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Solutions of boundary value problems on extended-Branciari b-distance

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Abstract

In this paper, we consider a new distance structure, extended Branciari b-distance, to combine and unify several distance notions and obtain fixed point results that cover several existing ones in the corresponding literature. As an application of our obtained result, we present a solution for a fourth-order differential equation boundary value problem.

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1 Introduction

The distance notion is as old as the history of finding the writing. On the other hand, the abstract formulation of the notion of distance is relatively new. It was formulated by Fréchet in 1905 for the distance of sets. In the setting of points, it was discovered by Pompeiu and Hausdorff in 1914 under the name of metric. After that it has been improved, generalized, and extended in several ways. Among them, we mention the distance that was proposed by Branciari.

Definition 1 (See e.g. [1]) Let $\mathcal{S}$ be a non-empty set, and let $b : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ such that, for all $x, y \in \mathcal{S}$ and all $u \neq v \in \mathcal{S} \setminus \{x, y\}$,

$(b1)$ $\quad b(x, y) = 0 \quad$ if and only if $\quad x = y \quad$ (self-distance/indistancy),

$(b2)$ $\quad b(x, y) = b(y, x) \quad$ (symmetry),   (1.1)

$(b3)$ $\quad b(x, y) \leq b(x, u) + b(u, v) + b(v, y) \quad$ (quadrilateral inequality).

In this case, the map b is called a Branciari distance. The pair $(\mathcal{S}, b)$ is called a Branciari distance space and abbreviated as "BDS".

In many sources it was called "generalized metric space" or "rectangular metric space", but these names do not reflect and indicate the notion (see e.g. [2–19]). Indeed, "Branciari

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distance" cannot reduce to the standard metric. Further, general topological properties are not compatible. That is why we prefer to use Branciari distance space. For a recent extension of such spaces, we refer to [20, 21] and [22–28].

Recently, in [29] a concept of Θ-contraction was proposed to extend some fixed point theorems in the context of Branciari distance space. For the sake of completeness, we recollect the notion of Θ-contraction here:

Let Θ be the set of all non-decreasing, continuous functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(†) for each sequence $\{s_n\} \subset (0, \infty)$, $\lim_{n \to \infty} \theta(s_n) = 1$ if and only if $\lim_{n \to \infty} s_n = 0^+$;

(‡) there exist $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{s \to 0^+} \frac{\theta(s) - 1}{s^q} = \ell$.

This definition has been refined and applied to several fixed point results, see e.g. [8–12, 20, 21, 30–48].

On the other hand, we recall the notion of extended b-metric space (simply, d_e-metric space) introduced by Kamran *et al.* [32], which is the most general form of the concept of metric. We recollect the definition as well.

Definition 2 ([32]) For a non-empty set S and a mapping $\omega : S \times S \to [1, \infty)$, we say that a function $\rho_e : S \times S \to [0, \infty)$ is called an extended b-metric (in short, ρ_e-metric) if it satisfies:

(i) $\rho_e(x, y) = 0$ if and only if $x = y$;

(ii) $\rho_e(x, y) = \rho_e(y, x)$;

(iii) $\rho_e(x, y) \leq \omega(x, y)[\rho_e(x, z) + \rho_e(z, y)]$

for all $x, y, z \in S$. The symbol (X, ρ_e) denotes a ρ_e-metric space.

We shall combine these two notions, *extended b-metric* and *Branciari distance*, under the name of an *extended Branciari b-distance space* by the following definition.

Definition 3 For a non-empty set S and a mapping $\omega : S \times S \to [1, \infty)$, we say that a function $d_e : S \times S \to [0, \infty)$ is called an extended Branciari b-distance if it satisfies:

(i) $d_e(x, y) = 0$ if and only if $x = y$;

(ii) $d_e(x, y) = d_e(y, x)$;

(iii) $d_e(x, y) \leq \omega(x, y)[d_e(x, u) + d_e(u, v) + d_e(v, y)]$

for all $x, y \in \mathcal{S}$ and all distinct $u, v \in \mathcal{S} \setminus \{x, y\}$. The couple of symbols (S, d_e) denotes an extended Branciari b-distance space (shortly, d_e-metric space).

Example 1 Let $S = l_p$, where $1 \leq p < \infty$, be defined by

$$l_p = \left\{ (x_n)_{n \geq 1} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define $d_e : S \times S \to \mathbb{R}^+$ by

$$d_e(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \quad \text{for all } x, y \in S.$$

The function $\omega : S \times S \to [1, \infty)$ is defined by $\omega(x, y) = 2^{\frac{1}{p}}$ for all $x, y \in S$. Then d_e satisfies all the conditions of an extended Branciari b-distance space. Indeed if $p = 1$, the quadrilateral

inequality trivially holds. So, let $p > 1$ and $x = (x_n)_{n \geq 1}$; $y = (y_n)_{n \geq 1}$; $z = (z_n)_{n \geq 1}$; $w = (w_n)_{n \geq 1}$ be sequences in S with $y, w \in S \setminus \{x, z\}$. Consider

$$
\begin{aligned}
|x_n - z_n|^p &\leq p|x_n - z_n|^p \\
&= p|x_n - z_n||x_n - z_n|^{p-1} \\
&= p\big[|x_n - y_n + y_n - w_n + w_n - z_n|\big]|x_n - z_n|^{p-1} \\
&\leq p\big[|x_n - y_n| + |y_n - w_n| + |w_n - z_n|\big]|x_n - z_n|^{p-1} \\
&= p\big[|x_n - y_n||x_n - z_n|^{p-1} + |y_n - w_n||x_n - z_n|^{p-1} \\
&\quad + |w_n - z_n||x_n - z_n|^{p-1}\big] \quad \text{for } n \in \mathbb{N},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}|x_n - z_n|^p &\leq p\Bigg\{\sum_{n=1}^{\infty}|x_n - y_n||x_n - z_n|^{p-1} \\
&\quad + \sum_{n=1}^{\infty}|y_n - w_n||x_n - z_n|^{p-1} \\
&\quad + \sum_{n=1}^{\infty}|w_n - z_n||x_n - z_n|^{p-1}\Bigg\} \\
&\leq p\Bigg\{\left(\sum_{n=1}^{\infty}|x_n - y_n|^p\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{n=1}^{\infty}|y_n - w_n|^p\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{n=1}^{\infty}|w_n - z_n|^p\right)^{\frac{1}{p}}\Bigg\}\left(\sum_{n=1}^{\infty}|x_n - z_n|^{p(p-1)}\right)^{\frac{1}{p}}.
\end{aligned}
$$

After simplifying we get

$$
\begin{aligned}
\left(\sum_{n=1}^{\infty}|x_n - z_n|^p\right)^{\frac{1}{p}} &\leq p\Bigg\{\left(\sum_{n=1}^{\infty}|x_n - y_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}|y_n - w_n|^p\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{n=1}^{\infty}|w_n - z_n|^p\right)^{\frac{1}{p}}\Bigg\} \\
&\leq 2^p\Bigg\{\left(\sum_{n=1}^{\infty}|x_n - y_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}|y_n - w_n|^p\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{n=1}^{\infty}|w_n - z_n|^p\right)^{\frac{1}{p}}\Bigg\} \quad \text{for } 1 \leq p < \infty.
\end{aligned}
$$

Thus $d_e(x, z) \leq \omega(x, z)[d_e(x, y) + d_e(y, w) + d_e(w, z)]$ and d_e is an extended Branciari b-distance space.

Example 2 Let $S = [0, 1]$. Define $d_e : S \times S \to \mathbb{R}$ by $\omega(x, y) = 5x + 5y + 3$, then (S, d_e) is an extended Branciari b-distance space.

We will prove the extended quadrilateral inequality only as the other conditions are clear.

$$
\begin{aligned}
d_e(x, y) &= |x - y|^2 \\
&= |x - z + z - w + w - y|^2 \\
&= |x - z|^2 + |z - w|^2 + |w - y|^2 + 2|x - z||z - w| \\
&\quad + 2|z - w||w - y| + 2|w - y||x - z| \\
&\le (5x + 5y + 3)\big[|x - z|^2 + |z - w|^2 + |w - y|^2\big] \\
&= \omega(x, y)\big[d_e(x, z) + d_e(z, w) + d_e(w, y)\big].
\end{aligned}
$$

Hence $d_e(x, y) \le \omega(x, y)[d_e(x, z) + d_e(z, w) + d_e(w, y)]$.

Remark 1 If $\theta(x, y) = s$ for $s \ge 1$, then we obtain the definition of Branciari b-distance and $s = 1$ yields the standard Branciari distance. As is known well, b-metric does not need to be continuous. Consequently, an extended Branciari b-distance is not necessarily continuous either. In this paper, we presume that each extended Branciari b-distance is continuous.

Definition 4 Let S be a non-empty set endowed with extended Branciari b-distance d_e.
  (a) A sequence $\{x_n\}$ in S converges to x if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_e(x_n, x) < \epsilon$ for all $n \ge N$. For this particular case, we write $\lim_{n \to \infty} x_n = x$.
  (b) A sequence $\{x_n\}$ in S is called Cauchy if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_e(x_m, x_n) < \epsilon$ for all $m, n \ge N$.
  (c) A d_e-metric space (S, d_e) is complete if every Cauchy sequence in S is convergent.

Motivated and inspired by the above concerns, we have organized the article as follows:
  • In Sect. 1, the concept of an extended Branciari b-distance space is introduced and needed definitions are presented.
  • In Sect. 2, various topics called Θ-Branciari contraction, Ćirić–Reich–Rus type Θ-Branciari contraction, and interpolative-Θ-Branciari contraction are introduced. By using these new contractions, we formulate and prove some fixed point theorems in the setting of extended Branciari b-distance spaces.
  • A supporting example is presented by using various sequences.
  • In Sect. 3, as an application, we present a solution for a fourth-order differential equation boundary value problem.

2 Main results

Now, we start this section by introducing the concept of Θ-Branciari contraction.

Definition 5 Let (S, d_e) be an extended Branciari b-distance space and $T : X \to X$ be a mapping. Then T is said to be a Θ-Branciari contraction if there exists a function $\theta \in \Theta$ such that

$$
\theta\big(d_e(Tx, Ty)\big) \le \big[\theta\big(d_e(x, y)\big)\big]^r \quad \text{if } d_e(Tx, Ty) \ne 0 \text{ for } x, y \in S, \tag{2.1}
$$

where $r \in (0, 1)$ is such that $\sup_{m \ge 1} \lim_{n \to \infty} \omega(x_n, x_m) < \frac{1}{r}$, where $x_n = T^n x_0$ for $x_0 \in S$.

Theorem 1 *Let (S, d_e) be a complete extended Branciari b-distance and $T : X \to X$ be a Θ-Branciari contraction. Then T has a unique fixed point in S.*

Proof For an arbitrary point $x_0 \in S$, we construct an iterative sequence $\{x_n\}_0$ as follows:

$$x_n = T^n x_0 \quad \text{for all } n \in \mathbb{N}.$$

Suppose that if $T^{n_*}x = T^{n_*+1}x$ for some $n_* \in \mathbb{N}$, then $T^{n_*}x$ is clearly a fixed point of T.

Hence, without loss of generality, we may assume that $d_e(T^n x, T^{n+1}x) > 0$ for all $n \in \mathbb{N}$. From Definition 5, we have

$$\begin{aligned}
\theta\big(d_e(x_n, x_{n+1})\big) = \theta\big(d_e(Tx_{n-1}, Tx_n)\big) \\
\leq \big[\theta\big(d_e(x_{n-1}, x_n)\big)\big]^r \\
\leq \big[\theta\big(d_e(x_{n-2}, x_{n-1})\big)\big]^{r^2}.
\end{aligned}$$

Recursively, we find that

$$\theta\big(d_e(x_n, x_{n+1})\big) \leq \big[\theta\big(d_e(x_0, x_1)\big)\big]^{r^n}. \tag{2.2}$$

Accordingly, we obtain that

$$1 < \theta\big(d_e(x_n, x_{n+1})\big) \leq \big[\theta\big(d_e(x_0, x_1)\big)\big]^{r^n} \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Letting $n \to \infty$ in (2.3), we get $\theta(d_e(x_n, x_{n+1})) \to 1$ as $n \to \infty$.

From (†), we have

$$\lim_{n \to \infty} d_e(x_n, x_{n+1}) = 0. \tag{2.4}$$

Similarly, we can easily deduce that

$$\lim_{n \to \infty} d_e(x_n, x_{n+2}) = 0. \tag{2.5}$$

From (‡), there exist $q \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n \to \infty} \frac{\theta(d_e(x_n, x_{n+1})) - 1}{[d_e(x_n, x_{n+1})]^q} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. Using limit definition, we pick $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d_e(x_n, x_{n+1})) - 1}{[d_e(x_n, x_{n+1})]^q} - l \right| \leq B$$

for all $n \geq n_0$.

This implies that $|\frac{\theta(d_e(x_n, x_{n+1})) - 1}{[d_e(x_n, x_{n+1})]^q}| \geq l - B = B$ for all $n \geq n_0$.

Then, we derive that

$$n\big[d_e(x_n, x_{n+1})^q\big] \leq n\left[\frac{\theta(d_e(x_n, x_{n+1})) - 1}{B}\right] \quad \text{for all } n \geq n_0.$$

Suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number.

Using the limit definition, we find $n_0 \in \mathbb{N}$ such that $\dfrac{\theta(d_e(x_n, x_{n+1})) - 1}{[d_e(x_n, x_{n+1})]^q} \geq B$ for all $n \geq n_0$.

This implies that

$$n\big[d_e(x_n, x_{n+1})\big]^q \leq n\left[\frac{\theta(d_e(x_n, x_{n+1})) - 1}{B}\right] \quad \text{for all } n \geq n_0.$$

Thus, in all cases, there exist $\frac{1}{B} > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\big[d_e(x_n, x_{n+1})\big]^q \leq n\left[\frac{\theta(d_e(x_n, x_{n+1})) - 1}{B}\right] \quad \text{for all } n \geq n_0.$$

Using Eq. (2.3), we obtain

$$n\big[d_e(x_n, x_{n+1})\big]^q \leq \big[\theta\big(d_e(x_0, x_1)\big)\big]^{r^n} - 1 \quad \text{for all } n \geq n_0.$$

If we let $n \to \infty$ in the above inequality, then we obtain

$$\lim_{n \to \infty} n\big[d_e(x_n, x_{n+1})\big]^q = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d_e(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{q}}} \quad \text{for all } n \geq n_1. \tag{2.6}$$

Let $N = \max\{n_0, n_1\}$. Due to the modified triangle inequality, we have two cases.

For all $n \geq 1$, we have two cases as follows.

Case 1: Let $x_n = x_m$ for some integers $n \neq m$. So, for $m > n$, we have $T^{m-n}(x_n) = x_n$. Choose $y = x_n$ and $p = m - n$. Then $T^p y = y$, that is, y is a periodic point of T. Thus, $d_e(y, Ty) = d_e(T^p y, T^{p+1} y)$. Thus, by the above argument, we can easily deduce that $d_e(y, Ty) = 0$, so $y = Ty$, that is, y is a fixed point of T.

Case 2: Suppose that $T^n x \neq T^m x$ for all integers $n \neq m$. Let $n < m$ be two natural numbers. To show that $\{x_n\}$ is a Cauchy sequence, we need to consider two subcases as follows.

Subcase 1: We claim that if $n - m$ is odd, then $d_e(x_n, x_m)$ converges to 0 as $n, m \to \infty$ To prove this, we may assume that $m = n + 2p + 1$. Thus,

$$d_e(x_n, x_{n+2p+1}) \leq w(x_n, x_{n+2p+1})\big[d_e(x_n, x_{n+1}) + d_e(x_{n+1}, x_{n+2}) + d_e(x_{n+2}, x_{n+2p+1})\big]$$

$$\leq w(x_n, x_{n+2p+1})d_e(x_n, x_{n+1}) + w(x_n, x_{n+2p+1})d_e(x_{n+1}, x_{n+2})$$

$$+ w(x_n, x_{n+2p+1})w(x_{n+2}, x_{n+2p+1})\big[d_e(x_{n+2}, x_{n+3})$$

$$+ d_e(x_{n+3}, x_{n+4}) + d_e(x_{n+4}, x_{n+2p+1})\big]$$

$$\leq w(x_n, x_{n+2p+1})d_e(x_n, x_{n+1}) + w(x_n, x_{n+2p+1})d_e(x_{n+1}, x_{n+2})$$

$$+ w(x_n, x_{n+2p+1})w(x_{n+2}, x_{n+2p+1})d_e(x_{n+2}, x_{n+3})$$

$$+ w(x_n, x_{n+2p+1})w(x_{n+2}, x_{n+2p+1})d_e(x_{n+3}, x_{n+4})$$

$$+ w(x_n, x_{n+2p+1})w(x_{n+2}, x_{n+2p+1})d_e(x_{n+4}, x_{n+2p+1})$$

$$\leq \cdots$$

$$\leq w(x_n, x_{n+2p+1})d_e(x_n, x_{n+1}) + w(x_n, x_{n+2p+1})d_e(x_{n+1}, x_{n+2})$$

$$+ w(x_n, x_{n+2p+1})w(x_{n+2}, x_{n+2p+1})d_e(x_{n+2}, x_{n+3})$$

$$+ w(x_n, x_{n+2p+1})w(x_{n+2}, x_{n+2p+1})d_e(x_{n+3}, x_{n+4}) + \cdots$$

$$+ w(x_n, x_{n+2p+1})w(x_{n+2}, x_{n+2p+1}) \cdots$$

$$\times w(x_{n+2p-2}, x_{n+2p+1})d_e(x_{n+2p}, x_{n+2p+1}).$$

This can be written as follows:

$$d_e(x_n, x_m) \leq \sum_{j=n}^{n+m-1} d_e(x_j, x_{j+1}) \prod_{i=n}^{n+m-1} \omega(x_i, x_m).$$

Since $\sup_{m\geq 1} \lim_{n\to\infty} \omega(x_n, x_m) < \frac{1}{r}$, we have

$$d_e(x_n, x_m) \leq \sum_{j=n}^{n+m-1} d_e(x_j, x_{j+1}) \prod_{i=n}^{n+m-1} \omega(x_i, x_m) \leq \frac{1}{r} \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{q}}},$$

which is convergent to 0 as $n, m \to \infty$ and $\frac{1}{q} > 1$.

Subcase 2: We claim that if $n - m$ is even, then $d_e(x_n, x_m)$ converges to 0 as $n, m \to \infty$. To prove this, we may assume that $m = n + 2p$. Thus,

$$d_e(x_n, x_{n+2p}) \leq w(x_n, x_{n+2p})\big[d_e(x_n, x_{n-2}) + d_e(x_{n-2}, x_{n+2p+1}) + d_e(x_{n+2p}, x_{n+2p+1})\big].$$

Hence, by the fact that $\sup_{m\geq 1} \lim_{n\to\infty} \omega(x_n, x_m) < \frac{1}{r}$, subcase 1, (2.4), and (2.5), we can easily deduce that $d_e(x_n, x_m)$ converges to 0 as $n, m \to \infty$. Thus, the sequence $\{x_n\}$ in S is a Cauchy sequence.

Since (S, d_e) is a complete extended Branciari b-distance, there exists a point η in S such that $\{x_n\}$ converges to η.

Next, we indicate that T is continuous. Suppose that if $Tx \neq Ty$, then from (2.2) we have

$$\ln\big[\theta d_e(Tx, Ty)\big] \leq r \ln\big[\theta d_e(x, y)\big]$$

$$\leq \ln\big[\theta d_e(x, y)\big].$$

Since θ is non-decreasing, we derive, from the above observation, that $d_e(Tx, Ty) \leq d_e(x, y)$ for all distinct $x, y \in S$.

From this inspection, we can get $d_e(x_{n+1}, T\eta) = d_e(Tx_n, T\eta) \leq d_e(x_n, \eta)$ for all $n \in \mathbb{N}$.

Letting $n \to \infty$ in the above inequality, we get $x_{n+1} \to T\eta$.

From the rectangle inequality, we have

$$d_e(\eta, T\eta) \leq \omega(\eta, T\eta)\big[d_e(\eta, x_n) + d_e(x_n, x_{n+1}) + d_e(x_{n+1}, T\eta)\big].$$

Taking limit as $n \to \infty$ and using (2.6) and Definition 3 of (i), we have $d_e(\eta, T\eta) = 0$, which implies that $T\eta = \eta$.

This contradicts the assumption that T does not have a periodic point. Hence, assume that η is a periodic point of T with period q.

Suppose that the set of fixed points of T is empty.

Then we have $d_e(z, Tz) > 0$ for all $z \in S$ and $d_e(z, T^q z) = 0$ for $q > 1$.

Using Definition 5, we get

$$\theta\big(d_e(z, Tz)\big) = \theta\big(d_e\big(T^q z, T^{q+1} z\big)\big) \leq \big[\theta\big(d_e(z, Tz)\big)\big]^{r^q}$$
$$< \theta\big(d_e(z, Tz)\big),$$

which leads to a contradiction. So, there exists a point $\eta \in S$ such that $T\eta = \eta$.

Suppose that f has another fixed point ζ such that $\eta \neq \zeta$. Then clearly $d_e(\eta, \zeta) = d_e(f\eta, f\zeta) \neq 0$.

Now, using condition (2.1), we get

$$\theta\big(d_e(\eta, \zeta)\big) = \theta\big(d_e(T\eta, T\zeta)\big)$$
$$= \theta\big(d_e\big(T^q \eta, T^q \zeta\big)\big) \leq \big[\theta\big(d_e(\eta, \zeta)\big)\big]^{r^q}$$
$$< \theta\big(d_e(\eta, \zeta)\big), \quad \text{a contradiction.}$$

Therefore, $\eta = \zeta$. This claims that T has a unique fixed point in S. $\qquad\qquad\square$

Example 3 Consider the sequence:

$$\tau_1 = 1 \times 2,$$
$$\tau_2 = 1 \times 2 + 2 \times 3,$$
$$\tau_3 = 1 \times 2 + 2 \times 3 + 3 \times 4, \qquad\qquad\qquad\qquad (2.7)$$
$$\tau_4 = 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5,$$
$$\tau_n = 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Let $S = \{\tau_n : n \in \mathbb{N}\}$. Define $d_e : S \times S \to [0, \infty)$ as $d_e(x, y) = |x - y|^2$ and $\omega : S \times S \to [1, \infty)$ as $\omega(x, y) = 4x + 2y + 3$. Then (S, d_e) is a complete extended b-Branciari distance space.

Define the mapping $T : S \to S$ by $T(\tau_1) = \tau_1$, $T(\tau_n) = \tau_{n-1}$ for all $n \geq 2$. Now we show that T is a Θ-Branciari contraction with $\theta(t) = e^t$.

Since $\theta(d_e(Tx, Ty)) \leq [\theta(d_e(x, y))]^r$, which yields $e^{(d_e(Tx, Ty))} \leq [e^{(d_e(x, y))}]^r$. Applying log on both sides, we get

$$d_e(Tx, Ty) \leq r d_e(x, y).$$

Thus to prove T is a Θ-Branciari contraction, it suffices to prove the above equation.

Case-1: For $n = 1$ and $m > 2$, we have

$$d_e(T\tau_1, T\tau_m) = d_e(\tau_1, \tau_{m-1})$$

$$= \left| \frac{m(m-1)(m+1) - 6}{3} \right|^2$$

and

$$d_e(\tau_1, \tau_m) = \left| \frac{m(m+1)(m+2) - 6}{3} \right|^2.$$

Now consider

$$\frac{d_e(T\tau_1, T\tau_m)}{d_e(\tau_1, \tau_m)} = \left| \frac{m(m-1)(m+1) - 6}{m(m+1)(m+2) - 6} \right|^2$$

$$< r, \quad \text{where } r \in (0, 1).$$

Case-2: For $m > n > 1$, we have

$$d_e(T\tau_n, T\tau_m) = d_e(\tau_{n-1}, \tau_{m-1})$$

$$= d_e\left(\frac{(n-1)n(n+1)}{3}, \frac{(m-1)m(m+1)}{3} \right)$$

$$= \left| \frac{(n-1)n(n+1)}{3} - \frac{(m-1)m(m+1)}{3} \right|^2$$

$$= \left| \frac{n^3 - n}{3} - \frac{m^3 - m}{3} \right|^2$$

$$= \left| \frac{n^3 - m^3 - (n - m)}{3} \right|^2$$

$$= \left| \frac{(n - m)(n^2 + nm + m^2) - (n - m)}{3} \right|^2$$

$$= \left| \frac{(n - m)(n^2 + nm + m^2 - 1)}{3} \right|^2$$

and

$$d_e(\tau_n, \tau_m) = d_e\left(\frac{n(n+1)(n+2)}{3}, \frac{m(m+1)(m+2)}{3} \right)$$

$$= \left| \frac{n(n+1)(n+2)}{3} - \frac{m(m+1)(m+2)}{3} \right|^2$$

$$= \left| \frac{n^3 + 3n^2 + 2n}{3} - \frac{m^3 + 3m^2 + 2m}{3} \right|^2$$

$$= \left| \frac{n^3 - m^3 - 3(n^2 - m^2) + 2(n - m)}{3} \right|^2$$

$$= \left| \frac{(n - m)(n^2 + nm + m^2) + 3(n + m) + 2}{3} \right|^2.$$

Now consider

$$\frac{d_e(T\tau_n, T\tau_m)}{d_e(\tau_n, \tau_m)} = \left| \frac{n^2 + nm + m^2 - 1}{n^2 + nm + m^2 + 3(n + m) + 2} \right|^2$$

$$< r, \quad \text{where } r \in (0, 1).$$

Thus T satisfies Θ-Branciari contraction with $\theta(t) = e^t$. Then, from Theorem 1, T has a unique fixed point τ_1.

If we take $\omega(x, y) = b > 1$ in Theorem 1, then we get the following corollary.

Corollary 1 *Let T be a self-map on a complete Branciari b-distance space (X, d). Suppose that there exist $\vartheta \in \Theta$ and $r \in (0, 1)$ such that*

$$\vartheta\left(d(Tx, Ty)\right) \leq \left[\vartheta\left(d(x, y)\right)\right]^r \quad \text{if } d(Tx, Ty) \neq 0 \text{ for } x, y \in S.$$

Then T has a unique fixed point in S.

If we take $\omega(x, y) = 1$ in the above theorem, then we get the following corollary.

Corollary 2 *Let T be a self-map on a complete Branciari distance space. If there exist $\vartheta \in \Theta$ and $r \in (0, 1)$ such that*

$$\vartheta\left(d(Tx, Ty)\right) \leq \left[\vartheta\left(d(x, y)\right)\right]^r \quad \text{if } d(Tx, Ty) \neq 0 \text{ for } x, y \in S,$$

then T possesses a unique fixed point in S.

Definition 6 Let (S, d_e) be a d_e-metric space. A mapping $f : S \to S$ is called Ćirić–Reich–Rus type Θ-Branciari contraction, in short, CRR-Θ-Branciari contraction, if there exist a function $\theta \in \Theta$ and a non-negative real number $r < 1$ such that

$$\theta\left(d_e(fx, fy)\right) \leq \left[M_{f,\theta}(x, y)\right]^r \tag{2.8}$$

for all $x, y \in S$, where

$$M_{f,\theta}(x, y) := \max\left\{\theta\left(d_e(x, y)\right), \theta\left(d_e(y, fy)\right)\right\},$$

where $\limsup_{n,m \to \infty} \omega(x_n, x_m) < \frac{1}{r}$, here $x_n = f^n x_0$ for $x_0 \in S$ and $r \in (0, 1)$.

Theorem 2 *Let (S, d_e) be a complete extended Branciari b-distance space and $f : S \to S$ be a CRR-Θ-Branciari contraction. Then f has a unique fixed point in S.*

Proof As in Theorem 1, we construct an iterative sequence $\{x_n\}_0^\infty$ by starting an arbitrary point $x_0 \in S$ as follows:

$$x_n = f^n x_0 \quad \text{for all } n \in \mathbb{N}.$$

Without loss of generality, we assume that $d_e(f^n x, f^{n+1} x) > 0$ for all $n \in \mathbb{N}$. Indeed, if $f^{n_*} x = f^{n_*+1} x$ for some $n_* \in \mathbb{N}$, then $f^{n_*} x$ will be a fixed point of T.

We prove that $\lim_{n \to \infty} d_e(x_n, x_{n+1}) = 0$.

Employing the contraction condition (2.8), we get

$$\theta\big(d_e(x_{n+1}, x_n)\big) \leq \big[M_{f,\theta}(x_n, x_{n-1})\big]^r, \tag{2.9}$$

where

$$M_{f,\theta}(x_n, x_{n-1}) = \max\big\{\theta\big(d_e(x_n, x_{n-1})\big), \theta\big(d_e(x_n, fx_n)\big), \theta\big(d_e(x_{n-1}, fx_{n-1})\big)\big\}$$

$$= \max\big\{\theta\big(d_e(x_n, x_{n-1})\big), \theta\big(d_e(x_n, x_{n+1})\big), \theta\big(d_e(x_{n-1}, x_n)\big)\big\}.$$

$$\leq \max\big\{\theta\big(d_e(x_n, x_{n-1})\big), \theta\big(d_e(x_n, x_{n+1})\big)\big\}.$$

If $M_{f,\theta}(x_n, x_{n-1}) = \theta(d_e(x_n, x_{n+1}))$, then inequality (2.9) becomes

$$\theta\big(d_e(x_{n+1}, x_n)\big) \leq \theta\big(d_e(x_n, x_{n+1})\big)^r \quad \Leftrightarrow \quad \ln\big(\theta\big(d_e(x_{n+1}, x_n)\big)\big) \leq r \ln\big(\theta\big(d_e(x_{n+1}, x_n)\big)\big),$$

which is a contradiction (since $r < 1$). Thus, we have $M_{f,\theta}(x_n, x_{n-1}) = \theta(d_e(x_{n-1}, x_n))$. It yields from (2.9) that

$$\theta\big(d_e(x_n, x_{n+1})\big) \leq \big[\theta\big(d_e(x_{n-1}, x_n)\big)\big]^r.$$

Iteratively, we find that

$$\theta\big(d_e(x_n, x_{n+1})\big) \leq \big[\theta\big(d_e(x_0, x_1)\big)\big]^{r^n}.$$

After this observation, by following the related lines in the proof of Theorem 2, we conclude that the sequence $\{x_n\}$ in S is a Cauchy sequence. Regarding that (S, d_e) is a complete extended Branciari b-distance, there exists a point η in S such that $\{x_n\}$ converges to η.

Without loss of generality, we assume that $f^n x \neq \eta$ for all n (or for large enough n). Assume that $d(\eta, T\eta) > 0$. Employing (2.8), we get

$$\theta\big(d_e(fx_n, f\eta)\big) \leq \big[M_{f,\theta}(x_n, \eta)\big]^r \tag{2.10}$$

for all $x, y \in S$, where

$$M_{f,\theta}(x_n, \eta) := \max\big\{\theta\big(d_e(x_n, \eta)\big), \theta\big(d_e(x_n, fx_n)\big), \theta\big(d_e(\eta, f\eta)\big)\big\}.$$

By taking $n \to \infty$ in the inequality above, we derive that

$$\theta\big(d_e(\eta, f\eta)\big) \leq \big[\theta\big(d_e(\eta, f\eta)\big)\big]^r < \theta\big(d_e(\eta, f\eta)\big),$$

a contradiction. Hence, $f\eta = \eta$.

That is, f has a fixed point in S.

Suppose that $\eta \neq \zeta$ are two fixed points of f. Then clearly $d_e(\eta, \zeta) = d_e(f\eta, f\zeta) \neq 0$.

Now, using condition (2.11), we get

$$
\begin{aligned}
1 < \theta\big(d_e(\eta,\zeta)\big) &= \theta\big(d_e(f\eta,f\zeta)\big) \\
&\leq \big[\max\{\theta\big(d_e(\eta,\zeta)\big), \theta\big(d_e(\eta,f\eta)\big), \theta\big(d_e(\zeta,f\zeta)\big)\}\big]^r \\
&< \theta\big(d_e(\eta,\zeta)\big),
\end{aligned}
$$

a contradiction. Accordingly, we have $\eta = \zeta$.

Thus f has a unique fixed point in S. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 7 Let (S, d_e) be an extended Branciari b-distance and $f : S \to S$ be a mapping. Then f is said to be an interpolative-Θ-Branciari contraction if there exist a function $\theta \in \Theta$ and non-negative real numbers r_1, r_2, r_3 with $r_1 + r_2 + r_3 < 1$ such that

$$
\theta\big(d_e(fx,fy)\big) \leq \big[\theta\big(d_e(x,y)\big)\big]^{r_1} \big[\theta\big(d_e(x,fx)\big)\big]^{r_2} \big[\theta\big(d_e(y,fy)\big)\big]^{r_3} \tag{2.11}
$$

for all $x, y \in S$, where $\limsup_{n,m\to\infty} \omega(x_n, x_m) < \frac{1}{r}$, here $x_n = f^n x_0$ for $x_0 \in S$ and $r \in (0, 1)$.

Theorem 3 *Let (S, d_e) be a complete extended Branciari b-distance where d_e is a continuous functional. If $f : S \to S$ is an interpolative-Θ-Branciari contraction, then f possesses a unique fixed point in S.*

We skip the proof since

$$
\begin{aligned}
&\big[\theta\big(d_e(x,y)\big)\big]^{r_1} \big[\theta\big(d_e(x,fx)\big)\big]^{r_2} \big[\theta\big(d_e(y,fy)\big)\big]^{r_3} \\
&\quad \leq \big[M_{\theta,f}(x,y)\big]^{r_1+r_2+r_3}.
\end{aligned}
$$

Thus, it is sufficient to choose $r := r_1 + r_2 + r_3 < 1$ in Theorem 2 to conclude the theorem above.

In Theorem 3, if we take $r_2 = 0$, $r_3 = 0$, then the above theorem reduces to the following.

Theorem 4 *Let f be a self-mapping on an extended Branciari b-distance space (S, d_e) and $\theta \in \Theta$. If there exists $r_1 \in [0, 1)$ such that*

$$
\theta\big(d_e(fx,fy)\big) \leq \big[\theta\big(d_e(x,y)\big)\big]^{r_1} \quad \text{for all } x, y \in S, \tag{2.12}
$$

where $r \in [0, 1)$ and $\limsup_{m,n\to\infty} \omega(x_n, x_m) < \frac{1}{r}$, then f has a unique fixed point in S.

In Theorem 3, if we take $r_1 = 0$, $r_2 = 0$, $r_3 = 0$, then the above theorem reduces to the following one.

Theorem 5 *Let (S, d_e) be an extended Branciari b-distance space such that d_e is a continuous functional and $\theta \in \Theta$, $f : S \to S$ be a mapping. Suppose that there exists $r_4 \in [0, 1)$ such that*

$$
\theta\big(d_e(fx,fy)\big) \leq \big[\theta\big(d_e(x,fy) + d_e(y,fx)\big)\big]^{r_4} \quad \text{for all } x, y \in S \tag{2.13}
$$

and $r \in [0, 1)$ such that $\limsup_{m,n\to\infty} \omega(x_n, x_m) < \frac{1}{r}$. Consequently, f possesses a unique fixed point in S.

3 Existence of a solution of fourth-order differential equation

We consider the problem

$$
\begin{cases}
\xi^4(t) = g(t, \xi(t), \xi', \xi'', \xi'''), \\
\xi(0) = \xi'(0) = \xi''(1) = \xi'''(1) = 0; \quad t \in [0,1],
\end{cases}
\tag{3.1}
$$

where $g : [0,1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a continuous function. This problem known as a boundary value problem (shortly, BVP) is employed to model such phenomena as deformations of an elastic beam in its equilibrium state, where one end-point is free while the other is fixed. In the discipline of mechanics, boundary value problem is said to be a Cantilever beam equation. Due to its significance in mathematics, the existence of solutions for such a problem plays a vital role. With this inspiration, we shall employ the fixed point technique to find the existence of solution of BVP.

In this section, we study the existence of solution of a fourth-order differential equation boundary value problem. Let $S = \mathcal{C}[0,1]$, where $\mathcal{C}[0,1]$ represents the space of all continuous functions defined on the closed interval $[0,1]$. An extended Branciari b-distance space on S is given by $d_e(\xi, y) = |\xi - y|^2$, where $d_e : S \times S \to \mathbb{R}$ and $\omega : S \times S \to [1, \infty)$ by $\omega(\xi, y) = 5\xi + 5y + 3$.

Note that the space $S = (\mathcal{C}[0,1], d_e)$ is a complete extended Branciari b-distance space. Now, we consider the above fourth-order ordinary differential equation boundary value problem. Then the problem BVP can be written in the following integral form:

$$
\xi(t) = \int_0^1 \mathcal{G}(t,s) g\big(s, \xi(s), \xi'(s)\big)\, ds, \quad \xi \in \mathcal{C}[0,1],
$$

where $\mathcal{G}(t,s)$ is Green's function of the homogenous linear problem $\xi^4(t) = 0, \xi(0) = \xi'(0) = \xi''(1) = \xi'''(1) = 0$, which is explicitly given by

$$
\mathcal{G}(t,s) = \begin{cases}
\frac{1}{6} t^2 (3s - t), & 0 \le t \le s \le 1, \\
\frac{1}{6} s^2 (3t - s), & 0 \le s \le t \le 1.
\end{cases}
\tag{3.2}
$$

From (3.2), we can easily check that $\mathcal{G}(t,s)$ has the following properties: $\frac{1}{3} t^2 s^2 \le \mathcal{G}(t,s) \le \frac{1}{2} t^2$ (or $\frac{1}{2} s^2$), $t, s \in [0,1]$.

Theorem 6 *Assume that the following conditions are satisfied*:

(1) $g : [0,1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *is continuous*.

(2) *There exists $\tau \in [1, \infty)$ such that the following condition holds for all $\xi, y \in S$*:

$$
\big|g\big(s, \xi, \xi'\big) - g\big(s, y, y'\big)\big| \le \sqrt{20}\, e^{\frac{-\tau}{2}} \big|\xi(s) - y(s)\big|, \quad s \in [0,1].
$$

(3) *There exists $\xi_0 \in X$ such that, for all $t \in [0,1]$, we have*

$$
\xi_0 t \le \int_0^1 \mathcal{G}(t,s) g\big(s, \xi_0(s), \xi_0'(s)\big)\, ds.
$$

Then the problem BVP has a solution in S.

Proof If we define the mapping $f : S \to S$ by

$$f(\xi)(t) = \int_0^1 \mathcal{G}(t,s)g\big(s,\xi(s),\xi'(s)\big)\,ds,$$

then $\xi = f\xi$, which yields that BVP has a unique solution. Consider

$$
\begin{aligned}
\big|f(\xi)(t) - f(y)(t)\big|^2 &= \left| \int_0^1 \mathcal{G}(t,s)g\big(s,\xi(s),\xi'(s)\big)\,ds - \int_0^1 \mathcal{G}(t,s)g\big(s,y(s),y'(s)\big)\,ds \right|^2 \\
&\leq \int_0^1 \big(\mathcal{G}(t,s)\big)^2 \big|g\big(s,\xi(s),\xi'(s)\big) - g(s,y(s),y'(s))\big|^2\,ds \\
&\leq \int_0^1 \frac{1}{4}s^4 20e^{-\tau}\big|\xi(s) - y(s)\big|^2\,ds \\
&\leq 20e^{-\tau}d_e(\xi,y)\int_0^1 \frac{1}{4}s^4\,ds \\
&\leq 20e^{-\tau}d_e(\xi,y)\frac{1}{20} \\
&= e^{-\tau}d_e(\xi,y),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&d_e\big(f(\xi),f(y)\big) \leq e^{-\tau}d_e(\xi,y) \\
&\sqrt{d_e\big(f(\xi),f(y)\big)} \leq \sqrt{e^{-\tau}d_e(\xi,y)} \\
&e^{\sqrt{d_e(f(\xi),f(y))}} \leq \big(e^{\sqrt{d_e(\xi,y)}}\big)^{\sqrt{e^{-\tau}}}, \quad \text{where } e^{-\tau} < 1 \text{ as } \tau \geq 1.
\end{aligned}
$$

Hence $e^{\sqrt{d_e(f(\xi),f(y))}} \leq (e^{\sqrt{d_e(\xi,y)}})^r$ with $r = \sqrt{e^{-\tau}}$, which gives

$$\theta\big(d_e(f\xi,fy)\big) \leq \big[\theta\big(d_e(\xi,y)\big)\big]^r, \quad \text{where } \theta(t) = e^{\sqrt{t}}.$$

As all the conditions of Theorem 4 are satisfied, f has a fixed point. Thus BVP has a solution in S. □

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Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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