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Exponential type convexity and some related inequalities

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Abstract

In this manuscript, we give and study the concept of exponential type convex functions and some of their algebraic properties. We prove two Hermite–Hadamard (H-H) type integral inequalities for the newly introduced class of functions. We also obtain some refinements of the H-H inequality for functions whose first derivative in absolute value at certain power is exponential type convex.

MSC: Primary 26D15; secondary 26A51

Keywords: Convex function; Exponential type convexity; Hermite–Hadamard inequality

1 Introduction

A function $f: I \to \mathbb{R}$ is said to be convex function if the following inequality holds:

$$f\bigl(km + (1-k)n\bigr) \le kf(m) + (1-k)f(n) \tag{1}$$

for all $m, n \in I$ and $k \in [0,1]$. If (1) reverses, then f is said to be concave on $I \neq \emptyset$. Convexity theory provides powerful principles and techniques for studying a class of problems in mathematics. See articles [4, 5, 7, 9–13] and the references therein.

Let $f: I \to \mathbb{R}$ be a convex function. Then the following inequalities hold:

$$f\left(\frac{m+n}{2}\right) \le \frac{1}{n-m}\int_m^n f(x)\,dx \le \frac{f(m)+f(n)}{2} \tag{2}$$

for all $m, n \in I$ with $m < n$. Inequality (2) is well known as the Hermite–Hadamard (H-H) integral inequality [6]. Some refinements of the H-H inequality for convex functions have been obtained [3, 15].

The aim of this study is to submit the concept of exponential type convex functions and find some results connected with the right-hand side of new inequalities similar to the H-H inequality for this type of functions.

Definition 1.1 ([14]) Let $h: J \to \mathbb{R}$ be a nonnegative function and $h \neq 0$. We say that $f: I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $m, n \in I$, $k \in [0,1]$ we have

$$f\big(km + (1-k)n\big) \leq h(k)f(m) + h(1-k)f(n). \tag{3}$$

If (3) is reversed, then f is said to be h-concave, i.e., $f \in SV(h, I)$. It is clear that, if $h(u) = u$, then the h-convexity reduces to convexity.

Readers can look at [1, 8] for studies on h-convexity.

2 Main results

In this section, we give a new definition, which is called exponential type convexity, and we give it by setting some algebraic properties for the exponential type convex functions, as follows.

Definition 2.1 A nonnegative function $f : I \to \mathbb{R}$ is called exponential type convex function if, for every $m, n \in I$ and $k \in [0,1]$,

$$f\big(km + (1-k)n\big) \leq \big(e^k - 1\big)f(m) + \big(e^{1-k} - 1\big)f(n). \tag{4}$$

The class of all exponential type convex functions on interval I is indicated by $EXPC(I)$.

Remark 2.1 The range of the exponential type convex functions is $[0, \infty)$.

Proof Let $m \in I$ be arbitrary. Using the definition of the exponential type convex function for $k = 1$, we have $f(m) \leq (e-1)f(m) \Longrightarrow 0 \leq (e-2)f(m) \Longrightarrow f(m) \geq 0$. □

We discuss some connections between the class of exponential type convex functions and other classes of generalized convex functions.

Lemma 2.1 *For all $k \in [0,1]$, the inequalities $e^k - 1 \geq k$ and $e^{1-k} - 1 \geq 1 - k$ hold.*

Proof The proof is obvious. □

Proposition 2.1 *Every nonnegative convex function is exponential type convex function.*

Proof According to Lemma 2.1, since $k \leq e^k - 1$ and $1 - k \leq e^{1-k} - 1$ for all $k \in [0,1]$, we obtain

$$f\big(km + (1-k)n\big) \leq kf(m) + (1-k)f(n) \leq \big(e^k - 1\big)f(m) + \big(e^{1-k} - 1\big)f(n). \qquad \square$$

Proposition 2.2 *Every exponential type convex function is an h-convex function with $h(k) = e^k - 1$.*

Proof If we substitute $e^k - 1 = h(k)$ and $e^{1-k} - 1 = h(1-k)$ in inequality (3), an h-convex function is easily obtained. □

Theorem 2.1 *Let $f, g : [a,b] \to \mathbb{R}$. If f and g are exponential type convex functions, then*

(i) *$f + g$ is exponential type convex,*

(ii) *for $c \geq 0$, cf is exponential type convex.*

Proof (i) Let f, g be exponential type convex, then

$$\begin{aligned}
&(f+g)\big(km+(1-k)n\big)\\
&\quad = f\big(km+(1-k)n\big)+g\big(km+(1-k)n\big)\\
&\quad \le \big(e^k-1\big)f(m)+\big(e^{1-k}-1\big)f(n)+\big(e^k-1\big)g(m)+\big(e^{1-k}-1\big)g(n)\\
&\quad = \big(e^k-1\big)\big[f(m)+g(m)\big]+\big(e^{1-k}-1\big)\big[f(n)+g(n)\big]\\
&\quad = \big(e^k-1\big)(f+g)(m)+\big(e^{1-k}-1\big)(f+g)(n).
\end{aligned}$$

(ii) Let f be exponential type convex and $c \in \mathbb{R}$ $(c \ge 0)$, then

$$\begin{aligned}
(cf)\big(km+(1-k)n\big) &\le c\big[\big(e^k-1\big)f(m)+\big(e^{1-k}-1\big)f(n)\big]\\
&= \big(e^k-1\big)cf(m)+\big(e^{1-k}-1\big)cf(n)\\
&= \big(e^k-1\big)(cf)(m)+\big(e^{1-k}-1\big)(cf)(n).
\end{aligned}$$

□

Remark 2.2 Theorem 2.1 follows from the known fact that the space of an h-convex function is a convex cone for each h (see [14], Proposition 9).

Theorem 2.2 *If $f: I \to J$ is convex and $g: J \to \mathbb{R}$ is an exponential type convex function and nondecreasing, then $g \circ f: I \to \mathbb{R}$ is an exponential type convex function.*

Proof For $m, n \in I$ and $k \in [0,1]$, we get

$$\begin{aligned}
(g\circ f)\big(km+(1-k)n\big) &= g\big(f\big(km+(1-k)n\big)\big)\\
&\le g\big(kf(m)+(1-k)f(n)\big)\\
&\le \big(e^k-1\big)g\big(f(m)\big)+\big(e^{1-k}-1\big)g\big(f(n)\big)\\
&= \big(e^k-1\big)(g\circ f)(m)+\big(e^{1-k}-1\big)(g\circ f)(n).
\end{aligned}$$

□

Remark 2.3 The above theorem can also be derived from Theorem 15 in [14].

Theorem 2.3 *Let $n > 0$ and $f_\alpha : [m,n] \to \mathbb{R}$ be an arbitrary family of exponential type convex functions, and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [m,n] : f(u) < \infty\}$ is nonempty, then J is an interval and f is an exponential type convex function on J.*

Proof Let $k \in [0,1]$ and $m, n \in J$ be arbitrary. Then

$$\begin{aligned}
f\big(km+(1-k)n\big) &= \sup_\alpha f_\alpha\big(km+(1-k)n\big)\\
&\le \sup_\alpha\big[\big(e^k-1\big)f_\alpha(m)+\big(e^{1-k}-1\big)f_\alpha(n)\big]\\
&\le \big(e^k-1\big)\sup_\alpha f_\alpha(m)+\big(e^{1-k}-1\big)\sup_\alpha f_\alpha(n)\\
&= \big(e^k-1\big)f(m)+\big(e^{1-k}-1\big)f(n) < \infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an exponential type convex function on J. □

Theorem 2.4 *If the function $f : [m,n] \to \mathbb{R}$ is exponential type convex, then f is bounded on $[m,n]$.*

Proof Let $K = \max\{f(m), f(n)\}$ and $x \in [m,n]$ be an arbitrary point. Then there exists $k \in [0,1]$ such that $x = km + (1-k)n$. Thus, since $e^k \le e$ and $e^{1-k} \le e$, we have

$$\begin{aligned} f(x) &\le f\bigl(km + (1-k)n\bigr) \\ &\le \bigl(e^k - 1\bigr)f(m) + \bigl(e^{1-k} - 1\bigr)f(n) \\ &\le \bigl(e^k + e^{1-k} - 2\bigr).K \\ &\le 2(e-1).K = M. \end{aligned}$$

Also, for every $x \in [m,n]$, there exists $\lambda \in [0, \frac{n-m}{2}]$ such that $x = \frac{m+n}{2} + \lambda$ or $x = \frac{m+n}{2} - \lambda$. Without loss of generality, we suppose $x = \frac{m+n}{2} + \lambda$. So, we have

$$\begin{aligned} f\left(\frac{m+n}{2}\right) &= f\left(\frac{1}{2}\left[\frac{m+n}{2} + \lambda\right] + \frac{1}{2}\left[\frac{m+n}{2} - \lambda\right]\right) \\ &\le (\sqrt{e} - 1)\left(f(x) + f\left(\frac{m+n}{2} - \lambda\right)\right). \end{aligned}$$

By using M as the upper bound, we get

$$\begin{aligned} f(x) &\ge f\left(\frac{m+n}{2}\right)\frac{1}{\sqrt{e}-1} - f\left(\frac{m+n}{2} - \lambda\right) \\ &\ge \frac{1}{\sqrt{e}-1} f\left(\frac{m+n}{2}\right) - M = m. \end{aligned}$$

□

3 Hermite–Hadamard inequality for exponential type convex functions

The aim of this section is to find some inequalities of H-H type for exponential type convex functions. In the next sections, we denote by $L[m,n]$ the space of (Lebesgue) integrable functions on the interval $[m,n]$.

Theorem 3.1 *Let $f : [m,n] \to \mathbb{R}$ be an exponential type convex function. If $m < n$ and $f \in L[m,n]$, then the following Hermite–Hadamard type inequalities hold:*

$$\frac{1}{2(\sqrt{e}-1)} f\left(\frac{m+n}{2}\right) \le \frac{1}{n-m}\int_m^n f(x)\,dx \le (e-2)\bigl[f(m) + f(n)\bigr]. \tag{5}$$

Proof Firstly, from the property of the exponential type convex function of f, we get

$$\begin{aligned} f\left(\frac{m+n}{2}\right) &= f\left(\frac{[km + (1-k)n] + [(1-k)m + kn]}{2}\right) \\ &= f\left(\frac{1}{2}\bigl[km + (1-k)n\bigr] + \frac{1}{2}\bigl[(1-k)m + kn\bigr]\right) \\ &\le \bigl(e^{\frac{1}{2}} - 1\bigr)f\bigl(km + (1-k)n\bigr) + \bigl(e^{1-\frac{1}{2}} - 1\bigr)f\bigl((1-k)m + kn\bigr). \end{aligned}$$

Now, if we take integral in the last inequality with respect to $k \in [0,1]$, we deduce that

$$\begin{aligned} f\left(\frac{m+n}{2}\right) &\leq \left[\left(e^{\frac{1}{2}}-1\right)\int_0^1 f\big(km+(1-k)n\big)\,dk + \left(e^{\frac{1}{2}}-1\right)\int_0^1 f\big((1-k)m+kn\big)\,dk\right] \\ &= \frac{2(\sqrt{e}-1)}{n-m}\int_m^n f(x)\,dx. \end{aligned}$$

Secondly, by using the property of the exponential type convex function f, if the variable is changed as $u = km + (1-k)n$, then

$$\begin{aligned} \frac{1}{n-m}\int_m^n f(u)\,du &= \int_0^1 f\big(km+(1-k)n\big)\,dk \\ &\leq \int_0^1 \big\{\big(e^k-1\big)f(m) + \big(e^{1-k}-1\big)f(n)\big\}\,dk \\ &= (e-2)\big[f(m)+f(n)\big]. \end{aligned}$$

□

4 Some new inequalities for exponential type convex functions

The aim of this section is to find new estimates that refine H-H inequality for functions whose first derivative in absolute value at certain power is exponential type convex. Dragomir and Agarwal [2] used the following lemma.

Lemma 4.1 ([2]) *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m,n \in I^\circ$ with $m < n$. If $f' \in L[m,n]$, then the following identity holds*:

$$\frac{f(m)+f(n)}{2} - \frac{1}{n-m}\int_m^n f(x)\,dx = \frac{n-m}{2}\int_0^1 (1-2k)f'\big(km+(1-k)n\big)\,dk.$$

Theorem 4.1 *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $m,n \in I^\circ$ with $m < n$, and assume that $f' \in L[m,n]$. If $|f'|$ is an exponential type convex function on $[m,n]$, then the inequality*

$$\left|\frac{f(m)+f(n)}{2} - \frac{1}{n-m}\int_m^n f(x)\,dx\right| \leq (n-m)\left(4\sqrt{e}-e-\frac{7}{2}\right)A\big(\big|f'(m)\big|,\big|f'(n)\big|\big)$$

holds for $k \in [0,1]$, where $A(u,v)$ is the arithmetic mean of u and v.

Proof From Lemma 4.1 and the inequality

$$\big|f'\big(km+(1-k)n\big)\big| \leq \big(e^k-1\big)\big|f'(m)\big| + \big(e^{1-k}-1\big)\big|f'(n)\big|,$$

we get

$$\begin{aligned} &\left|\frac{f(m)+f(n)}{2} - \frac{1}{n-m}\int_m^n f(x)\,dx\right| \\ &\quad\leq \frac{n-m}{2}\int_0^1 |1-2k|\big|f'\big(km+(1-k)n\big)\big|\,dk \end{aligned}$$

$$
\begin{aligned}
&\le \frac{n-m}{2}\int_0^1 |1-2k|\left[\left(e^k-1\right)\left|f'(m)\right|+\left(e^{1-k}-1\right)\left|f'(n)\right|\right]dk \\
&= \frac{n-m}{2}\left[\left|f'(m)\right|\int_0^1 |1-2k|\left(e^k-1\right)dk+\left|f'(n)\right|\int_0^1 |1-2k|\left(e^{1-k}-1\right)dk\right] \\
&= (n-m)\left(4\sqrt{e}-e-\frac{7}{2}\right)A\left(\left|f'(m)\right|,\left|f'(n)\right|\right),
\end{aligned}
$$

where

$$
\int_0^1 |1-2k|\left(e^k-1\right)dk=\int_0^1 |1-2k|\left(e^{1-k}-1\right)dk=4\sqrt{e}-e-\frac{7}{2}. \qquad \square
$$

Theorem 4.2 *Let $f: I \to \mathbb{R}$ be a differentiable function on I°, $m,n \in I^\circ$ with $m<n$, $q>1$, and assume that $f' \in L[m,n]$. If $|f'|^q$ is an exponential type convex function on $[m,n]$, then the inequality*

$$
\left|\frac{f(m)+f(n)}{2}-\frac{1}{n-m}\int_m^n f(x)\,dx\right| \le \frac{n-m}{2}\left[2(e-2)\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}A^{\frac{1}{q}}\left(\left|f'(m)\right|^q,\left|f'(n)\right|^q\right)
$$

holds for $k \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$ and A is the arithmetic mean.

Proof From Lemma 4.1, Hölder's integral inequality, and the following inequality:

$$
\left|f'\left(km+(1-k)n\right)\right|^q \le \left(e^k-1\right)\left|f'(m)\right|^q+\left(e^{1-k}-1\right)\left|f'(n)\right|^q,
$$

which is the exponential type convex function of $|f'|^q$, we get

$$
\begin{aligned}
&\left|\frac{f(m)+f(n)}{2}-\frac{1}{n-m}\int_m^n f(x)\,dx\right| \\
&\quad\le \frac{n-m}{2}\int_0^1 |1-2k|\left|f'\left(km+(1-k)n\right)\right|dk \\
&\quad\le \frac{n-m}{2}\left(\int_0^1 |1-2k|^p\,dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(km+(1-k)n\right)\right|^q dk\right)^{\frac{1}{q}} \\
&\quad\le \frac{n-m}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left[\left(e^k-1\right)\left|f'(m)\right|^q+\left(e^{1-k}-1\right)\left|f'(n)\right|^q\right]dk\right)^{\frac{1}{q}} \\
&\quad= \frac{n-m}{2}\left[2(e-2)\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}A^{\frac{1}{q}}\left(\left|f'(m)\right|^q,\left|f'(n)\right|^q\right). \qquad \square
\end{aligned}
$$

Theorem 4.3 *Let $f: I \to \mathbb{R}$ be a differentiable function on I°, $m,n \in I^\circ$ with $m<n$, $q \ge 1$, and assume that $f' \in L[m,n]$. If $|f'|^q$ is an exponential type convex function on $[m,n]$, then the inequality*

$$
\begin{aligned}
&\left|\frac{f(m)+f(n)}{2}-\frac{1}{n-m}\int_m^n f(x)\,dx\right| \\
&\quad\le \frac{n-m}{2^{2-\frac{1}{q}}}\left[2\left(4\sqrt{e}-e-\frac{7}{2}\right)\right]^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(m)\right|^q,\left|f'(n)\right|^q\right) \qquad (6)
\end{aligned}
$$

holds for $k \in [0,1]$.

Proof Assume first that $q > 1$. By using Lemma 4.1, Hölder's inequality, and the property of the exponential type convex function of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(m)+f(n)}{2} - \frac{1}{n-m}\int_m^n f(x)\,dx\right| \\
&\quad\le \frac{n-m}{2}\int_0^1 |1-2k|\bigl|f'\bigl(km+(1-k)n\bigr)\bigr|\,dk \\
&\quad\le \frac{n-m}{2}\left(\int_0^1 |1-2k|\,dk\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2k|\bigl|f'\bigl(km+(1-k)n\bigr)\bigr|^q\,dk\right)^{\frac{1}{q}} \\
&\quad= \frac{n-m}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2k|\bigl[\bigl(e^k-1\bigr)\bigl|f'(m)\bigr|^q + \bigl(e^{1-k}-1\bigr)\bigl|f'(n)\bigr|^q\bigr]\,dk\right)^{\frac{1}{q}} \\
&\quad= \frac{n-m}{2^{2-\frac{1}{q}}}\left[2\left(4\sqrt{e}-e-\frac{7}{2}\right)\right]^{\frac{1}{q}} A^{\frac{1}{q}}\bigl(\bigl|f'(m)\bigr|^q,\bigl|f'(n)\bigr|^q\bigr).
\end{aligned}$$

For $q = 1$, we consider the estimates from the proof of Theorem 4.1, which also follows step by step the above estimates. □

Remark 4.1 Under the assumptions of Theorem 4.3 with $q = 1$, we get the conclusion of Theorem 4.1.

5 Applications for special means

Throughout this section, for the sake of simplicity, the following notations are used for special means of two nonnegative numbers m, n ($n > m$):

1. The arithmetic mean

$$A := A(m,n) = \frac{m+n}{2}, \quad m,n \ge 0.$$

2. The geometric mean

$$G := G(m,n) = \sqrt{mn}, \quad m,n \ge 0.$$

3. The harmonic mean

$$H := H(m,n) = \frac{2mn}{m+n}, \quad m,n > 0.$$

4. The logarithmic mean

$$L := L(m,n) = \begin{cases} \frac{n-m}{\ln n - \ln m}, & m \ne n, \\ m, & m = n; \end{cases} \quad m,n > 0.$$

5. The p-logarithmic mean

$$L_p := L_p(m,n) = \begin{cases} \left(\frac{n^{p+1}-m^{p+1}}{(p+1)(n-m)}\right)^{\frac{1}{p}}, & m \ne n, p \in \mathbb{R}\setminus\{-1,0\}, \\ m, & m = n; \end{cases} \quad m,n > 0.$$

6. The identric mean

$$I := I(m,n) = \frac{1}{e}\left(\frac{n^n}{m^m}\right)^{\frac{1}{n-m}}, \quad m,n > 0.$$

It is well known that L_p is monotonically increasing over $p \in \mathbb{R}$. Moreover, $L_0 = I, L_{-1} = L$.

Proposition 5.1 *Let $m,n \in [0,\infty)$ with $m < n$ and $r \in (-\infty,0) \cup [1,\infty) \setminus \{-1\}$. Then the inequalities*

$$\frac{A^r(m,n)}{2(\sqrt{e}-1)} \le L_r^r(m,n) \le 2(e-2)A\left(m^r,n^r\right)$$

hold.

Proof It is easily seen from inequalities (5) for the function

$$f(x) = x^r, \quad x \in [0,\infty).$$ □

Proposition 5.2 *Let $m,n \in (0,\infty)$ with $m < n$. Then the inequalities*

$$\frac{A^{-1}(m,n)}{2(\sqrt{e}-1)} \le L^{-1}(m,n) \le 2(e-2)H^{-1}(m,n)$$

hold.

Proof It is easily seen from inequalities (5) for the function

$$f(x) = x^{-1}, \quad x \in (0,\infty).$$ □

Proposition 5.3 *Let $m,n \in (0,1]$ with $m < n$. Then the inequalities*

$$2(e-2)\ln G(m,n) \le \ln I(m,n) \le \frac{\ln A(m,n)}{2(\sqrt{e}-1)}$$

hold.

Proof It is easily seen from inequalities (5) for the function

$$f(x) = -\ln x, \quad x \in (0,1].$$ □

6 Conclusion

In this paper, we studied the concept of exponential type convex functions, which is a new concept. We proved some new Hermite–Hadamard type integral inequalities for the newly introduced class of functions using an identity together with Hölder's integral inequality. Especially, we would like to emphasize that different types of integral inequalities can be obtained using this new definition.

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