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A new error estimate on uniform norm of a parabolic variational inequality with nonlinear source terms via the subsolution concepts

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This presented work is in memory of the first author's father (1910–1999) Mr. Mahmoud ben Mouha Boulaaras

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Abstract

This paper deals with the numerical analysis of parabolic variational inequalities with nonlinear source terms, where the existence and uniqueness of the solution is provided by using Banach's fixed point theorem. In addition, an optimally L^∞-asymptotic behavior is proved using Euler time scheme combined with the finite element spatial approximation. The approach is based on Bensoussan–Lions algorithm for evolutionary free boundary problems using the concepts of subsolutions.

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Keywords: PVIs; Nonlinear source terms; Subsolution concepts; Existence and uniqueness; Bensoussan–Lions algorithm

1 Introduction

We consider the following parabolic variational inequality: Find $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u - f(u) \leq 0; \quad u \leq \psi, \\ (\frac{\partial u}{\partial t} + \mathcal{L}u - f(u))(u-\psi) = 0 \quad \text{in } Q_T :=]0,T[\times \Omega, \\ u = 0 \quad \text{on } \sum_T :=]0,T[\times \partial\Omega, \\ u(0,\cdot) = u_0, \quad \text{on } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth and regular domain of $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\Omega$; the $f(\cdot)$ and $u_0 = u_0(x)$ are given data; the ψ is a regular function in $L^2(0,T;W^{2,\infty}(\Omega))$, and the $\mathcal{L}$ is a second-order, uniformly elliptic operator of the form

$$\mathcal{L} = -\sum_{j,k=1}^{d} a_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + a_0(x), \tag{1.2}$$

Parabolic variational inequality (1.1) has arisen from many scientific, engineering, and economic problems such as heat control problem, Stefan problem, and American option problem (see [3, 5–13, 16, 18, 19, 21]).

In this paper, we give an L^∞-error estimate for the numerical approximation of the solution of problem (1.1). From [2] (see also [8]), we know that (1.1) can be approximated by the following parabolic variational inequality with nonlinear source terms (PVI): Find $u(x,t)$ such that $u \in L^2(0,T;H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega))$, and

$$\begin{cases} (\frac{\partial u}{\partial t}, v-u) + a(u,v-u) \geq (f(u), v-u), & \text{for all } v \in H_0^1(\Omega), t \in (0,T], \\ u \leq \psi, \qquad v \leq \psi, \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.3}$$

where $a(\cdot,\cdot)$ is a bilinear form continuous on $H^1(\Omega) \times H^1(\Omega)$ corresponding to elliptic operator $\mathcal{L}$ of second order defined as follows:

$$a(u,v) = \int_\Omega \left(\sum_{j,k=1}^{d} a_{jk}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^{d} b_k(x) \frac{\partial u}{\partial x_k} v + a_0(x) uv \right) dx, \tag{1.4}$$

with $a_{jk}(\cdot)$, $b_j(\cdot)$, $a_0(\cdot)$, smooth coefficients satisfying the following conditions:

$$\begin{cases} a_{jk}(x) = a_{kj}(x), \\ a_0(x) \geq \beta > 0, \quad \beta \text{ is a constant}, \end{cases} \tag{1.5}$$

and for each $\xi \in \mathbb{R}^d$ and for almost every $x \in \Omega$,

$$\sum_{j,k=1}^{d} a_{jk}(x) \xi_j \xi_k \geq \alpha_0 |\xi|^2 \quad \text{with constant } \alpha_0 > 0. \tag{1.6}$$

According to Theorem 2.3 in [8], there exists $\gamma > 0$ such that

$$a(\varphi,\varphi) + \lambda \|\varphi\|^2_{L^2(\Omega)} \geq \gamma \|\varphi\|^2_{H_0^1(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega), \text{with } \gamma > 0. \tag{1.7}$$

The function $f(\cdot)$ is a nondecreasing and Lipschitz continuous nonlinearity

$$f \in L^2\big(0,T;L^\infty(\Omega)\big) \cap C^1\big(0,T;H^{-1}(\Omega)\big), \quad f \geq 0, \tag{1.8}$$

with Lipschitz constant $\alpha > 0$, satisfying the following assumption:

$$\alpha < \beta, \tag{1.9}$$

where β is the constant defined in (1.3). The symbol $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$.

Error estimates for piecewise linear finite element approximations of parabolic variational inequalities with linear source terms have been established in various papers: in [20] and [9] an L^2-error estimate is given by using a backward differencing in time. Also an L^2-error estimate is given in [23] by using a general finite difference discretization in time. Reference [4] gives an L^2-error estimate using the discretized truncation method. In [1] and [22] a posteriori error estimates have been proved. An L^∞-error estimate has been proved in [15] and [17]. Recently an L^∞-asymptotic behavior has been considered in [2] by using a semi-implicit time scheme combined with the finite element spatial approximation.

In this paper, we introduce a new approach to derive optimal L^∞-asymptotic behavior for parabolic variational inequality with nonlinear source terms. This approach is based on Bensoussan–Lions algorithm for evolutionary free boundary problems using the concepts of subsolutions.

The paper is organized as follows. In Sect. 2, we state the continuous problem and study some qualitative properties. In Sect. 3, we consider the discrete problem and set up analogous discrete qualitative properties. In Sect. 4, we derive an L^∞-error estimate of the approximation and we give the main result of the paper.

2 Semi continuous problem

2.1 Time discretization

In order to obtain a full discretization of (1.3), we consider a uniform mesh for the time variable t and define

$$t_n = n\Delta t, \quad n = 0, 1, \ldots, \mathcal{N}, \tag{2.1}$$

$\Delta t > 0$ being the time-step, and $\mathcal{N} = [\frac{T}{\Delta t}]$, the integral part of $\frac{T}{\Delta t}$.

Next, we replace the time derivative by means of suitable difference quotients, thus constructing a sequence $u^n(x) \in H_0^1(\Omega)$ that approaches $u(t_n, x)$. For simplicity, we confine ourselves to the so-called semi-implicit scheme, which consists in replacing (1.3) by the following scheme: Find $u^n \in H_0^1(\Omega)$ such that

$$\begin{cases} (\frac{1}{\Delta t}(u^n - u^{n-1}), v - u^n) + a(u^n, v - u^n) \geq (f(u^n), v - u^n), \quad v \in H_0^1(\Omega), \\ u^n \leq \psi, \qquad v \leq \psi, \quad n = 1, \ldots, \mathcal{N}, \end{cases} \tag{2.2}$$

where

$$\Delta t = \frac{T}{\mathcal{N}}. \tag{2.3}$$

By adding $(\frac{u^{n-1}}{\Delta t}, v - u^n)$ to both parties of inequalities (2.2), we get

$$\begin{cases} \frac{1}{\Delta t}(u^n, v - u^n) + a(u^n, v - u^n) \geq (f(u^n) + \frac{1}{\Delta t}u^{n-1}, v - u^n), \\ u^n \leq \psi, \qquad v \leq \psi, \quad n = 0, 1, \ldots, \mathcal{N}. \end{cases} \tag{2.4}$$

As the bilinear form $a(\cdot,\cdot)$ is noncoercive in $H_0^1(\Omega)$.

Set

$$b(u, v) = a(u, v) + \lambda(u, v). \tag{2.5}$$

Then the bilinear form $b(u,v)$ is an elliptic, and therefore (2.4) can be written as the following coercive elliptic variational inequalities: Find $u^n \in H_0^1(\Omega)$ such that

$$\begin{cases} b(u^n, v-u^n) \geq (f(u^n) + \lambda u^{n-1}, v-u^n), & v \in H_0^1(\Omega), \\ u^n \leq \psi, \quad v \leq \psi, \quad n = 0,1,\dots,\mathcal{N}, \end{cases} \tag{2.6}$$

where

$$\begin{cases} b(u^n, v-u^n) = a(u^n, v-u^n) + \lambda(u^n, v-u^n), & v \in H_0^1(\Omega), \\ \lambda = \frac{1}{\Delta t} > 0. \end{cases} \tag{2.7}$$

Remark 1 Equation (2.6) is called the coercive continuous problem of elliptic variational inequalities (VI).

Notation 1 We denote by $u^n = \partial(f(u^n), \psi)$ the solution of problem (2.6).

2.2 Existence and uniqueness

Next, using the preceding assumptions, we prove the existence of a unique solution for problem (2.6) by means of Banach's fixed point theorem.

2.2.1 A fixed point mapping associated with continuous problem (2.6)

We consider the following mapping:

$$\begin{aligned} &\mathbb{T} : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \\ &\quad w \rightarrow \mathbb{T}w = \zeta^n, \end{aligned} \tag{2.8}$$

where $\xi^n = \sigma(f(w), \psi)$ is the solution to the following variational inequalities:

$$\begin{cases} b(\zeta^n, v-\zeta^n) \geq (f(w) + \lambda w, v-\zeta^n), & v \in H_0^1(\Omega), \\ \zeta^n \leq \psi, \quad v \leq \psi. \end{cases} \tag{2.9}$$

Problem (2.10) being a coercive VI, thanks to [3] and [10], has one and only one solution.

Theorem 1 *Under the preceding hypotheses and notation, the mapping $\mathbb{T}$ is a contraction in $L^\infty(\Omega)$ with a contraction constant $(\frac{\alpha\Delta t+1}{\beta\Delta t+1})$. Therefore, $\mathbb{T}$ admits a unique fixed point which coincides with the solution of problem* (2.6).

Proof In [13], by taking $\lambda = \frac{1}{\Delta t}$, we can easily get

$$\|\mathbb{T}w - \mathbb{T}\tilde{w}\|_\infty \leq \left(\frac{\alpha\Delta t + 1}{\beta\Delta t + 1}\right)\|w - \tilde{w}\|_\infty. \qquad \square$$

The mapping $\mathbb{T}$ clearly generates the following continuous algorithm.

2.3 A continuous iterative scheme

A continuous iterative scheme for the solution of problem (2.6) is given as follows.

Starting from $u^0 = u_0$ the solution of the following equation:

$$b(u_0, v) = \big(f(u_0) + \lambda u_0, v\big), \quad \forall v \in H_0^1(\Omega). \tag{2.10}$$

Now, we give the following algorithm:

$$u^n = \mathbb{T}u^{n-1}, \quad n = 1, \ldots, \mathcal{N} - 1, \tag{2.11}$$

where u^n is the solution to (2.6).

Making use of the propriety of mapping $\mathbb{T}$, we have the following geometric convergence result.

Proposition 1 *Let* $\rho = \frac{\alpha \Delta t + 1}{\beta \Delta t + 1}$, *under conditions of Theorem* 1, *we have*

$$\|u^n - u^\infty\|_\infty \le \rho^n \|u_0 - u^\infty\|_\infty, \tag{2.12}$$

where u^∞ *is the asymptotic solution of the problem of variational inequalities*: *Find* $u^\infty \in H_0^1(\Omega)$ *such that*

$$\begin{cases} b(u^\infty, v - u^\infty) \ge (f(u^\infty) + \lambda u^\infty, v - u^\infty), & v \in H_0^1(\Omega), \\ u^\infty \le \psi, \qquad v \le \psi. \end{cases} \tag{2.13}$$

Proof We adapt [2]. □

In what follows, we give some qualitative properties of the solution of problem (2.6).

2.4 Some qualitative properties of the solution of (2.6)

The solution u^n of (2.6) possesses the following properties.

2.4.1 A monotonicity property

Let $u^n = \partial(F(u^n), \psi)$ (resp. $\tilde{u}^n = \partial(\tilde{F}(\tilde{u}^n), \tilde{\psi})$) be the solution of problem (2.6) with right-hand side $F(u^n) = f(u^n) + \lambda u^{n-1}$ (resp. $\tilde{F}(\tilde{u}^n) = \tilde{f}(\tilde{u}^n) + \lambda \tilde{u}^{n-1}$). Then we have the following.

Lemma 1 (cf. [6] and [10]) *If* $F(u^n) \ge \tilde{F}(\tilde{u}^n)$ *and* $\psi \ge \tilde{\psi}$, *then*

$$\partial\big(F(u^n), \psi\big) \ge \partial\big(\tilde{F}(\tilde{u}^n), \tilde{\psi}\big). \tag{2.14}$$

2.4.2 A continuous L^∞-stability property

Proposition 2 *Under conditions of Lemma* 1, *we have*

$$\big\|\partial\big(F(u^n), \psi\big) - \partial\big(\tilde{F}(\tilde{u}^n), \tilde{\psi}\big)\big\|_\infty \le \frac{1}{\beta}\big\|f(u^n) - \tilde{f}(\tilde{u}^n)\big\|_\infty. \tag{2.15}$$

Proof Let

$$\phi = \frac{1}{\lambda + \beta}\big\|f(u^n) - \tilde{f}(\tilde{u}^n)\big\|_\infty.$$

Then, from (1.5), it is easy to see that

$$\tilde{F}(\tilde{u}^n) \le F(u^n) + \frac{\lambda + a_0}{\lambda + \beta}\|f(u^n) - \tilde{f}(\tilde{u}^n)\|_\infty = F(u^n) + (\lambda + a_0).\phi.$$

So, due to Lemma 1, it follows that

$$\partial(\tilde{F}(\tilde{u}^n), \tilde{\psi}) \le \partial(F(u^n) + (\lambda + a_0).\phi, \psi + \phi) \le \partial(F(u^n), \psi) + \phi,$$

hence

$$\partial(\tilde{F}(\tilde{u}^n), \tilde{\psi}) - \partial(F(u^n), \psi) \le \phi.$$

Interchanging the role of $F(u^n)$ and $\tilde{F}^n$, we also get

$$\partial(F(u^n), \psi) - \partial(\tilde{F}^n, \tilde{\psi}) \le \phi.$$

Then, from (2.8), it is easy to see that

$$\begin{aligned}
\|\partial(F(u^n), \psi) - \partial(\tilde{F}(\tilde{u}^n), \tilde{\psi})\|_\infty &\le \frac{\Delta t}{\beta\Delta t + 1}\|f(u^n) - \tilde{f}(\tilde{u}^n)\|_\infty \\
&\le \frac{1}{\beta(1 + \frac{1}{\beta\Delta t})}\|f(u^n) - \tilde{f}(\tilde{u}^n)\|_\infty \\
&\le \frac{1}{\beta}\|f(u^n) - \tilde{f}(\tilde{u}^n)\|_\infty,
\end{aligned}$$

which completes the proof. □

2.4.3 The concept of continuous subsolution property

Definition 1 $z^n \in H_0^1(\Omega)$ is said to be a continuous subsolution for the problem of VI (2.6) if

$$\begin{cases} b(z^n, v) \le (f(z^n) + \lambda z^n, v), & v \in H_0^1(\Omega), \\ z^n \le \psi, \qquad v \ge 0, & n = 1, \dots, \mathcal{N} - 1. \end{cases} \tag{2.16}$$

Theorem 2 (cf. [6]) *Let $\mathbb{X}$ denote the set of such subsolutions, then the solution of* (2.6) *is the least upper bound of* $\mathbb{X}$.

3 The discrete problem

Let Ω be decomposed into triangles, and let τ_h denote the set of all those elements; $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider ϕ_l, $l = 1, 2, \dots, m(h)$, the usual basis of affine functions defined by $\phi_l(M_s) = \delta_{l,s}$, where M_s is a vertex of the considered triangulation.

Let us $\mathbb{V}_h$ denote the standard piecewise linear finite element space such that

$$\mathbb{V}_h = \left\{ \begin{array}{l} v_h \in C^0(\bar{\Omega}), v_h = 0 \text{ on } \partial\Omega \text{ such that:} \\ v_h|_{K^i} \in P_1, K \in \tau_h, v_h \le r_h\psi, v_h(\cdot, 0) = v_{0h} \text{ in } \Omega \end{array} \right\}. \tag{3.1}$$

The interpolation operator is applied to the function v continuous, defined by

$$r_h v = \sum_{l=1}^{m(h)} v(M_l)\phi_l(x) \tag{3.2}$$

and $\mathbb{B}$ is the matrix with generic entries

$$(\mathbb{B})_{l,s} = a(\phi_l, \phi_s), \quad 1 \leq l, s \leq m(h). \tag{3.3}$$

In the sequel of the paper, we use the discrete maximum assumption (d.m.p.). In other words, we assume that the matrix $\mathbb{B}$ is an M-matrix (cf. [14]).

Remark 2 Under the d.m.p., we achieve a similar study to that devoted to the continuous problem; therefore the qualitative properties and results stated in the continuous case are conserved in the discrete case.

As in the continuous situation, one can tackle the discrete problem by considering the equivalent formulation: Find $u_h^n \in \mathbb{V}_h$ such that

$$\begin{cases} b(u_h^n, v_h - u_h^n) \geq (f(u_h^n) + \lambda u_h^n, v_h - u_h^n), & v_h \in \mathbb{V}_h, \\ u_h^n \leq r_h\psi, \qquad v_h \leq r_h\psi. \end{cases} \tag{3.4}$$

Notation 2 We denote by $u_h^n = \partial_h(f^n(u_h^n), r_h\psi)$ the solution of problem (3.4).

Existence and uniqueness of a solution of problem (3.4) can be shown similarly to that of the continuous case provided the discrete maximum principle is satisfied.

3.1 Existence and uniqueness

3.1.1 A fixed point mapping associated with discrete problem (3.4)

We consider the following mapping:

$$\begin{aligned} &\mathbb{T} : L^\infty(\Omega) \longrightarrow \mathbb{V}_h, \\ &\quad w \rightarrow \mathbb{T}_h w = \xi_h^n, \end{aligned} \tag{3.5}$$

where $\xi_h^n = \sigma_h(f^n(w), r_h\psi)$ is a solution of the following discrete coercive VI:

$$\begin{cases} b(\xi_h^n, v - \xi_h^n) \geq (f(w) + \lambda w, v - \xi_h^n), & v_h \in \mathbb{V}_h, \\ \xi_h^n \leq r_h\psi, \qquad v \leq r_h\psi. \end{cases} \tag{3.6}$$

Theorem 3 *Under the d.m.p. assumption and the preceding hypotheses and notation, the mapping $\mathbb{T}_h$ is a contraction in $L^\infty(\Omega)$ with a contraction constant $(\frac{\alpha\Delta t+1}{\beta\Delta t+1})$. Therefore, $\mathbb{T}_h$ admits a unique fixed point which coincides with the solution of problem* (3.4).

As in the continuous situation, one can define the following discrete iterative scheme.

3.2 A discrete iterative scheme

A discrete iterative scheme for the solution of problem (3.4) is given as follows.

Starting from $u_h^0 = u_{0,h}$, the solution of the following equation:

$$b(u_{0,h}, v_h) = \big(f(u_{0,h}) + \lambda u_{0,h}, v_h\big), \quad v_h \in \mathbb{V}_h. \tag{3.7}$$

Now, we give the following algorithm:

$$u_h^n = \mathbb{T}_h u_h^{n-1}, \quad n = 1, \dots, \mathcal{N} - 1, \tag{3.8}$$

where u_h^n is a solution of problem (3.4).

Using the above result, we are able to establish the following geometric convergence of sequence u_h^n.

Proposition 3 *Let $\rho = \frac{\alpha\Delta t+1}{\beta\Delta t+1}$, under the d.m.p. assumption and Theorem 3, we have*

$$\big\|u_h^n - u_h^\infty\big\|_\infty \le \rho^n \big\|u_{h,0} - u_h^\infty\big\|_\infty, \tag{3.9}$$

where u_h^∞ is the asymptotic solution of problem of variational inequalities: Find $u_h^\infty \in \mathbb{V}_h$ such that

$$\begin{cases} b(u_h^\infty, v - u_h^\infty) \ge (f(u_h^\infty) + \lambda u_h^\infty, v_h - u_h^\infty), & v_h \in \mathbb{V}_h, \\ u_h^\infty \le r_h\psi, \qquad v_h \le r_h\psi. \end{cases} \tag{3.10}$$

Proof It is very similar to that of the continuous case. □

Under the d.m.p., the solution of discrete problem (3.4) possesses analogous properties to those of the continuous problem.

3.3 Some qualitative properties of the solution of (3.4)

As in the continuous situation, the solution u_h^n of system (3.4) possesses the following properties.

3.3.1 A monotonicity property

Let $u_h^n = \partial_h(F^n, r_h\psi)$ (resp. $\tilde{u}_h^n = \partial_h(\tilde{F}^n, r_h\tilde{\psi})$) the solution to (3.4) with right-hand side F^n.

Lemma 2 *If $F^n \ge \tilde{F}^n$ and $\psi \ge \tilde{\psi}$, then*

$$\partial_h\big(F^n, r_h\psi\big) \ge \partial_h\big(\tilde{F}^n, r_h\tilde{\psi}\big). \tag{3.11}$$

3.3.2 A discrete L^∞-stability

Proposition 4 *Under the d.m.p. assumption and conditions of Lemma 2, we have*

$$\big\|\partial_h\big(F(u^n), r_h\psi\big) - \partial_h\big(\tilde{F}(\tilde{u}^n), r_h\tilde{\psi}\big)\big\|_\infty \le \frac{1}{\beta}\big\|f\big(u_h^n\big) - \tilde{f}\big(\tilde{u}_h^n\big)\big\|_\infty. \tag{3.12}$$

Proof It is very similar to that of the continuous case. □

3.3.3 The concept of discrete subsolution

Definition 2 $z_h^n \in \mathbb{V}_h$ is said to be a discrete subsolution for the system of quasi-variational inequalities (3.4) if

$$\begin{cases} b(z_h, \varphi_s) \le (f(z_h) + \lambda z_h, \varphi_s), & \forall s = 1, \dots, m(h), \\ z_h \le r_h \psi, \qquad \varphi_s \ge 0. \end{cases} \tag{3.13}$$

Theorem 4 *Let $\mathbb{X}_h$ be the set of such subsolutions, then under the d.m.p., the solution of* (3.4) *is the least upper bound of the set $\mathbb{X}_h$.*

4 Finite element error analysis

This section is devoted to deriving an error estimate, in the maximum norm, between the nth iterates u^n and their finite element counterpart u_h^n. For that we first introduce two auxiliary sequences.

4.1 Two auxiliary sequences

4.1.1 A discrete sequence

We define the following discrete sequence $\{\bar{u}_h^n\}_{n\ge 1}$, where $\bar{u}_h^n$ is a solution to the following discrete problem of variational inequalities (VI):

$$\begin{cases} b(\bar{u}_h^n, v_h - \bar{u}_h^n) \ge (f(u^n) + \lambda.u^n, v_h - \bar{u}_h^n), & v_h \in \mathbb{V}_h, \\ \bar{u}_h^n \le r_h \psi, \qquad v_h \le r_h \psi, \end{cases} \tag{4.1}$$

where u^n is the solution to (2.6).

Lemma 3 (cf. [13]) *There exists a constant C independent of h, n, and Δt such that*

$$\left\| \bar{u}_h^n - u^n \right\|_\infty \le C h^2 |\log h|^2. \tag{4.2}$$

Proposition 5 *There exists a sequence of discrete subsolutions $\{\alpha_h^n\}_{n\ge 1}$ such that*

$$\begin{cases} \alpha_h^n \le u_h^n, \\ \textit{and} \\ \|\alpha_h^n - u^n\|_\infty \le C h^2 |\log h|^2, \end{cases} \tag{4.3}$$

where the constant C is independent of h, Δt, and n.

Proof For $n = 1$, we consider the discrete problem of VI:

$$\begin{cases} b(\bar{u}_h^1, v_h - \bar{u}_h^1) \ge (f(u_0) + \lambda u_0, v_h - \bar{u}_h^1), & v_h \in \mathbb{V}_h, \\ \bar{u}_h^1 \le r_h \psi, \qquad v_h \le r_h \psi. \end{cases}$$

Then as $\bar{u}_h^1$ is a solution to a discrete VI, it is also a subsolution, i.e.,

$$\begin{cases} b(\bar{u}_h^1, \varphi_s) \le (f(u_0) + \lambda u_0, \varphi_s), & \forall \varphi_s, \\ \bar{u}_h^1 \le r_h \psi \end{cases}$$

or

$$\begin{cases} b(\bar{u}_h^1, \varphi_s) \le (f(u_0) + f(u_{0,h}) - f(u_{0,h}) + \lambda u_0 - \lambda u_{0,h} + \lambda u_{0,h}, \varphi_s), \\ \bar{u}_h^1 \le r_h \psi. \end{cases}$$

Then

$$\begin{cases} b(\bar{u}_h^1, \varphi_s) \le (f(u_{0,h}) + \|f(u_0) - f(u_{0,h})\|_\infty + \lambda \|u_0 - u_{0,h}\|_\infty + \lambda u_{0,h}, \varphi_s), \\ \bar{u}_h^1 \le r_h \psi. \end{cases}$$

Using the Lipschitz continuity of $f(\cdot)$, we have

$$\begin{cases} b(\bar{u}_h^1, \varphi_s) \le (f(u_{0,h}) + \alpha \|u_0 - u_{0,h}\|_\infty + \lambda \|u_0 - u_{0,h}\|_\infty + \lambda u_{0,h}, \varphi_s), \\ \bar{u}_h^1 \le r_h \psi. \end{cases}$$

On the other hand, due to [11]

$$\|u_0 - u_{0,h}\|_\infty \le C h^2 |\log h|.$$

Then

$$\begin{cases} b(\bar{u}_h^1, \varphi_s) \le (f(u_{0,h}) + C h^2 |\log h| + \lambda u_{0,h}, \varphi_s), \\ \bar{u}_h^1 \le r_h \psi. \end{cases}$$

So, $\bar{u}_h^1$ is a discrete subsolution for the VI whose solution is $\bar{U}_h^1 = \partial_h(f(u_{0,h}) + C h^2 |\log h|, r_h \psi)$. Then $u_h^1 = \partial_h(f(u_{0,h}), r_h \psi)$, and making use of Proposition 4, we have

$$\begin{aligned} \|\bar{U}_h^1 - u_h^1\|_\infty &\le \frac{1}{\beta} \|f(u_{0,h}) + C h^2 |\log h| - f(u_{0,h})\|_\infty \\ &\le C h^2 |\log h|. \end{aligned}$$

Hence, making use of Theorem 4, we have

$$\bar{u}_h^1 \le \bar{U}_h^1 \le u_h^1 + C h^2 |\log h|.$$

Putting

$$\alpha_h^1 = \bar{u}_h^1 - C h^2 |\log h|,$$

we get

$$\alpha_h^1 \le u_h^1$$

and

$$\begin{aligned} \|\alpha_h^1 - u^1\|_\infty &= \|\bar{u}_h^1 - C h^2 |\log h| - u^1\|_\infty \\ &\le \|\bar{u}_h^1 - u^1\|_\infty + C h^2 |\log h|. \end{aligned}$$

Using Lemma 3, we get

$$\left\|\alpha_h^1 - u^1\right\|_\infty \le Ch^2|\log h|^2 + Ch^2|\log h|.$$

For $n+1$, let us now assume that

$$\begin{cases} \alpha_h^n \le u_h^n, \\ \text{and} \\ \|\alpha_h^n - u^n\|_\infty \le Ch^2|\log h|^2, \end{cases}$$

and we consider the discrete problem

$$\begin{cases} b(\bar{u}_h^{n+1}, v_h - \bar{u}_h^n) \ge (f(u^n) + \lambda u^n, v_h - \bar{u}_h^{n+1}), & v_h \in \mathbb{V}_h, \\ \bar{u}_h^{n+1} \le r_h\psi, & v_h \le r_h\psi. \end{cases}$$

Then

$$\begin{cases} b(\bar{u}_h^{n+1}, \varphi_s) \le (f(u^n) + \lambda u^n, \varphi_s), & \forall \varphi_s, \\ \bar{u}_h^{n+1} \le r_h\psi \end{cases}$$

or

$$\begin{cases} b(\bar{u}_h^{n+1}, \varphi_s) \le (f(u^n) + f(\bar{u}_h^n) - f(\bar{u}_h^n) + \lambda u^n - \lambda\bar{u}_h^n + \lambda\bar{u}_h^n, \varphi_s), \\ \bar{u}_h^{n+1} \le r_h\psi. \end{cases}$$

Then

$$\begin{cases} b(\bar{u}_h^{n+1}, \varphi_s) \le (f(\bar{u}_h^n) + \|f(u^n) - f(\bar{u}_h^n)\|_\infty + \lambda\|u^n - \bar{u}_h^n\|_\infty + \lambda\bar{u}_h^n, \varphi_s), \\ \bar{u}_h^{n+1} \le r_h\psi. \end{cases}$$

Using the Lipschitz continuity of $f(\cdot)$, we have

$$\begin{cases} b(\bar{u}_h^{n+1}, \varphi_s) \le (f(\bar{u}_h^n) + \alpha\|u^n - \bar{u}_h^n\|_\infty + \lambda\|u^n - \bar{u}_h^n\|_\infty + \lambda\bar{u}_h^n, \varphi_s), \\ \bar{u}_h^{n+1} \le r_h\psi. \end{cases}$$

Using (4.2), we have

$$\begin{cases} b(\bar{u}_h^{n+1}, \varphi_s) \le (f(\bar{u}_h^n) + Ch^2|\log h|^2 + \lambda\bar{u}_h^n, \varphi_s), \\ \bar{u}_h^{n+1} \le r_h\psi. \end{cases}$$

So, $\bar{u}_h^{n+1}$ is a discrete subsolution for the VI whose solution is $\bar{U}_h^{n+1} = \partial_h(f(\bar{u}_h^n) + Ch^2|\log h|^2, r_h\psi)$. Then $u_h^{n+1} = \partial_h(f(\bar{u}_h^n), r_h\psi)$, making use of Proposition 4, we have

$$\begin{aligned} \left\|\bar{u}_h^{n+1} - u_h^{n+1}\right\|_\infty &\le \frac{1}{\beta}\left\|f(\bar{u}_h^n) + Ch^2|\log h|^2 - f(\bar{u}_h^n)\right\|_\infty \\ &\le Ch^2|\log h|^2. \end{aligned}$$

Hence, applying Theorem 4, we get

$$\bar{u}_h^{n+1} \le u_h^{n+1} + Ch^2|\log h|^2.$$

Putting

$$\alpha_h^{n+1} = \bar{u}_h^{n+1} - Ch^2|\log h|^2,$$

we get

$$\alpha_h^{n+1} \le u_h^{n+1}$$

and

$$\begin{aligned} \left\|\alpha_h^{n+1} - u^{n+1}\right\|_\infty &= \left\|\bar{u}_h^{n+1} - Ch^2|\log h|^2 - u^{n+1}\right\|_\infty \\ &\le \left\|\bar{u}_h^{n+1} - u^{n+1}\right\|_\infty + Ch^2|\log h|^2. \end{aligned}$$

Using Lemma 3, we obtain

$$\left\|\alpha_h^{n+1} - u^{n+1}\right\|_\infty \le Ch^2|\log h|^2,$$

which completes the proof. □

4.1.2 A continuous sequence

We define the following continuous sequence $\{\bar{u}_{(h)}^n\}_{n\ge1}$, where $\bar{u}_{(h)}^n$ is a solution to the following continuous problem of variational inequalities (VI):

$$\begin{cases} b(\bar{u}_{(h)}^n, v - \bar{u}_{(h)}^n) \ge (f(u_h^n) + \lambda.u_h^n, v - \bar{u}_{(h)}^n), & v \in H_0^1(\Omega), \\ \bar{u}_{(h)}^n \le \psi, \qquad v \le \psi, \end{cases} \tag{4.4}$$

where u_h^n is the solution of discrete problem (3.4).

Lemma 4 (cf. [13]) *There exists a constant C independent of h, k, and n such that*

$$\left\|\bar{u}_{(h)}^n - u_h^n\right\|_\infty \le Ch^2|\log h|^2, \tag{4.5}$$

where the constant C is independent of h, n, and Δt.

Proposition 6 *There exists a sequence of continuous subsolutions* $\{\beta_{(h)}^n\}_{n\ge1}$ *such that*

$$\begin{cases} \beta_{(h)}^n \le u^n \\ \text{and} \\ \|\beta_{(h)}^n - u_h^n\|_\infty \le Ch^2|\log h|^2, \end{cases} \tag{4.6}$$

where the constant C is independent of h, Δt, *and n.*

Proof For $n = 1$, we consider the continuous problem of VI

$$\begin{cases} b(\bar{u}^1_{(h)}, v - \bar{u}^1_{(h)}) \geq (f(u_{0,h}) + \lambda u_{0,h}, v - \bar{u}^1_{(h)}), & v \in H^1_0(\Omega), \\ \bar{u}^1_{(h)} \leq \psi, & v \leq \psi. \end{cases}$$

Then, as $\bar{u}^1_{(h)}$ is a solution to a continuous VI, it is also a subsolution, i.e.,

$$\begin{cases} b(\bar{u}^1_{(h)}, v) \leq (f(u_{0,h}) + \lambda u_{0,h}, v), \\ \bar{u}^1_{(h)} \leq \psi \end{cases}$$

or

$$\begin{cases} b(\bar{u}^1_{(h)}, v) \leq (f(u_{0,h}) + f(u_0) - f(u_0) + \lambda u_{0,h} - \lambda u_0 + \lambda u_0, v), \\ \bar{u}^1_{(h)} \leq \psi. \end{cases}$$

Then

$$\begin{cases} b(\bar{u}^1_{(h)}, v) \leq (f(u_0) + \|f(u_0) - f(u_{0,h})\|_\infty + \lambda \|u_0 - u_{0,h}\|_\infty + \lambda u_0, v), \\ \bar{u}^1_{(h)} \leq \psi. \end{cases}$$

Using the Lipschitz continuity of $f(\cdot)$, we have

$$\begin{cases} b(\bar{u}^1_{(h)}, v) \leq (f(u_0) + \alpha \|u_0 - u_{0,h}\|_\infty + \lambda \|u_0 - u_{0,h}\|_\infty + \lambda u_0, v), \\ \bar{u}^1_{(h)} \leq \psi. \end{cases}$$

On the other hand, due to [11]

$$\|u_0 - u_{0,h}\|_\infty \leq Ch^2 |\log h|.$$

Then

$$\begin{cases} b(\bar{u}^1_{(h)}, v) \leq (f(u_0) + Ch^2 |\log h| + \lambda u_0, v), \\ \bar{u}^1_{(h)} \leq \psi. \end{cases}$$

So, $\bar{u}^1_{(h)}$ is a continuous subsolution for the VI whose solution is $\bar{U}^1_{(h)} = \partial(f(u_0) + Ch^2|\log h|, \psi)$. Then $u^1 = \partial(f(u_0), \psi)$, and making use of Proposition 2, we have

$$\begin{aligned} \|\bar{U}^1_{(h)} - u^1\| &\leq \frac{1}{\beta} \|f(u_0) + Ch^2 |\log h| - f(u_0)\|_\infty \\ &\leq Ch^2 |\log h|. \end{aligned}$$

Hence, making use of Theorem 2, we have

$$\bar{u}^1_{(h)} \leq \bar{U}^1_{(h)} \leq u^1 + Ch^2 |\log h|^2.$$

Putting

$$\beta^1_{(h)} = \bar{u}^1_{(h)} - Ch^2|\log h|^2,$$

we get

$$\beta^1_{(h)} \leq u^1$$

and

$$\begin{aligned}\|\beta^1_{(h)} - u^1_h\|_\infty &= \|\bar{u}^1_{(h)} - Ch^2|\log h|^2 - u^1_h\|_\infty \\ &\leq \|\bar{u}^1_{(h)} - u^1_h\|_\infty + Ch^2|\log h|^2.\end{aligned}$$

Using Lemma 4, we obtain

$$\|\beta^1_{(h)} - u^1_h\|_\infty \leq Ch^2|\log h|^2.$$

For $n+1$, let us now assume that

$$\begin{cases}\beta^n_{(h)} \leq u^n \\ \text{and} \\ \|\beta^n_{(h)} - u^n_h\|_\infty \leq Ch^2|\log h|^2\end{cases}$$

and consider the continuous problem

$$\begin{cases} b(\bar{u}^{n+1}_{(h)}, v - \bar{u}^{n+1}_{(h)}) \geq (f(u^n_h) + \lambda u^n_h, v - \bar{u}^{n+1}_{(h)}), \quad v \in H^1_0(\Omega), \\ \bar{u}^{n+1}_{(h)} \leq \psi, \qquad v \leq \psi. \end{cases}$$

Then

$$\begin{cases} b(\bar{u}^{n+1}_{(h)}, v) \leq (f(u^n_h) + \lambda u^n_h, v), \quad v \in H^1_0(\Omega), \\ \bar{u}^{n+1}_{(h)} \leq \psi \end{cases}$$

or

$$\begin{cases} b(\bar{u}^{n+1}_{(h)}, v) \leq (f(u^n_h) + f(\bar{u}^n_{(h)}) - f(\bar{u}^n_{(h)}) + \lambda \bar{u}^n_{(h)} - \lambda \bar{u}^n_{(h)} + \lambda u^n_h, v), \\ \bar{u}^{n+1}_{(h)} \leq \psi. \end{cases}$$

Then

$$\begin{cases} b(\bar{u}^{n+1}_{(h)}, v) \leq (f(\bar{u}^n_{(h)}) + \|f(u^n_h) - f(\bar{u}^n_{(h)})\|_\infty + \lambda \|\bar{u}^n_{(h)} - u^n_h\|_\infty + \lambda \bar{u}^n_{(h)}, v), \\ \bar{u}^{n+1}_{(h)} \leq \psi. \end{cases}$$

Using the Lipschitz continuity of $f(\cdot)$, we have

$$\begin{cases} b(\bar{u}^{n+1}_{(h)}, v) \leq (f(\bar{u}^n_{(h)}) + \alpha \|\bar{u}^n_{(h)} - u^n_h\|_\infty + \lambda \|\bar{u}^n_{(h)} - u^n_h\|_\infty + \lambda \bar{u}^n_{(h)}, v), \\ \bar{u}^{n+1}_{(h)} \leq \psi. \end{cases}$$

Using (4.4), we have

$$\begin{cases} b(\bar{u}_{(h)}^{n+1}, v) \leq (f(\bar{u}_{(h)}^{n}) + Ch^2|\log h|^2 + \lambda \bar{u}_{(h)}^{n}, v), \\ \bar{u}_{(h)}^{n+1} \leq \psi. \end{cases}$$

So, $\bar{u}_{(h)}^{n+1}$ is a continuous subsolution for the VI whose solution is $\bar{U}_{(h)}^{n+1} = \partial(f(\bar{u}_{(h)}^{n}) + Ch^2|\log h|^2, \psi)$. Then $u^{n+1} = \partial(f(\bar{u}_{(h)}^{n}), \psi)$, and making use of Proposition 2, we have

$$\begin{aligned} \bar{u}_{(h)}^{n+1} - u^{n+1} &\leq C\big(\big\|f\big(\bar{u}_{(h)}^{n}\big) + Ch^2|\log h|^2 - f\big(\bar{u}_{(h)}^{n}\big)\big\|_\infty\big) \\ &\leq Ch^2|\log h|^2 \end{aligned}$$

and, making use of Theorem 2, we obtain

$$\bar{u}_{(h)}^{n+1} \leq u^{n+1} + Ch^2|\log h|^2.$$

Now, taking

$$\beta_{(h)}^{n+1} = \bar{u}_{(h)}^{n+1} - Ch^2|\log h|^2,$$

we have

$$\beta_{(h)}^{n+1} \leq u^{n+1}$$

and

$$\begin{aligned} \big\|\beta_{(h)}^{n+1} - u_h^{n+1}\big\|_\infty &= \big\|\bar{u}_{(h)}^{n+1} - Ch^2|\log h|^2 - u_h^{n+1}\big\|_\infty \\ &\leq \big\|\bar{u}_{(h)}^{n+1} - u_h^{n+1}\big\|_\infty + Ch^2|\log h|^2. \end{aligned}$$

Using Lemma 4, we obtain

$$\big\|\beta_{(h)}^{n+1} - u_h^{n+1}\big\|_\infty \leq Ch^2|\log h|^2,$$

which completes the proof. □

4.2 L^∞-Error estimate

Now, guided by Propositions 5 and 6, we are in a position to prove the following.

Theorem 5 *Under the conditions of Propositions 5 and 6, we have*

$$\big\|u^n - u_h^n\big\|_\infty \leq Ch^2|\log h|^2, \tag{4.7}$$

where the constant C is independent of h, Δt, and n.

Proof Using (4.3), we have

$$\begin{aligned} u^n &\leq \alpha_h^n + Ch^2|\log h|^2 \\ &\leq u_h^n + Ch^2|\log h|^2, \end{aligned}$$

thus

$$u^n - u_h^n \le Ch^2|\log h|^2,$$

and using (4.6), we have

$$\begin{aligned} u_h^n &\le \beta_{(h)}^n + Ch^2|\log h|^2 \\ &\le u^n + Ch^2|\log h|^2. \end{aligned}$$

Thus, we get

$$u_h^n - u^n \le Ch^2|\log h|^2.$$

Therefore

$$\|u^n - u_h^n\|_\infty \le Ch^2|\log h|^2,$$

which completes the proof. □

Corollary 1 *In* (4.7), *passing to the limit, as* $n \to +\infty$, *we get*

$$\|u^\infty - u_h^\infty\|_\infty \le Ch^2|\log h|^2. \tag{4.8}$$

4.3 L^∞-Asymptotic behavior

Now we estimate the order of the difference between $u_h(T,\cdot)$, the discrete solution calculated at the moment $T = n\Delta t$, and u^∞, the solution of problem (2.13).

Theorem 6 (The main result) *Under the conditions of Proposition* 3 *and Corollary* 1, *the following inequality holds*:

$$\|u_h(T,\cdot) - u^\infty(\cdot)\|_\infty \le C\left(h^2|\log h|^2 + \left(\frac{\alpha\Delta t + 1}{\beta\Delta t + 1}\right)^N\right). \tag{4.9}$$

Proof We have

$$u_h^n(t,\cdot) = u_h(t,\cdot) \quad \text{for all } t \in \bigl((n-1)\Delta t, n\Delta t\bigr),$$

thus

$$\begin{aligned} \|u_h(T,\cdot) - u^\infty(\cdot)\|_\infty &= \|u_h^N(\cdot) - u^\infty(\cdot)\|_\infty \\ &\le \|u_h^N - u_h^\infty\|_\infty + \|u_h^\infty - u^\infty\|_\infty. \end{aligned}$$

Indeed, applying the previous results of Proposition 3 and Corollary 1, we get

$$\|u_h(T,\cdot) - u^\infty(\cdot)\|_\infty \le \left(\frac{\alpha\Delta t + 1}{\beta\Delta t + 1}\right)^N \|u_h^0 - u_h^\infty\|_\infty + Ch^2|\log h|^2.$$

Then the following result can be deduced:

$$\left\| u_h(T,x) - u^{\infty}(x) \right\|_{\infty} \leq C\left(h^2 |\log h|^2 + \left(\frac{\alpha \Delta t + 1}{\beta \Delta t + 1} \right)^N \right),$$

which completes the proof. □

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