|X_{ni}^{(4)}\big|^{\lambda} + \left(\sum_{i=1}^{n} E\big|X_{ni}^{(4)}\big|^{2}\right)^{\lambda/2}\right\} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \sum_{i=1}^{n} E|a_{ni}X_i|^{\lambda} I\big(|a_{ni}X_i| > n^{1/p}\big) \\
&\quad + C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \left[\sum_{i=1}^{n} E|a_{ni}X_i|^{2} I\big(|a_{ni}X_i| > n^{1/p}\big)\right]^{\lambda/2} \\
&:= I_{41} + I_{42}.
\end{aligned}$$

From Lemma 2.7 and (2.1)–(2.3), we obtain $I_{41} < \infty$.

The proof of $I_{42} < \infty$ will mainly be conducted under the following two cases.

Case a: $\alpha \le rp$. Taking $q = \max\{(r-1)\beta, \lambda\} > 2$, by (2.1)–(2.2), (3.2), we have $E|X|^q < \infty$ and

$$\begin{aligned}
I_{42} &\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \left[\sum_{i=1}^{n} E|a_{ni}X_i|^{q} n^{(2-q)/p} I\big(|a_{ni}X_i| > n^{1/p}\big)\right]^{\lambda/2} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \big[n^{q/\alpha} E|X|^{q} n^{(2-q)/p}\big]^{\lambda/2} \\
&= C\sum_{n=1}^{\infty} n^{r-2-q\lambda/2\beta} \big[E|X|^{q}\big]^{\lambda/2} < \infty. \qquad (3.12)
\end{aligned}$$

Case b: $\alpha > rp$. Letting $q = \max\{rp, \lambda\} > 2$, it follows from (2.3) that $E|X|^q < \infty$. If $\alpha \ge q$, by (3.1), we have

$$\begin{aligned}
I_{42} &\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \left[\sum_{i=1}^{n} E|a_{ni}X_i|^{q} n^{(2-q)/p} I\big(|a_{ni}X_i| > n^{1/p}\big)\right]^{\lambda/2} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p} \big[nE|X|^{q} n^{(2-q)/p}\big]^{\lambda/2} \\
&= C\sum_{n=1}^{\infty} n^{r-2-(q-p)\lambda/2p} \big[E|X|^{q}\big]^{\lambda/2} < \infty. \qquad (3.13)
\end{aligned}$$

If $\alpha < q$, then $(r-1)\beta < rp < \alpha < q$, by (3.2), we have

$$\begin{aligned} I_{42} &\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p}\left[\sum_{i=1}^{n} E|a_{ni}X_i|^q n^{(2-q)/p} I\left(|a_{ni}X_i| > n^{1/p}\right)\right]^{\lambda/2} \\ &\le C\sum_{n=1}^{\infty} n^{r-2-\lambda/p}\left[n^{q/\alpha}E|X|^{\nu} n^{(2-q)/p}\right]^{\lambda/2} \\ &= C\sum_{n=1}^{\infty} n^{r-2-q\lambda/2\beta}\left[E|X|^q\right]^{\lambda/2} < \infty. \end{aligned} \tag{3.14}$$

Then it follows from (3.10)–(3.14) that $I_4 < \infty$.

Similar to the proof of $I_2 < \infty$ and $I_4 < \infty$, we can get $I_3 < \infty$ and $I_5 < \infty$, too.

Necessity. By (2.4), we have

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/p}\right) < \infty, \quad \forall \varepsilon > 0. \tag{3.15}$$

Set $a_{ni} = 1$ for $\{1 \le i \le n, n \ge 1\}$, then (3.15) can be rewritten as follows:

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon n^{1/p}\right) < \infty, \quad \forall \varepsilon > 0, \tag{3.16}$$

which implies that $EX = 0$, $E|X|^{rp} < \infty$ (see Theorem 2 in Peligard and Gut [24]). Take $a_{ni} = 0$ for $1 \le i \le n-1$, $n \ge 1$, and $a_{nn} = n^{1/\alpha}$, then (3.15) can be rewritten as follows:

$$\sum_{n=1}^{\infty} n^{r-2} P\left(|X_n| > \varepsilon n^{1/\beta}\right) < \infty, \quad \forall \varepsilon > 0, \tag{3.17}$$

which is equivalent to $E|X|^{(r-1)\beta} < \infty$. The proof is completed. □

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Authors' contributions
The authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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