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Taylor theory associated with Hahn difference operator

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Abstract

In this paper, we establish Taylor theory based on Hahn's difference operator $D_{q,\omega}$ which is defined by $D_{q,\omega}f(t) = \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}$, $t \neq \frac{\omega}{1-q}$, where $q \in (0,1)$ and ω is a positive number.

MSC: 39A10; 39A13; 39A70; 41A58; 47B39

Keywords: Taylor series; Hahn difference operator $D_{q,\omega}$

1 Introduction and preliminaries

Let $q \in (0,1)$, $\omega > 0$ and $\omega_0 := \frac{\omega}{1-q}$. Let f be a function defined on an interval I of $\mathbb{R}$ which contains ω_0. Hahn [10] introduced his difference operator which is defined by

$$D_{q,\omega}f(t) := \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, \quad \text{if } t \neq \omega_0, \tag{1.1}$$

and $D_{q,\omega}f(\omega_0) := f'(\omega_0)$, provided that f is differentiable at ω_0 in the usual sense. In this case we call $D_{q,\omega}f$ the q,ω-derivative and that f is q,ω-differentiable at t whenever $D_{q,\omega}f(t)$ exists. Finally, we say that f is q,ω-differentiable, i.e., throughout I if $D_{q,\omega}f(\omega_0)$ exists.

Hahn difference operator unifies the two most well-known quantum difference operators: the Jackson q-difference operator [11–13], which is defined by

$$D_q f(t) = \frac{f(qt)-f(t)}{t(q-1)}, \quad \text{if } t \neq 0, 0 < q < 1; \tag{1.2}$$

and the forward difference Δ_ω, which is defined by

$$\Delta_\omega f(t) = \frac{f(t+\omega)-f(t)}{\omega}, \quad t \in \mathbb{R}, \omega > 0, \tag{1.3}$$

see [4, 5, 14, 15]. Hahn operator has attracted the attention of several researchers and a variety of results can be found in papers [1, 2, 6, 16–22]. In [3] Annaby and Mansour proved analytically the q-Taylor series associated with D_q, introduced by Jackson [12], of an analytic function in some complex domain. In the present paper, we establish an overarching

q,ω-Taylor theory associated with Hahn difference operator $D_{q,\omega}$. In this theory the Hahn difference operator $D_{q,\omega}$ replaces the differentiation operator in the usual Taylor series.

First, we introduce some preliminary results and some notations. Let f, g be q,ω-differentiable at $t \in I$, then

$$D_{q,\omega}(f+g)(t) = D_{q,\omega}f(t) + D_{q,\omega}g(t), \tag{1.4}$$

$$D_{q,\omega}(fg)(t) = D_{q,\omega}\big(f(t)\big)g(t) + f(qt+\omega)D_{q,\omega}g(t), \tag{1.5}$$

$$D_{q,\omega}(f/g)(t) = \frac{D_{q,\omega}(f(t))g(t) - f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)} \tag{1.6}$$

provided that in (1.6), $g(t)g(qt+\omega) \neq 0$ [1, 2]. Also, for $n \in \mathbb{N}$, the following relations hold:

$$D_{q,\omega}(\alpha t+\beta)^n = \alpha \sum_{k=0}^{n-1} \big(\alpha(qt+\omega)+\beta\big)^k (\alpha t+\beta)^{n-k-1}, \tag{1.7}$$

$$D_{q,\omega}(\alpha t+\beta)^{-n} = -\alpha \sum_{k=0}^{n-1} \big(\alpha(qt+\omega)+\beta\big)^{-n+k} (\alpha t+\beta)^{-k-1}, \tag{1.8}$$

where $\alpha, \beta \in \mathbb{R}$, see [1, 2].

The q-shifted factorial $(b;q)_n$ for a complex number b and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is defined to be

$$(b;q)_n = \begin{cases} \prod_{j=1}^{n}(1-bq^{j-1}), & \text{if } n \in \mathbb{N}, \\ 1, & \text{if } n = 0. \end{cases}$$

The limit $\lim_{n\to\infty}(b;q)_n$ is denoted by $(b;q)_\infty$. Moreover $(b;q)_n$ has the representation [9]

$$(b;q)_n = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix}_q q^{\frac{k(k-1)}{2}} b^k. \tag{1.9}$$

The q-binomial coefficients [9]

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

satisfy the following property:

$$\begin{pmatrix} n+1 \\ k \end{pmatrix}_q = \begin{pmatrix} n \\ k \end{pmatrix}_q q^k + \begin{pmatrix} n \\ k-1 \end{pmatrix}_q = \begin{pmatrix} n \\ k \end{pmatrix}_q + \begin{pmatrix} n \\ k-1 \end{pmatrix}_q q^{n+1-k}. \tag{1.10}$$

For $n \in \mathbb{N}_0$ and $0 < q < 1$, the q-analogues of the natural numbers of the factorial function and of the semifactorial function [7, 13] are defined by

$$[n]_q = \frac{1-q^n}{1-q}, \quad n \in \mathbb{N}_0, 0 < q < 1, \tag{1.11}$$

and

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \qquad [0]_q! := 1, \quad 0 < q < 1. \tag{1.12}$$

$[x-a]_n$ is defined by

$$[x-a]_n = (x-a)(x-aq)\big(x-aq^2\big)\cdots\big(x-aq^{n-1}\big), \quad n \geq 1, \qquad [x-a]_0 = 1. \tag{1.13}$$

The following formula was obtained by Euler [8]:

$$[x-a]_n = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} x^{n-k}(-a)^k. \tag{1.14}$$

The q-gamma function [9] is defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad 0 < q < 1,$$

where $z \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}_0\}$. Here, we take the principal values of q^z and $(1-q)^{1-z}$. In particular

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

It is known that, for $x > 0$, $\Gamma_q(x)$ is the unique logarithmically convex function that satisfies the functional equation:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \qquad \Gamma_q(1) = 1.$$

In [1], Aldowah introduced the q,ω-integral of f from a to b as follows.

Definition 1.1 Let I be any interval of $\mathbb{R}$ containing ω_0. Assume that $f : I \to \mathbb{R}$ is a function, and let $a, b \in I$ such that $a < b$. The q,ω-integral of f from a to b is defined by

$$\int_a^b f(t)\, d_{q,\omega}t := \int_{\omega_0}^b f(t)\, d_{q,\omega}t - \int_{\omega_0}^a f(t)\, d_{q,\omega}t, \tag{1.15}$$

where

$$\int_{\omega_0}^x f(t)\, d_{q,\omega}t := \big(x(1-q)-\omega\big) \sum_{k=0}^{\infty} q^k f\big(xq^k + \omega[k]_q\big), \quad x \in I, \tag{1.16}$$

provided that the series converges at $x = a$ and $x = b$. In this case f is called q,ω-integrable over $[a,b]$ for all $a, b \in I$.

Lemma 1.2 ([1, 2]) *Let $f, g : I \to \mathbb{R}$ be q,ω-integrable on I, $k \in \mathbb{R}$ and $a, b, c \in I$, $a < c < b$. Then*

(i) $\int_a^a f(t)\,d_{q,\omega}t = 0$,
(ii) $\int_a^b kf(t)\,d_{q,\omega}t = k\int_a^b f(t)\,d_{q,\omega}t$,
(iii) $\int_a^b f(t)\,d_{q,\omega}t = -\int_b^a f(t)\,d_{q,\omega}t$,
(iv) $\int_a^b f(t)\,d_{q,\omega}t = \int_a^c f(t)\,d_{q,\omega}t + \int_c^b f(t)\,d_{q,\omega}t$,
(v) $\int_a^b (f(t)+g(t))\,d_{q,\omega}t = \int_a^b f(t)\,d_{q,\omega}t + \int_a^b g(t)\,d_{q,\omega}t$.

Lemma 1.3 ([1, 2]) *If $f : I \to \mathbb{R}$ is continuous at ω_0, then $\{f(sq^k + \omega[k]_q)\}_{k\in\mathbb{N}}$ converges uniformly to $f(\omega_0)$ on I.*

Corollary 1.4 ([1, 2]) *If $f : I \to \mathbb{R}$ is continuous at ω_0, then $\sum_{k=0}^{\infty} |f((sq^k)+\omega[k]_q)|$ converges uniformly on I, and consequently f is q,ω-integrable over I.*

Lemma 1.5 ([1, 2]) *If $f, g : I \to \mathbb{R}$ are continuous at ω_0, then*

$$\int_a^b f(t)D_{q,\omega}\bigl(g(t)\bigr)\,d_{q,\omega}t = f(t)g(t)|_a^b - \int_a^b D_{q,\omega}\bigl(f(t)\bigr)g(qt+\omega)\,d_{q,\omega}t, \quad a,b \in I. \tag{1.17}$$

Theorem 1.6 ([1, 2]) *Assume that $f : I \to \mathbb{R}$ is continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t)\,d_{q,\omega}t.$$

Then F is continuous at ω_0. Furthermore, $D_{q,\omega}F(x)$ exists for every $x \in I$ and $D_{q,\omega}F(x) = f(x)$. Conversely,

$$\int_a^b D_{q,\omega}f(t)\,d_{q,\omega}t = f(b) - f(a), \quad a,b \in I.$$

2 Main results

We define the q,ω-derivative of higher order in the usual way. That is, the nth q,ω-derivative, $n \in \mathbb{N}$, of $f : I \to \mathbb{R}$ is the function $D^n_{q,\omega}f : I \to \mathbb{R}$ given by $D^n_{q,\omega}f := D_{q,\omega}(D^{n-1}_{q,\omega}f)$, provided $D^{n-1}_{q,\omega}f$ is q,ω-differentiable on I and $D^0_{q,\omega}f = f$. We consider the following linear spaces:

$$\begin{aligned} C^n &= C^n(I,\mathbb{R}) \\ &:= \bigl\{f : I \to \mathbb{R} \mid f \text{ is differentiable } n\text{-times and } f^{(i)} \text{ are continuous}, i = 1,2,\ldots,n\bigr\}, \\ C^n_{q,\omega} &= C^n_{q,\omega}(I,\mathbb{R}) \\ &:= \bigl\{f : I \to \mathbb{R} \mid f \text{ is } q,\omega\text{-differentiable } n\text{-times and } D^n_{q,\omega}f \text{ is continuous at } \omega_0\bigr\}, \end{aligned}$$

and

$$\begin{aligned} C^\infty_{q,\omega} &= C^\infty_{q,\omega}(I,\mathbb{R}) \\ &:= \{f : I \to \mathbb{R} \mid f \text{ is } q,\omega\text{-differentiable infinitely many times at } \omega_0\}. \end{aligned}$$

Our target is to obtain Taylor expansion of a function f defined on an interval I that contains ω_0 associated with Hahn difference operator. We need the following lemmas in proving our main results.

Lemma 2.1 *Let f be a function defined on I. Then, for $x \neq \omega_0$, the nth q,ω derivative $(D^n_{q,\omega}f)(x)$ can be expressed as*

$$\bigl(D^n_{q,\omega}f\bigr)(x) = \bigl(x(q-1)+\omega\bigr)^{-n} q^{-\frac{n(n-1)}{2}} \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} f\bigl(xq^{n-k}+\omega[n-k]_q\bigr). \quad (2.1)$$

Proof For $n = 1$, the formula above yields (1.1). Assume that formula (2.1) is true for $n = m$. By relations (1.5), (1.8), and (1.10), we have

$$\begin{aligned}
\bigl(D^{m+1}_{q,\omega}f\bigr)(x) &= D_{q,\omega}\Biggl[\bigl(x(q-1)+\omega\bigr)^{-m} q^{-\frac{m(m-1)}{2}} \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} \\
&\quad \times f\bigl(xq^{m-k}+\omega[m-k]_q\bigr)\Biggr] \\
&= -(q-1)\sum_{j=0}^{m-1}\bigl((qx+\omega)(q-1)+\omega\bigr)^{-m+j}\bigl(x(q-1)+\omega\bigr)^{-j-1} \\
&\quad \times q^{-\frac{m(m-1)}{2}} \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} f\bigl(xq^{m-k}+\omega[m-k]_q\bigr) \\
&\quad + \bigl((qx+\omega)(q-1)+\omega\bigr)^{-m} q^{-\frac{m(m-1)}{2}} \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} \\
&\quad \times D_{q,\omega} f\bigl(xq^{m-k}+\omega[m-k]_q\bigr) \\
&= q^{-\frac{m(m-1)}{2}} q^{-m}\Biggl[-(q-1)\sum_{j=0}^{m-1} q^j\bigl(x(q-1)+\omega\bigr)^{-m-1} \\
&\quad \times \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} f\bigl(xq^{m-k}+\omega[m-k]_q\bigr) \\
&\quad + \bigl(x(q-1)+\omega\bigr)^{-m-1} \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} \\
&\quad \times \Bigl(f\bigl(xq^{m-k+1}+\omega[m-k+1]_q\bigr) - f\bigl(xq^{m-k}+\omega[m-k]_q\bigr)\Bigr)\Biggr].
\end{aligned}$$

This implies that

$$\begin{aligned}
\bigl(D^{m+1}_{q,\omega}f\bigr)(x) &= q^{-\frac{m(m-1)}{2}} q^{-m}\bigl(x(q-1)+\omega\bigr)^{-m-1}\Biggl[-(q-1)\sum_{j=0}^{m-1} q^j \\
&\quad \times \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} f\bigl(xq^{m-k}+\omega[m-k]_q\bigr) \\
&\quad + \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} \Bigl(f\bigl(xq^{m-k+1}+\omega[m-k+1]_q\bigr) \\
&\quad - f\bigl(xq^{m-k}+\omega[m-k]_q\bigr)\Bigr)\Biggr]
\end{aligned}$$

$$
\begin{aligned}
&= q^{-\frac{m(m+1)}{2}}\big(x(q-1)+\omega\big)^{-m-1}\Bigg[-(q-1)\frac{q^m-1}{q-1}\sum_{k=0}^{m}\binom{m}{k}_q \\
&\quad \times (-1)^k q^{\frac{k(k-1)}{2}} f\big(xq^{m-k}+\omega[m-k]_q\big)+\sum_{k=0}^{m}\binom{m}{k}_q(-1)^k \\
&\quad \times q^{\frac{k(k-1)}{2}}\Big(f\big(xq^{m-k+1}+\omega[m-k+1]_q\big)-f\big(xq^{m-k}+\omega[m-k]_q\big)\Big)\Bigg] \\
&= q^{-\frac{m(m+1)}{2}}\big(x(q-1)+\omega\big)^{-m-1}\Bigg[-q^m\sum_{k=0}^{m}\binom{m}{k}_q(-1)^k q^{\frac{k(k-1)}{2}} \\
&\quad \times f\big(xq^{m-k}+\omega[m-k]_q\big)+\sum_{k=0}^{m}\binom{m}{k}_q(-1)^k q^{\frac{k(k-1)}{2}} \\
&\quad \times f\big(xq^{m-k+1}+\omega[m-k+1]_q\big)\Bigg] \\
&= q^{-\frac{m(m+1)}{2}}\big(x(q-1)+\omega\big)^{-m-1}\Bigg[-q^m\sum_{k=1}^{m+1}\binom{m}{k-1}_q(-1)^{k-1} \\
&\quad \times q^{\frac{(k-1)(k-2)}{2}} f\big(xq^{m-k+1}+\omega[m-k+1]_q\big) \\
&\quad +\sum_{k=0}^{m}\binom{m}{k}_q(-1)^k q^{\frac{k(k-1)}{2}} f\big(xq^{m-k+1}+\omega[m-k+1]_q\big)\Bigg] \\
&= q^{-\frac{m(m+1)}{2}}\big(x(q-1)+\omega\big)^{-m-1}\Bigg[\sum_{k=1}^{m+1}\binom{m}{k-1}_q q^{m-k+1}(-1)^k \\
&\quad \times q^{\frac{k(k-1)}{2}} f\big(xq^{m-k+1}+\omega[m-k+1]_q\big) \\
&\quad +\sum_{k=0}^{m}\binom{m}{k}_q(-1)^k q^{\frac{k(k-1)}{2}} f\big(xq^{m-k+1}+\omega[m-k+1]_q\big)\Bigg] \\
&= q^{-\frac{m(m+1)}{2}}\big(x(q-1)+\omega\big)^{-m-1}\Bigg[(-1)^{m+1}q^{\frac{m(m+1)}{2}} f(x) \\
&\quad +\sum_{k=1}^{m}\left(\binom{m}{k-1}_q q^{m-k+1}+\binom{m}{k}_q\right) \\
&\quad \times (-1)^k q^{\frac{k(k-1)}{2}} f\big(xq^{m-k+1}+\omega[m-k+1]_q\big) \\
&\quad + f\big(xq^{m+1}+\omega[m+1]_q\big)\Bigg].
\end{aligned}
$$

That is,

$$
\begin{aligned}
\big(D_{q,\omega}^{m+1}f\big)(x) &= q^{-\frac{m(m+1)}{2}}\big(x(q-1)+\omega\big)^{-m-1}\Bigg[(-1)^{m+1}q^{\frac{m(m+1)}{2}} f(x) \\
&\quad +\sum_{k=1}^{m}\binom{m+1}{k}_q(-1)^k q^{\frac{k(k-1)}{2}} f\big(xq^{m-k+1}+\omega[m-k+1]_q\big)
\end{aligned}
$$

$$+f\left(xq^{m+1}+\omega[m+1]_q\right)\Bigg]$$
$$=q^{-\frac{m(m+1)}{2}}\left(x(q-1)+\omega\right)^{-m-1}\sum_{k=0}^{m+1}\left[\binom{m+1}{k}_q(-1)^k q^{\frac{k(k-1)}{2}}\right.$$
$$\left.\times f\left(xq^{m-k+1}+\omega[m-k+1]_q\right)\right].$$

Therefore relation (2.1) is true at $n=m+1$ and by induction it is true for every $n\in\mathbb{N}$. □

In the following result, a formula of the nth derivative of a power series of center zero is given.

Lemma 2.2 *Assume that a function f has the power series expansion $f(x)=\sum_{k=0}^{\infty}a_k x^k$, $x\in I$. Then*

$$\left(D^n_{q,\omega}f\right)(x)=(1-q)^{-n}\sum_{k=0}^{\infty}\frac{a_{n+k}}{(1-q)^k}\sum_{m=0}^{k}(-1)^m\binom{n+k}{n+m}$$
$$\times\left(x(q-1)+\omega\right)^m(\omega)^{k-m}\left(q^{m+1};q\right)_n,\quad x\neq\omega_0, n\in\mathbb{N}_0. \tag{2.2}$$

Proof It is clear that Eq. (2.2) is true for $n=0$. From Eq. (2.1) and relation (1.9), we have, for $n\in\mathbb{N}$,

$$\left(D^n_{q,\omega}f\right)(x)=\left(x(q-1)+\omega\right)^{-n}q^{-\frac{n(n-1)}{2}}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\frac{k(k-1)}{2}}$$
$$\times\sum_{j=0}^{\infty}a_j\left(xq^{n-k}+\omega[n-k]_q\right)^j$$
$$=\left(x(q-1)+\omega\right)^{-n}q^{-\frac{n(n-1)}{2}}\sum_{j=0}^{\infty}\frac{a_j}{(1-q)^j}\sum_{r=0}^{j}(-1)^r\binom{j}{r}q^{nr}$$
$$\times\left(x(q-1)+\omega\right)^r(\omega)^{j-r}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\frac{k(k-1)}{2}}q^{-kr}$$
$$=\left(x(q-1)+\omega\right)^{-n}q^{-\frac{n(n-1)}{2}}\sum_{j=0}^{\infty}\frac{a_j}{(1-q)^j}\sum_{r=0}^{j}(-1)^r\binom{j}{r}q^{nr}$$
$$\times\left(x(q-1)+\omega\right)^r(\omega)^{j-r}\left(q^{-r};q\right)_n.$$

Then

$$\left(D^n_{q,\omega}f\right)(x)=(-1)^n\left(x(q-1)+\omega\right)^{-n}q^{-\frac{n(n-1)}{2}}\sum_{j=n}^{\infty}\frac{a_j}{(1-q)^j}\sum_{r=n}^{j}(-1)^r q^{nr}\binom{j}{r}$$
$$\times\left(x(q-1)+\omega\right)^r(\omega)^{j-r}q^{-rn+\frac{n(n-1)}{2}}\left(q^{r-n+1};q\right)_n$$
$$=(-1)^n\left(x(q-1)+\omega\right)^{-n}\sum_{j=n}^{\infty}\frac{a_j}{(1-q)^j}\sum_{r=n}^{j}(-1)^r\binom{j}{r}$$

$$
\begin{aligned}
&\times \bigl(x(q-1)+\omega\bigr)^{r}(\omega)^{j-r}\bigl(q^{r-n+1};q\bigr)_{n}\\
&= (-1)^{n}\bigl(x(q-1)+\omega\bigr)^{-n}\sum_{k=0}^{\infty}\frac{a_{n+k}}{(1-q)^{n+k}}\sum_{r=n}^{n+k}(-1)^{r}\begin{pmatrix} n+k\\ r\end{pmatrix}\\
&\times \bigl(x(q-1)+\omega\bigr)^{r}(\omega)^{n+k-r}\bigl(q^{r-n+1};q\bigr)_{n}\\
&= (-1)^{n}\bigl(x(q-1)+\omega\bigr)^{-n}\sum_{k=0}^{\infty}\frac{a_{n+k}}{(1-q)^{n+k}}\sum_{m=0}^{k}(-1)^{n+m}\begin{pmatrix} n+k\\ n+m\end{pmatrix}\\
&\times \bigl(x(q-1)+\omega\bigr)^{n+m}(\omega)^{k-m}\bigl(q^{m+1};q\bigr)_{n}\\
&= (1-q)^{-n}\sum_{k=0}^{\infty}\frac{a_{n+k}}{(1-q)^{k}}\sum_{m=0}^{k}(-1)^{m}\begin{pmatrix} n+k\\ n+m\end{pmatrix}\bigl(x(q-1)+\omega\bigr)^{m}\\
&\times (\omega)^{k-m}\bigl(q^{m+1};q\bigr)_{n}\\
&= (1-q)^{-n}\sum_{k=0}^{\infty}\frac{a_{n+k}}{(1-q)^{k}}\Biggl[(-1)^{k}\bigl(x(q-1)+\omega\bigr)^{k}\bigl(q^{k+1};q\bigr)_{n}\\
&+\sum_{m=0}^{k-1}(-1)^{m}\begin{pmatrix} n+k\\ n+m\end{pmatrix}\bigl(x(q-1)+\omega\bigr)^{m}(\omega)^{k-m}\bigl(q^{m+1};q\bigr)_{n}\Biggr]. \qquad \square
\end{aligned}
$$

The following result includes a useful formula for the nth derivative of a power series of center ω_0.

Lemma 2.3 *Assume that a function f has the power series expansion $f(x)=\sum_{k=0}^{\infty}a_k(x-\omega_0)^k$, $x\in I$. Then*

$$
D_{q,\omega}^{n}f(x)=\bigl(x(1-q)-\omega\bigr)^{-n}\sum_{k=0}^{\infty}a_{n+k}(x-\omega_0)^{n+k}\bigl(q^{k+1};q\bigr)_{n},\quad x\neq\omega_0. \tag{2.3}
$$

Proof It is clear that Eq. (2.3) is true for $n=0$. From Eq. (2.1) and relation (1.9), we have, for $n\in\mathbb{N}$,

$$
\begin{aligned}
\bigl(D_{q,\omega}^{n}f\bigr)(x) &= \bigl(x(q-1)+\omega\bigr)^{-n}q^{-\frac{n(n-1)}{2}}\sum_{k=0}^{n}\Biggl[\begin{pmatrix} n\\ k\end{pmatrix}_{q}(-1)^{k}q^{\frac{k(k-1)}{2}}\\
&\times\sum_{j=0}^{\infty}a_j\bigl(xq^{n-k}+\omega[n-k]_q-\omega_0\bigr)^{j}\Biggr].
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
\bigl(D_{q,\omega}^{n}f\bigr)(x) &= \bigl(x(q-1)+\omega\bigr)^{-n}q^{-\frac{n(n-1)}{2}}\sum_{k=0}^{n}\Biggl[\begin{pmatrix} n\\ k\end{pmatrix}_{q}(-1)^{k}q^{\frac{k(k-1)}{2}}\\
&\times\sum_{j=0}^{\infty}a_jq^{nj-kj}(x-\omega_0)^{j}\Biggr]
\end{aligned}
$$

$$
\begin{aligned}
&= \bigl(x(q-1)+\omega\bigr)^{-n} q^{-\frac{n(n-1)}{2}} \sum_{j=0}^{\infty} \Biggl[a_j q^{nj} (x-\omega_0)^j \\
&\quad \times \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} q^{-kj} \Biggr] \\
&= \bigl(x(q-1)+\omega\bigr)^{-n} q^{-\frac{n(n-1)}{2}} \sum_{j=0}^{\infty} a_j q^{nj} (x-\omega_0)^j \bigl(q^{-j};q\bigr)_n \\
&= \bigl(x(q-1)+\omega\bigr)^{-n} q^{-\frac{n(n-1)}{2}} \sum_{j=n}^{\infty} \bigl[a_j q^{nj} (x-\omega_0)^j (-1)^n q^{-nj+\frac{n(n-1)}{2}} \\
&\quad \times \bigl(q^{j-n+1};q\bigr)_n \bigr] \\
&= \bigl(x(1-q)-\omega\bigr)^{-n} \sum_{k=0}^{\infty} a_{n+k} (x-\omega_0)^{n+k} \bigl(q^{k+1};q\bigr)_n. \qquad \square
\end{aligned}
$$

One of the important questions: Is there a relation between the nth q,ω derivative and the usual nth derivative? The answer is in the following lemma.

Lemma 2.4 *If $f \in C^{n+1}$, then*

(i) *$D_{q,\omega}^m f$ exists on I and is continuous at ω_0 for all $m = 1, 2, \dots, n+1$;*

(ii) *for $1 \le m \le n+1$,*

$$D_{q,\omega}^m f(\omega_0) = \frac{[m]_q!}{m!} f^{(m)}(\omega_0), \tag{2.4}$$

where $f^{(m)}$ is the usual mth derivative of f.

Proof The proof is by induction. The q,ω derivative $D_{q,\omega} f$ exists and $D_{q,\omega} f(\omega_0) = f'(\omega_0)$. Also $D_{q,\omega} f$ is continuous at ω_0. Indeed,

$$\lim_{x\to\omega_0} D_{q,\omega} f(x) = \lim_{t\to\omega_0} \frac{f(qx+\omega)-f(x)}{x(q-1)+\omega} = f'(\omega_0) = D_{q,\omega} f(\omega_0).$$

Now, we assume that (i) and (ii) hold for all $m = 1, 2, \dots, l$, where $l \le n$ and we want to prove that they are true at $m = l+1$. By Lemma 2.1, we conclude that

$$
\begin{aligned}
\lim_{x\to\omega_0} D_{q,\omega}^{l+1} f(x) &= \lim_{x\to\omega_0} \frac{1}{(x(q-1)+\omega)^{l+1} q^{\frac{l(l+1)}{2}}} \Biggl[\sum_{k=0}^{l+1} \begin{pmatrix} l+1 \\ k \end{pmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} \\
&\quad \times f\bigl(xq^{l-k+1} + \omega[l-k+1]_q\bigr) \Biggr] \\
&= \lim_{x\to\omega_0} \sum_{k=0}^{l+1} \Biggl[\frac{\binom{l+1}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} q^{(l+1)(l-k+1)}}{(q-1)^{l+1} (xq^{l-k+1} + \omega[l-k+1]_q - \omega_0)^{l+1} q^{\frac{l(l+1)}{2}}} \\
&\quad \times f\bigl(xq^{l-k+1} + \omega[l-k+1]_q\bigr) \Biggr].
\end{aligned}
$$

Applying L'Hopital rule $l+1$ times and using relations (1.12), (1.13), and (1.14), we get

$$\begin{aligned}
\lim_{x\to\omega_0} D_{q,\omega}^{l+1} f(x) &= \lim_{x\to\omega_0} \frac{1}{(q-1)^{l+1}(l+1)!q^{\frac{l(l+1)}{2}}} \sum_{k=0}^{l+1} \left[\begin{pmatrix} l+1 \\ k \end{pmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} \right. \\
&\quad \left. \times q^{(l+1)(l-k+1)} f^{(l+1)}\bigl(xq^{l-k+1} + \omega[l-k+1]_q\bigr) \right] \\
&= \frac{\sum_{k=0}^{l+1} \binom{l+1}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} (q^{l+1})^{l-k+1} f^{(l+1)}(\omega_0)}{(q-1)^{l+1}(l+1)!q^{\frac{l(l+1)}{2}}} \\
&= \frac{[q^{l+1}-1]_{l+1} f^{(l+1)}(\omega_0)}{(q-1)^{l+1}(l+1)!q^{\frac{l(l+1)}{2}}} \\
&= \frac{(q^{l+1}-1)(q^{l+1}-q)(q^{l+1}-q^2)\cdots(q^{l+1}-q^l) f^{(l+1)}(\omega_0)}{(q-1)^{l+1}(l+1)!q^{0+1+2+\cdots+(l-1)+l}} \\
&= \frac{(q^{l+1}-1)(q^l-1)(q^{l-1}-1)\cdots(q-1) f^{(l+1)}(\omega_0)}{(q-1)^{l+1}(l+1)!} \\
&= \frac{[1]_q[2]_q\cdots[l]_q[l+1]_q f^{(l+1)}(\omega_0)}{(l+1)!} \\
&= \frac{[l+1]_q!}{(l+1)!} f^{(l+1)}(\omega_0).
\end{aligned}$$

On the other hand, we conclude that

$$\begin{aligned}
D_{q,\omega}^{l+1} f(\omega_0) &= \lim_{x\to\omega_0} \frac{D_{q,\omega}^{l} f(x) - D_{q,\omega}^{l} f(\omega_0)}{x-\omega_0} \\
&= \lim_{x\to\omega_0} \frac{d}{dx} \left[\frac{\sum_{k=0}^{l} \binom{l}{k}_q (-1)^k q^{\frac{k(k-1)}{2}} f(xq^{l-k} + \omega[l-k]_q)}{(x(q-1)+\omega)^l q^{\frac{l(l-1)}{2}}} \right] \\
&= \lim_{x\to\omega_0} \frac{1}{(x(q-1)+\omega)^{l+1} q^{\frac{l(l-1)}{2}}} \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} \\
&\quad \times \bigl[\bigl(x(q-1)+\omega\bigr) q^{l-k} f'\bigl(xq^{l-k} + \omega[l-k]_q\bigr) - l(q-1) f\bigl(xq^{l-k} + \omega[l-k]_q\bigr) \bigr].
\end{aligned}$$

Again, applying L'Hopital rule $l+1$ times and using relations (1.12), (1.13), and (1.14), we get

$$\begin{aligned}
D_{q,\omega}^{l+1} f(\omega_0) &= \lim_{x\to\omega_0} \frac{1}{(q-1)^{l+1}(l+1)!q^{\frac{l(l-1)}{2}}} \sum_{k=0}^{l} \left[\begin{pmatrix} l \\ k \end{pmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} q^{(l+1)(l-k)} \right. \\
&\quad \left. \times (q-1) f^{(l+1)}\bigl(xq^{l-k} + \omega[l-k]_q\bigr) \right] \\
&= \frac{[q^{l+1}-1]_l (q-1) f^{(l+1)}(\omega_0)}{(q-1)^{l+1}(l+1)!q^{\frac{l(l-1)}{2}}} \\
&= \frac{[l+1]_q!}{(l+1)!} f^{(l+1)}(\omega_0).
\end{aligned}$$

Therefore,

$$\lim_{x\to\omega_0} D_{q,\omega}^{l+1} f(x) = D_{q,\omega}^{l+1} f(\omega_0) = \frac{[l+1]_q!}{(l+1)!} f^{(l+1)}(\omega_0). \qquad \square$$

Corollary 2.5 *Assume that f has the power series expansion*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-\omega_0)^n, \quad x \in I.$$

Then

$$a_n = \frac{D_{q,\omega}^n f(\omega_0)}{[n]_q!}, \quad n \in \mathbb{N}. \tag{2.5}$$

Proof By Lemma 2.4, we have

$$a_n = \frac{f^{(n)}(\omega_0)}{n!} = \frac{D_{q,\omega}^n f(\omega_0)}{[n]_q!}. \qquad \square$$

Now we define the two variable polynomials $H_n(x,t)$, $x, t \in I$, to be

$$H_0(x,t) := 1, \qquad H_n(x,t) := \prod_{j=0}^{n-1} \bigl(x - h^j(t)\bigr), \tag{2.6}$$

where $h^j(t) = tq^j + \omega[j]_q, t \in I$ is the jth order iteration of $h(t) = qt + \omega$, which uniformly converges to ω_0 on I.

Lemma 2.6 *For $n \in \mathbb{N}$ and $x, t \in I$, we have*

$$_tD_{q,\omega} H_n(x,t) = -[n]_q H_{n-1}\bigl(x, h(t)\bigr), \tag{2.7}$$

$$_xD_{q,\omega} H_n(x,t) = [n]_q H_{n-1}(x,t), \tag{2.8}$$

where $_tD_{q,\omega}$ is the q,ω-derivative with respect to t,

$$I_{q,\omega}^n(1) = \frac{H_n(x,a)}{\Gamma_q(n+1)},$$

where $I_{q,\omega}^n$ is the q,ω-integral

$$I_{q,\omega}^n f(x) := \int_a^x \int_a^{x_{n-1}} \int_a^{x_{n-2}} \cdots \int_a^{x_1} f(s)\, d_{q,\omega}s\, d_{q,\omega}x_1 \cdots d_{q,\omega}x_{n-2}\, d_{q,\omega}x_{n-1}.$$

Now, we establish Taylor's theorem based on Hahn difference operator.

Theorem 2.7 *Let f be a function defined on I. If $f \in C_{q,\omega}^n$ for some $n \in \mathbb{N}$, then for $x, a \in I$,*

$$f(x) = \sum_{k=0}^{n-1} \frac{D_{q,\omega}^k f(a)}{[k]_q!} H_k(x,a) + R_n(x,a), \tag{2.9}$$

where

$$R_n(x,a) = \int_a^x \frac{D^n_{q,\omega} f(t)}{[n-1]_q!} H_{n-1}\big(x,h(t)\big)\, d_{q,\omega}t. \tag{2.10}$$

Proof We prove relation (2.9) by induction. The right-hand side (R.H.S) of (2.9) at $n = 1$ is

$$\begin{aligned} R.H.S &= f(a)H_0(x,a) + R_1(x,a) \\ &= f(a) + \int_a^x D_{q,\omega} f(t)\, d_{q,\omega}t = f(x). \end{aligned}$$

Assume that relation (2.9) is true for $n = m$, that is,

$$f(x) = \sum_{k=0}^{m-1} \frac{D^k_{q,\omega} f(a)}{[k]_q!} H_k(x,a) + R_m(x,a),$$

where $R_m(x,a) = \int_a^x \frac{D^m_{q,\omega} f(t)}{[m-1]_q!} H_{m-1}(x,h(t))\, d_{q,\omega}t$. We integrate by parts in the remainder term $R_m(x,a)$. We obtain

$$\begin{aligned} R_m(x,a) &= \int_a^x \frac{D^m_{q,\omega} f(t)}{[m-1]_q!} H_{m-1}\big(x,h(t)\big)\, d_{q,\omega}t \\ &= -\int_a^x \frac{D^m_{q,\omega} f(t)}{[m-1]_q!} \frac{{}_tD_{q,\omega} H_m(x,t)}{[m]_q}\, d_{q,\omega}t \\ &= -\frac{D^m_{q,\omega} f(t)}{[m]_q!} H_m(x,t)\Big|_a^x + \int_a^x \frac{D^{m+1}_{q,\omega} f(t)}{[m]_q!} H_m\big(x,h(t)\big)\, d_{q,\omega}t \\ &= D^m_{q,\omega} f(a) \frac{H_m(x,a)}{[m]_q!} + R_{m+1}(x,a). \end{aligned}$$

Then

$$f(x) = \sum_{k=0}^{m} \frac{D^k_{q,\omega} f(a)}{[k]_q!} H_k(x,a) + R_{m+1}(x,a).$$

Therefore, relation (2.9) is true for $n = m + 1$, then it is true for every $n \in \mathbb{N}$. □

As a direct consequence of the previous theorem, we deduce the following theorem.

Theorem 2.8 *Let* $f \in C^\infty_{q,\omega}$. *If for* $x, a \in I$, $\lim_{n\to\infty} R_n(x,a) = 0$, *then* $f(x)$ *has the following expansion*:

$$f(x) = \sum_{k=0}^{\infty} \frac{D^k_{q,\omega} f(a)}{[k]_q!} H_k(x,a). \tag{2.11}$$

Furthermore, if $\lim_{n\to\infty} R_n(x,a) = 0$ *uniformly with respect to* x *in some subinterval of* I, *then the series given by* (2.11) *is uniformly convergent in this subinterval.*

Corollary 2.9 *Let* $f \in C^{\infty}_{q,\omega}$. *If for* $x \in I$, $\lim_{n\to\infty} R_n(x,\omega_0) = 0$, *then* $f(x)$ *has the following expansion*:

$$f(x) = \sum_{k=0}^{\infty} \frac{D^k_{q,\omega} f(\omega_0)}{[k]_q!}(x-\omega_0)^k.$$

Theorem 2.10 *Let* $f \in C^{\infty}_{q,\omega}$. *Assume that there is a nonnegative sequence* $\{M_n\}$ *such that*
(i) $|D^n_{q,\omega} f(h^m(y))| \le CM_n$, $n, m \in \mathbb{N}_0$, $y \in I$, *for some* $C > 0$;
(ii) $\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = M$ *exists*.
Then f has the q,ω-*Taylor expansion*

$$f(x) = \sum_{k=0}^{\infty} \frac{D^k_{q,\omega} f(a)}{[k]_q!} H_k(x,a) \tag{2.12}$$

for every $x \in (\omega_0 - \frac{1}{M(1-q)}, \omega_0 + \frac{1}{M(1-q)})$ *when* $M > 0$ (*respectively* $x \in I$ *when* $M = 0$).

Proof We can write $R_n(x,a)$ as follows:

$$R_n(x,a) = R_{1,n}(x,\omega_0) - R_{2,n}(x;a,\omega_0),$$

where

$$R_{1,n}(x,\omega_0) := \frac{1}{\Gamma_q(n)} \int_{\omega_0}^{x} H_{n-1}\bigl(x,h(t)\bigr) D^n_{q,\omega} f(t)\, d_{q,\omega}t$$

and

$$R_{2,n}(x;a,\omega_0) := \frac{1}{\Gamma_q(n)} \int_{\omega_0}^{a} H_{n-1}\bigl(x,h(t)\bigr) D^n_{q,\omega} f(t)\, d_{q,\omega}t.$$

From (1.16), we have

$$\begin{aligned} R_{1,n}(x,\omega_0) &= \bigl(x(1-q)-\omega\bigr) \sum_{m=0}^{\infty} q^m \frac{1}{\Gamma_q(n)} H_{n-1}\bigl(x,h^{m+1}(x)\bigr) D^n_{q,\omega} f\bigl(h^m(x)\bigr) \\ &= \frac{1}{\Gamma_q(n)}\bigl(x(1-q)-\omega\bigr) \sum_{m=0}^{\infty} \Biggl[q^m \prod_{r=0}^{n-2} \bigl(x - \bigl[xq^{m+1+r} + [m+1+r]_q \omega\bigr]\bigr) \\ &\quad \times D^n_{q,\omega} f\bigl(h^m(x)\bigr) \Biggr] \\ &= \frac{(1-q)(x-\omega_0)}{[n-1]_q!} \sum_{m=0}^{\infty} q^m (x-\omega_0)^{n-1} \prod_{r=0}^{n-2} \bigl(1-q^{m+r+1}\bigr) D^n_{q,\omega} f\bigl(h^m(x)\bigr) \\ &= \frac{(1-q)(x-\omega_0)^n}{[n-1]_q!} \sum_{m=0}^{\infty} q^m \bigl(q^{m+1};q\bigr)_{n-1} D^n_{q,\omega} f\bigl(h^m(x)\bigr) \\ &= \frac{(1-q)^n(x-\omega_0)^n}{(q;q)_{n-1}} \sum_{m=0}^{\infty} q^m \bigl(q^{m+1};q\bigr)_{n-1} D^n_{q,\omega} f\bigl(h^m(x)\bigr). \end{aligned}$$

Consequently,

$$\begin{aligned}\bigl|R_{1,n}(x,\omega_0)\bigr| &\le \frac{C}{(q;q)_\infty} M_n\bigl[(1-q)|x-\omega_0|\bigr]^n \sum_{m=0}^{\infty} q^m \\ &\le \frac{CM_n[(1-q)|x-\omega_0|]^n}{(q;q)_\infty(1-q)}.\end{aligned}$$

Then $\lim_{n\to\infty} R_{1,n}(x,\omega_0) = 0$, $x \in (\omega_0 - \frac{1}{M(1-q)}, \omega_0 + \frac{1}{M(1-q)})$, when $M > 0$ (respectively $x \in I$, when $M = 0$). On the other hand, for $a \in I$, we have

$$R_{2,n}(x;a,\omega_0) = \frac{(a(1-q)-\omega)}{\Gamma_q(n)} \sum_{m=0}^{\infty} q^m H_{n-1}\bigl(x, h^{m+1}(a)\bigr) D^n_{q,\omega} f\bigl(h^m(a)\bigr).$$

Simple calculations show that

$$\begin{aligned}\bigl|H_{n-1}\bigl(x, h^{m+1}(a)\bigr)\bigr| &= \left|\prod_{r=0}^{n-2}\bigl(x - h^{m+r+1}(a)\bigr)\right| \\ &\le \prod_{r=0}^{n-2}\bigl[|x-\omega_0| + q^{m+r+1}|a-\omega_0|\bigr] \\ &\le |x-\omega_0|^{n-1} e^{\sum_{r=0}^{\infty} q^{m+r+1}\frac{|a-\omega_0|}{|x-\omega_0|}} \\ &\le |x-\omega_0|^{n-1} e^{\frac{|a-\omega_0|}{(1-q)|x-\omega_0|}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\bigl|R_{2,n}(x,a,\omega_0)\bigr| &\le \frac{|x-\omega_0|^{n-1}(1-q)|a-\omega_0|}{[n-1]_q!} CM_n e^{\frac{|a-\omega_0|}{(1-q)|x-\omega_0|}} \sum_{m=0}^{\infty} q^m \\ &\le \frac{C|a-\omega_0|M_n[(1-q)|x-\omega_0|]^{n-1}}{(q,q)_\infty} e^{\frac{|a-\omega_0|}{(1-q)|x-\omega_0|}}.\end{aligned}$$

This implies that $\lim_{n\to\infty} R_{2,n}(x;a,\omega_0) = 0$, $x \in (\omega_0 - \frac{1}{M(1-q)}, \omega_0 + \frac{1}{M(1-q)})$, when $M > 0$ (respectively $x \in I$, when $M = 0$). Therefore

$$\lim_{n\to\infty} R_n(x,a) = \lim_{n\to\infty}\bigl[R_{1,n}(x,\omega_0) - R_{2,n}(x;a,\omega_0)\bigr] = 0,$$

$x \in (\omega_0 - \frac{1}{M(1-q)}, \omega_0 + \frac{1}{M(1-q)})$, when $M > 0$ (respectively $x \in I$, when $M = 0$). □

Theorem 2.11 *Assume that f has the power series expansion $f(x) = \sum_{n=0}^{\infty} a_n(x-\omega_0)^n$ with interval of convergence $I_r = (\omega_0 - r, \omega_0 + r)$, $r > 0$. Then, for any $a \in I_r$, f has the q,ω-Taylor expansion*

$$f(x) = \sum_{k=0}^{\infty} \frac{D^k_{q,\omega} f(a)}{[k]_q!} H_k(x,a), \tag{2.13}$$

in any closed subinterval $\overline{I_\alpha}$, $\alpha < r$, where the series is absolutely and uniformly convergent on $\overline{I_\alpha}$, $\alpha < r$.

Proof For $n, m \in \mathbb{N}$ and by Lemma 2.3, we get

$$\begin{aligned} D^n_{q,\omega} f\big(h^m(y)\big) &= \big(h^m(y)(1-q)-\omega\big)^{-n} \sum_{k=0}^{\infty} a_{n+k} \big(h^m(y)-\omega_0\big)^{n+k} \big(q^{k+1};q\big)_n \\ &= q^{-mn}\big(y(1-q)-\omega\big)^{-n} \sum_{k=0}^{\infty} a_{n+k} q^{mn+mk} (y-\omega_0)^{n+k} \big(q^{k+1};q\big)_n \\ &= \frac{1}{(1-q)^n} \sum_{k=0}^{\infty} a_k q^{mk} (y-\omega_0)^k \big(q^{k+1};q\big)_n. \end{aligned}$$

Consequently, for $\alpha < r$,

$$\begin{aligned} \big|D^n_{q,\omega} f\big(h^m(y)\big)\big| &\le \frac{1}{(1-q)^n} \sum_{k=0}^{\infty} \big|a_k (y-\omega_0)^k\big| q^{mk} \\ &\le \frac{1}{(1-q)^n} \sum_{k=0}^{\infty} \big|a_k \alpha^k\big| q^{mk} \\ &\le \frac{1}{(1-q)^n} C, y \in \overline{I_\alpha}, \end{aligned}$$

where $C = \sum_{k=0}^{\infty} |a_k \alpha^k|$. Then, by Theorem 2.10, f has the q,ω-Taylor expansion (2.13). □

Now, we establish some properties of the q,ω-exponential functions $e_{q,\omega}(t)$ and $E_{q,\omega}(t)$ for $t \in \mathbb{R}$, $|t-\omega_0| < \frac{1}{1-q}$, where

$$\begin{aligned} e_{q,\omega}(t) &= \frac{1}{\prod_{k=0}^{\infty}(1-q^k(t(1-q)-\omega))} \\ &= \frac{1}{((t(1-q)-\omega);q)_\infty} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} E_{q,\omega}(t) &= \prod_{k=0}^{\infty}\big(1+q^k\big(t(1-q)-\omega\big)\big) \\ &= \big(-\big(t(1-q)-\omega\big);q\big)_\infty. \end{aligned} \tag{2.15}$$

Simple calculations show that the following inequalities are true:

$$\frac{e^{-\frac{q}{1-q}}}{(1-(t(1-q)-\omega))} < e_{q,\omega}(t) < \frac{e^A}{1-(t(1-q)-\omega)}, \quad |t-\omega_0| < \frac{1}{1-q} \tag{2.16}$$

and

$$\big(1+\big(t(1-q)-\omega\big)\big)e^{-A} < E_{q,\omega}(t) < \big(1+\big(t(1-q)-\omega\big)\big)e^{\frac{q}{1-q}}, \quad |t-\omega_0| < \frac{1}{1-q}, \tag{2.17}$$

where $A = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}$.

Finally, we can prove the following power series expansions for $e_{q,\omega}$ and $E_{q,\omega}$.

Example 2.12 The exponential functions $e_{q,\omega}$ and $E_{q,\omega}$ defined in (2.14) and (2.15) have the following power series expansions of center $a \in I$:

$$e_{q,\omega}(x) = \sum_{k=0}^{\infty} \frac{e_{q,\omega}(a)}{[k]_q!} H_k(x,a), \quad |x - \omega_0| < \frac{1}{1-q} \tag{2.18}$$

and

$$E_{q,\omega}(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} E_{q,\omega}(h^k(a))}{[k]_q!} H_k(x,a), \quad x \in I, \tag{2.19}$$

and have the following power series expansions of center ω_0:

$$e_{q,\omega}(t) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} (t - \omega_0)^k \tag{2.20}$$

and

$$E_{q,\omega}(t) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{[k]_q!} (t - \omega_0)^k. \tag{2.21}$$

Furthermore, both $e_{q,\omega}$ and $E_{q,\omega}$ are continuous.

Proof For $n \in \mathbb{N}_0$, we have

$$D^n_{q,\omega} e_{q,\omega}(t) = e_{q,\omega}(t).$$

Inequality (2.16) shows that $e_{q,\omega}(t)$ is positive and bounded on every compact subinterval of $(\omega_0 - \frac{1}{1-q}, \omega_0 + \frac{1}{1-q})$. For fixed $t \in (\omega_0 - \frac{1}{1-q}, \omega_0 + \frac{1}{1-q})$, there exists $0 < \alpha \leq 1$ such that $|t(1-q) - \omega| < \alpha$, which implies that

$$\left| D^n_{q,\omega} e_{q,\omega}(t) \right| \leq \frac{e^A}{1-\alpha}, \quad n \in \mathbb{N}_0.$$

By Theorem 2.10, the q,ω-Taylor expansion of $e_{q,\omega}(t)$ at a is given by

$$e_{q,\omega}(t) = \sum_{k=0}^{\infty} \frac{e_{q,\omega}(a)}{[k]_q!} H_k(t,a). \tag{2.22}$$

Since $D^n_{q,\omega} e_{q,\omega}(\omega_0) = 1$, the q,ω-Taylor expansion of $e_{q,\omega}(t)$ at ω_0 is given by

$$e_{q,\omega}(t) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} (t - \omega_0)^k. \tag{2.23}$$

The series in (2.23) is uniformly convergent on every compact subinterval of $(\omega_0 - \frac{1}{1-q}, \omega_0 + \frac{1}{1-q})$ by Weierstrass M-test, and consequently $e_{q,\omega}(t)$ is continuous.

Let $t \in \mathbb{R}$, $|t-\omega_0| < \frac{1}{1-q}$. First, we show that

$$D^n_{q,\omega} E_{q,\omega}(t) = q^{\frac{n(n-1)}{2}} E_{q,\omega}\big(h^n(t)\big), \quad n \in \mathbb{N}_0 \tag{2.24}$$

by induction. For $n = 1$, we have

$$\begin{aligned} D_{q,\omega} E_{q,\omega}(t) &= \frac{1}{t(q-1)+\omega} \Bigg[\prod_{k=0}^{\infty} \big(1 + q^k (qt+\omega)(1-q) - \omega\big) \\ &\quad - \prod_{k=0}^{\infty} \big(1 + q^k \big(t(1-q) - \omega\big)\big) \Bigg] \\ &= \frac{\prod_{k=0}^{\infty} (1 + q^{k+1}(t(1-q)-\omega))}{t(q-1)+\omega} \big[1 - \big(1 + t(1-q) - \omega\big)\big] \\ &= E_{q,\omega}\big(h(t)\big). \end{aligned}$$

Assume that formula (2.24) is true for $n = m$. We have

$$\begin{aligned} D^{m+1}_{q,\omega} E_{q,\omega}(t) &= D_{q,\omega}\big(D^m_{q,\omega} E_{q,\omega}(t)\big) \\ &= q^{\frac{m(m-1)}{2}} D_{q,\omega} E_{q,\omega}\big(h^m(t)\big) \\ &= q^{\frac{m(m-1)}{2}} \frac{1}{t(q-1)+\omega} \Bigg[\prod_{k=0}^{\infty} \big(1 + q^{k+m+1}\big(t(1-q)-\omega\big)\big) \\ &\quad - \prod_{k=0}^{\infty} \big(1 + q^{k+m}\big(t(1-q)-\omega\big)\big) \Bigg] \\ &= q^{\frac{m(m-1)}{2}} \frac{\prod_{k=0}^{\infty} (1 + q^{k+m+1}(t(1-q)-\omega))}{t(q-1)+\omega} \\ &\quad \times \big[1 - \big(1 + q^m \big(t(1-q)-\omega\big)\big)\big] \\ &= q^{\frac{m(m+1)}{2}} \prod_{k=0}^{\infty} \big(1 + q^{k+m+1}\big(t(1-q)-\omega\big)\big) \\ &= q^{\frac{m(m+1)}{2}} E_{q,\omega}\big(h^{m+1}(t)\big). \end{aligned}$$

Inequality (2.17) shows that $E_{q,\omega}(t)$ is positive and is bounded on every compact subinterval of $(\omega_0 - \frac{1}{1-q}, \omega_0 + \frac{1}{1-q})$. Also we can see that

$$\begin{aligned} \big|E_{q,\omega}\big(h^n(t)\big)\big| &\le \prod_{k=0}^{\infty} \big|1 + q^{k+n}\big(t(1-q)-\omega\big)\big| \\ &\le \prod_{k=0}^{\infty} \big[1 + q^{k+n}(1-q)|t-\omega_0|\big] \\ &\le \prod_{k=0}^{\infty} \big[1 + q^{k+n}\big] \\ &\le e^{\frac{1}{1-q}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left|D_{q,\omega}^{n} E_{q,\omega}(t)\right| &\leq q^{\frac{n(n-1)}{2}} \left|E_{q,\omega}\left(h^{n}(t)\right)\right| \\ &\leq q^{\frac{n(n-1)}{2}} e^{\frac{1}{1-q}}. \end{aligned}$$

By Theorem 2.10, the q,ω-Taylor expansion of $E_{q,\omega}(t)$ at a is given by

$$E_{q,\omega}(t) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} E_{q,\omega}(h^{k}(a))}{[k]_{q}!} H_{k}(t,a).$$

Since $D_{q,\omega}^{n} f(\omega_0) = q^{\frac{n(n-1)}{2}}$, the q,ω-Taylor expansion of $E_{q,\omega}(t)$ at ω_0 is given by

$$E_{q,\omega}(t) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{[k]_{q}!} (t-\omega_0)^{k}. \tag{2.25}$$

The series in (2.25) is uniformly convergent on every compact subinterval of $(\omega_0 - \frac{1}{1-q}, \omega_0 + \frac{1}{1-q})$ and consequently $E_{q,\omega}(t)$ is continuous. □

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