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Lyapunov-type inequalities for higher-order half-linear difference equations

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Abstract
In this paper, we will establish some new Lyapunov-type inequalities for some higher-order superlinear–sublinear difference equations with boundary conditions. Our results not only complement the existing results established in the literature, but also furnish a handy tool for the study of qualitative properties of solutions of some difference equations.

Keywords: Lyapunov-type inequality; Difference equation; Higher-order; Boundary conditions

1 Introduction

In recent years, there has been an increasing interest in obtaining various classes of inequalities, which play an important role in qualitative analysis of solutions to differential and difference equations; see [1–27]. In the field of inequalities, the Lyapunov-type inequality is one of the fundamental inequalities, which was initially investigated by Lyapunov in 1907. After having been discovered, the Lyapunov inequality and its various generalizations were extensively studied by numerous mathematicians. This is due to the fact that these inequalities have proved to be useful tools in the study of oscillation theory, disconjugacy, eigenvalue problems, and many directions of mathematics research areas. For some recent work, the reader is referred to [28–45] and the references therein. In particular, Liu and Tang [37] studied the following m-order p-Laplace difference equation:

$$\left|\Delta^m u(n)\right|^{p-2}\Delta^m u(n) + r(n)\left|u(n)\right|^{p-2}u(n) = 0, \tag{1}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and $r(n)$ is a real-valued function defined on $\mathbb{Z}$, $p > 1$ is a constant, Δ denotes the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, and $u(n)$ satisfies the following anti-periodic boundary conditions:

$$\Delta^i u(a) + \Delta^i u(b) = 0, \quad i = 0, 1, \ldots, m-1; \qquad u(n) \not\equiv 0, \quad n \in \mathbb{Z}[a,b]. \tag{2}$$

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Recently, Liu [43] established a new discrete Lyapunov-type inequality for the following generalized m-order p-Laplace difference equation with mixed non-linearities:

$$\left|\Delta^m u(n)\right|^{p-2}\Delta^m u(n)+\sum_{i=0}^{m-1} r_i(n)\left|\Delta^i u(n)\right|^{p-2}\Delta^i u(n)=0, \tag{3}$$

with the anti-periodic boundary conditions (2), where $m\in\mathbb{N}$, $n\in\mathbb{Z}$, $p>1$ is a constant and $r_i(n)$ $(i=0,1,\dots,m-1)$ are real-valued functions defined on $\mathbb{Z}$.

However, to the best of our knowledge, Lyapunov-type inequalities for the superlinear–sublinear difference equation have received less attention. The main goal of this paper is to use the Hölder inequality and other inequalities to establish Lyapunov-type inequalities for superlinear–sublinear difference equation of the form

$$\begin{aligned}&\left|\Delta^m u(n)\right|^{\alpha-2}\Delta^m u(n)+q(n)\left|\Delta^m u(n)\right|^{\beta-2}\Delta^m u(n)-r(n)\left|u(n)\right|^{\gamma-2}u(n)=0,\\&n\in\mathbb{Z}[a,b],\end{aligned} \tag{4}$$

with the anti-periodic boundary conditions (2), and superlinear–sublinear difference equation of the form

$$\begin{aligned}&\left|\Delta^{2m} u(n)\right|^{\alpha-2}\Delta^{2m} u(n)+q(n)\left|\Delta^{2m} u(n)\right|^{\beta-2}\Delta^{2m} u(n)-r(n)\left|u(n)\right|^{\gamma-2}u(n)=0,\\&n\in\mathbb{Z}[a,b],\end{aligned} \tag{5}$$

with the following boundary conditions:

$$\Delta^{2i}u(a)=\Delta^{2i}u(b)=0,\quad i=0,1,\dots,m-1;\qquad u(n)\not\equiv 0,\quad n\in\mathbb{Z}[a,b], \tag{6}$$

where $m\in\mathbb{N}$, $1<\alpha<\gamma<\beta$ are constants, $r(n)$ and $q(n)$ are real-valued functions defined on $\mathbb{Z}$ with $q(n)>0$.

Our results not only complement the existing results established in the literature, such as those in [37, 39, 43], but also furnish a handy tool for the study of qualitative properties of solutions of some difference equations.

2 Main results

In what follows, we always assume that $a,b\in\mathbb{N}$, $\mathbb{Z}[a,b]=\{a,a+1,\dots,b-1,b\}$ and $\mathbb{Z}(a,b)=\{a+1,a+2,\dots,b-2,b-1\}$.

Lemma 2.1 *Let $M>0$, $N>0$, $\lambda>0$ and $\theta>0$ be given and $1<\lambda<\theta$. Then, for each $x\geq 0$,*

$$Mx^{\lambda}-Nx^{\theta}\leq\frac{M(\theta-\lambda)}{\theta-1}\left(\frac{(\theta-1)N}{(\lambda-1)M}\right)^{(\lambda-1)/(\lambda-\theta)}x \tag{7}$$

holds.

Proof If $x=0$, then it is easy to see that the inequality (7) holds. So we only prove the inequality (7) holds in the case of $x>0$. Set $F(x)=Mx^{\lambda-1}-Nx^{\theta-1}$, $x>0$. Let $F'(x)=0$, we get $x_0 = (\frac{M(\lambda-1)}{N(\theta-1)})^{1/(\theta-\lambda)}$. Since $\forall x \in (0, x_0)$, $F'(x) > 0$; $\forall x \in (x_0, +\infty)$, $F'(x) < 0$, F obtains its maximum at $x_0 = (\frac{M(\lambda-1)}{N(\theta-1)})^{1/(\theta-\lambda)}$ and $F_{\max} = F(x_0) = \frac{M(\theta-\lambda)}{\theta-1}(\frac{(\theta-1)N}{(\lambda-1)M})^{(\lambda-1)/(\lambda-\theta)}$. Hence we get

$$Mx^{\lambda-1} - Nx^{\theta-1} \le \frac{M(\theta-\lambda)}{\theta-1}\left(\frac{(\theta-1)N}{(\lambda-1)M}\right)^{(\lambda-1)/(\lambda-\theta)},$$

i.e.,

$$Mx^{\lambda} - Nx^{\theta} \le \frac{M(\theta-\lambda)}{\theta-1}\left(\frac{(\theta-1)N}{(\lambda-1)M}\right)^{(\lambda-1)/(\lambda-\theta)} x,$$

then (7) holds. The proof is complete. □

Lemma 2.2 ([39]) *Assume that $u(n)$ is a real-valued function on $\mathbb{Z}[a,b]$, $u(a) = u(b) = 0$. Then*

$$\left|u(n)\right| \le \frac{(n-a)(b-n)}{b-a}\sum_{s=a}^{b-1}\left|\Delta^2 u(s)\right|, \quad \forall n \in \mathbb{Z}[a, b-1], \tag{8}$$

$$\sum_{n=a}^{b-1}\left|u(n)\right| \le \frac{1}{2}\sum_{n=a}^{b-1}\left[(n-a+1)(b-n-1)\left|\Delta^2 u(n)\right|\right] \le \frac{(b-a)^2}{8}\sum_{n=a}^{b-1}\left|\Delta^2 u(n)\right|, \quad \forall n \in \mathbb{Z}[a, b-1]. \tag{9}$$

Theorem 2.1 *If $u(n)$ is a nonzero solution of Eq.* (4) *satisfying the anti-periodic boundary conditions* (2), *then*

$$1 \le \Theta\left(\frac{\gamma-\alpha}{q_0(\beta-\alpha)}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{(\beta-\gamma)}{\beta-\alpha}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)(\beta-\alpha)/[\alpha(\beta-\gamma)]}, \tag{10}$$

where

$$q_0 = \min_{n\in\mathbb{Z}[a,b]}\left\{q(n)\right\}, \tag{11}$$

$$\Theta = (b-a)^{(\beta-\alpha)(m\alpha\gamma-m\alpha+1-\alpha)/[\alpha(\beta-\gamma)]}2^{m(\gamma-1)(\alpha-\beta)/(\beta-\gamma)}. \tag{12}$$

Proof Since the nonzero solution $u(n)$ of Eq. (4) satisfies the anti-periodic boundary conditions (2), then $u(a) + u(b) = 0$. For $n \in \mathbb{Z}[a,b]$, we have

$$\begin{aligned} u(n) &= u(n) - \frac{1}{2}\left[u(a)+u(b)\right] = \frac{1}{2}\sum_{k=a}^{n-1}\left[u(k+1)-u(k)\right] - \frac{1}{2}\sum_{k=n}^{b-1}\left[u(k+1)-u(k)\right] \\ &= \frac{1}{2}\sum_{k=a}^{n-1}\Delta u(k) - \frac{1}{2}\sum_{k=n}^{b-1}\Delta u(k). \end{aligned} \tag{13}$$

Then

$$\left|u(n)\right| \le \frac{1}{2}\sum_{k=a}^{b-1}\left|\Delta u(k)\right|. \tag{14}$$

Similarly, we get

$$\left|\Delta^i u(n)\right| \le \frac{1}{2}\sum_{k=a}^{b-1}\left|\Delta^{i+1}u(k)\right|, \quad i=1,2,\ldots,m-1. \tag{15}$$

Then, from (14) and (15), we have

$$\left|u(n)\right| \le \left(\frac{1}{2}\right)^m (b-a)^{m-1}\sum_{k=a}^{b-1}\left|\Delta^m u(k)\right|. \tag{16}$$

Multiplying (4) by $\Delta^m u(n)$, we obtain

$$\left|\Delta^m u(n)\right|^\alpha = r(n)\left|u(n)\right|^{\gamma-2}u(n)\Delta^m u(n) - q(n)\left|\Delta^m u(n)\right|^\beta, \quad n\in\mathbb{Z}[a,b]. \tag{17}$$

Then we get

$$\begin{aligned}\left|\Delta^m u(n)\right|^\alpha &= r(n)\left|u(n)\right|^{\gamma-2}u(n)\Delta^m u(n) - q(n)\left|\Delta^m u(n)\right|^\beta\\ &\le \left|r(n)\right|\left|u(n)\right|^{\gamma-1}\left|\Delta^m u(n)\right| - q(n)\left|\Delta^m u(n)\right|^\beta.\end{aligned} \tag{18}$$

Summing (18) from a to $b-1$, we have

$$\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha \le \sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n)\right|^{\gamma-1}\left|\Delta^m u(n)\right| - \sum_{n=a}^{b-1}q(n)\left|\Delta^m u(n)\right|^\beta. \tag{19}$$

For the first summation on the right-hand side of (19), from (16) we obtain

$$\begin{aligned}&\sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n)\right|^{\gamma-1}\left|\Delta^m u(n)\right|\\ &\quad\le \left(\frac{1}{2}\right)^{m(\gamma-1)}(b-a)^{(m-1)(\gamma-1)}\left(\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|\right)^{\gamma-1}\sum_{n=a}^{b-1}\left|r(n)\right|\left|\Delta^m u(n)\right|.\end{aligned} \tag{20}$$

On the other hand, from the discrete Hölder inequality,

$$\sum_{n=a}^{b-1}\left|f(n)g(n)\right| \le \left(\sum_{n=a}^{b-1}\left|f(n)\right|^\rho\right)^{1/\rho}\left(\sum_{n=a}^{b-1}\left|g(n)\right|^\nu\right)^{1/\nu}, \tag{21}$$

with $f(n)=1$, $g(n)=|\Delta^m u(n)|$, $\rho=\alpha/(\alpha-1)$ and $\nu=\alpha$, we have

$$\begin{aligned}\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right| &\le \left(\sum_{n=a}^{b-1}1^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha\right)^{1/\alpha}\\ &= (b-a)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha\right)^{1/\alpha},\end{aligned} \tag{22}$$

and, with $f(n) = |r(n)|$, $g(n) = |\Delta^m u(n)|$, $\rho = \alpha/(\alpha - 1)$ and $\nu = \alpha$, we get

$$\sum_{n=a}^{b-1} |r(n)| |\Delta^m u(n)| \le \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \right)^{1/\alpha}. \tag{23}$$

Then, from (20), (22) and (23), we obtain

$$\begin{aligned}
&\sum_{n=a}^{b-1} |r(n)| |u(n)|^{\gamma-1} |\Delta^m u(n)| \\
&\quad \le \left(\frac{1}{2}\right)^{m(\gamma-1)} (b-a)^{(m-1/\alpha)(\gamma-1)} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \right)^{(\gamma-1)/\alpha} \\
&\qquad \cdot \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \right)^{1/\alpha} \\
&\quad = \left(\frac{1}{2}\right)^{m(\gamma-1)} (b-a)^{(m-1/\alpha)(\gamma-1)} \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \right)^{\gamma/\alpha}.
\end{aligned} \tag{24}$$

Combining (11), (19) with (24), we get

$$\begin{aligned}
&\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \\
&\quad \le \left(\frac{1}{2}\right)^{m(\gamma-1)} (b-a)^{(m-1/\alpha)(\gamma-1)} \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \right)^{\gamma/\alpha} \\
&\qquad - \sum_{n=a}^{b-1} q(n) |\Delta^m u(n)|^\beta \\
&\quad \le \left(\frac{1}{2}\right)^{m(\gamma-1)} (b-a)^{(m-1/\alpha)(\gamma-1)} \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \right)^{\gamma/\alpha} \\
&\qquad - q_0 \sum_{n=a}^{b-1} |\Delta^m u(n)|^\beta.
\end{aligned} \tag{25}$$

On the other hand, by Hölder inequality (21) with $f(n) = 1$, $g(n) = |\Delta^m u(n)|^\alpha$, $\rho = \beta/(\beta - \alpha)$ and $\nu = \beta/\alpha$, we have

$$\begin{aligned}
\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha &\le \left(\sum_{n=a}^{b-1} 1^{\beta/(\beta-\alpha)} \right)^{(\beta-\alpha)/\beta} \left(\sum_{n=a}^{b-1} \left(|\Delta^m u(n)|^\alpha \right)^{\beta/\alpha} \right)^{\alpha/\beta} \\
&= (b-a)^{(\beta-\alpha)/\beta} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\beta \right)^{\alpha/\beta}.
\end{aligned} \tag{26}$$

Therefore,

$$(b-a)^{(\alpha-\beta)/\alpha} \left(\sum_{n=a}^{b-1} |\Delta^m u(n)|^\alpha \right)^{\beta/\alpha} \le \sum_{n=a}^{b-1} |\Delta^m u(n)|^\beta. \tag{27}$$

From (25) and (27), we get

$$\begin{aligned}
&\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha \\
&\quad\le \left(\frac{1}{2}\right)^{m(\gamma-1)}(b-a)^{(m-1/\alpha)(\gamma-1)}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha\right)^{\gamma/\alpha} \\
&\qquad - (b-a)^{(\alpha-\beta)/\alpha}q_0\left(\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha\right)^{\beta/\alpha}. \qquad (28)
\end{aligned}$$

For the right-hand of (28), from the inequality (7) in Lemma 2.1, with

$$M = \left(\frac{1}{2}\right)^{m(\gamma-1)}(b-a)^{(m-1/\alpha)(\gamma-1)}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha},$$

$x = \sum_{n=a}^{b-1}|\Delta^m u(n)|^\alpha$, $N = (b-a)^{(\alpha-\beta)/\alpha}q_0$, $\lambda = \frac{\gamma}{\alpha}$, and $\theta = \frac{\beta}{\alpha}$, we get

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{m(\gamma-1)}(b-a)^{(m-1/\alpha)(\gamma-1)}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\frac{\alpha}{\alpha-1}}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha\right)^{\gamma/\alpha} \\
&\qquad - (b-a)^{(\alpha-\beta)/\alpha}q_0\left(\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha\right)^{\beta/\alpha} \\
&\quad\le \Theta\left(\frac{\gamma-\alpha}{q_0(\beta-\alpha)}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{(\beta-\gamma)}{\beta-\alpha} \\
&\qquad\cdot\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)(\beta-\alpha)/[\alpha(\beta-\gamma)]}\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha, \qquad (29)
\end{aligned}$$

where Θ is defined as in (12). From (28) and (29), we have

$$\begin{aligned}
&\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha \\
&\quad\le \Theta\left(\frac{\gamma-\alpha}{q_0(\beta-\alpha)}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{(\beta-\gamma)}{\beta-\alpha} \\
&\qquad\cdot\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)(\beta-\alpha)/[\alpha(\beta-\gamma)]}\sum_{n=a}^{b-1}\left|\Delta^m u(n)\right|^\alpha. \qquad (30)
\end{aligned}$$

Now, we claim that $\sum_{n=a}^{b-1}|\Delta^m u(n)|^\alpha > 0$. In fact, if the above inequality is not true, we have $\sum_{n=a}^{b-1}|\Delta^m u(n)|^\alpha = 0$. From (16) and (22), we obtain $u(n) = 0$ for $n \in \mathbb{Z}[a,b]$, which contradicts $u(n) \not\equiv 0$, $n \in \mathbb{Z}[a,b]$. Thus dividing both sides of (30) by $\sum_{n=a}^{b-1}|\Delta^m u(n)|^\alpha$, we obtain (10) holds. This completes the proof of Theorem 2.1. □

Next, we establish a Lyapunov-type inequality for Eq. (5) under the boundary condition (6).

Theorem 2.2 *If $u(n)$ is a nonzero solution of Eq.* (5) *satisfying the boundary conditions* (6), *then*

$$1 \leq \Upsilon \left(\frac{\gamma - \alpha}{q_0(\beta - \alpha)} \right)^{(\gamma-1)/(\beta-\gamma)} \frac{\beta - \gamma}{\beta - \alpha} \cdot \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)(\beta-1)/[\alpha(\beta-\gamma)]}, \tag{31}$$

where q_0 is defined as in (11) *and*

$$\Upsilon = \frac{(b-a)^{(\gamma-1)(2m-2m\beta+\alpha-1)/[\alpha(\gamma-\beta)]}}{2^{(3m-1)(\gamma-1)(\beta-1)/(\beta-\gamma)}}. \tag{32}$$

Proof Choose $c \in \mathbb{Z}[a,b]$ such that $|u(c)| = \max_{n\in\mathbb{Z}[a,b]} |u(n)|$. Since (6), it follows from Lemma 2.2 that

$$|u(c)| \leq \frac{(c-a)(b-c)}{b-a} \sum_{n=a}^{b-1} |\Delta^2 u(n)| \leq \frac{b-a}{4} \sum_{n=a}^{b-1} |\Delta^2 u(n)| \tag{33}$$

and

$$\sum_{n=a}^{b-1} |\Delta^{2i} u(n)| \leq \frac{(b-a)^2}{8} \sum_{n=a}^{b-1} |\Delta^{2i+2} u(n)|, \quad i = 1, 2, \ldots, m-1. \tag{34}$$

From (33) and (34), we obtain

$$\begin{aligned} |u(c)| &\leq \frac{b-a}{4} \sum_{n=a}^{b-1} |\Delta^2 u(n)| \\ &\leq \frac{b-a}{4} \frac{(b-a)^2}{8} \sum_{n=a}^{b-1} |\Delta^4 u(n)| \\ &\leq \frac{b-a}{4} \frac{(b-a)^4}{8^2} \sum_{n=a}^{b-1} |\Delta^6 u(n)| \\ &\leq \cdots \\ &\leq \frac{b-a}{4} \frac{(b-a)^{2(m-1)}}{8^{m-1}} \sum_{n=a}^{b-1} |\Delta^{2m} u(n)|. \end{aligned} \tag{35}$$

Multiplying (5) by $\Delta^{2m} u(n)$, we have

$$|\Delta^{2m} u(n)|^{\alpha} = r(n)|u(n)|^{\gamma-2} u(n) \Delta^{2m} u(n) - q(n) |\Delta^{2m} u(n)|^{\beta}, \quad n \in \mathbb{Z}[a,b]. \tag{36}$$

Then we get

$$\begin{aligned} |\Delta^{2m} u(n)|^{\alpha} &= r(n)|u(n)|^{\gamma-2} u(n) \Delta^{2m} u(n) - q(n) |\Delta^{2m} u(n)|^{\beta} \\ &\leq |r(n)| |u(n)|^{\gamma-1} |\Delta^{2m} u(n)| - q(n) |\Delta^{2m} u(n)|^{\beta}. \end{aligned} \tag{37}$$

Summing (37) from a to $b-1$, we have

$$\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha} \le \sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n)\right|^{\gamma-1}\left|\Delta^{2m}u(n)\right| - \sum_{n=a}^{b-1}q(n)\left|\Delta^{2m}u(n)\right|^{\beta}. \tag{38}$$

For the first summation on the right-hand side of (38), from (35) we obtain

$$\begin{aligned}&\sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n)\right|^{\gamma-1}\left|\Delta^{2m}u(n)\right| \\ &\quad\le \left(\frac{b-a}{4}\right)^{\gamma-1}\frac{(b-a)^{2(m-1)(\gamma-1)}}{8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|\right)^{\gamma-1}\sum_{n=a}^{b-1}\left|r(n)\right|\left|\Delta^{2m}u(n)\right|.\end{aligned} \tag{39}$$

On the other hand, from the discrete Hölder inequality (21) with $f(n) = 1, g(n) = |\Delta^{2m}u(n)|$, $\rho = \alpha/(\alpha-1)$ and $\nu = \alpha$, we have

$$\begin{aligned}\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right| &\le \left(\sum_{n=a}^{b-1}1^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{1/\alpha} \\ &= (b-a)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{1/\alpha},\end{aligned} \tag{40}$$

and with $f(n) = |r(n)|$, $g(n) = |\Delta^{2m}u(n)|$, $\rho = \alpha/(\alpha-1)$ and $\nu = \alpha$, we get

$$\sum_{n=a}^{b-1}\left|r(n)\right|\left|\Delta^{2m}u(n)\right| \le \left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{1/\alpha}. \tag{41}$$

From (39)–(41), we have

$$\begin{aligned}&\sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n)\right|^{\gamma-1}\left|\Delta^{2m}u(n)\right| \\ &\quad\le \frac{(b-a)^{(\gamma-1)[2m-1+(\alpha-1)/\alpha]}}{4^{\gamma-1}8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{(\gamma-1)/\alpha} \\ &\qquad\cdot\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{1/\alpha} \\ &\quad= \frac{(b-a)^{(\gamma-1)[2m-1+(\alpha-1)/\alpha]}}{4^{\gamma-1}8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\gamma/\alpha}.\end{aligned} \tag{42}$$

By (38) and (42), we get

$$\begin{aligned}&\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha} \\ &\quad\le \frac{(b-a)^{(\gamma-1)[2m-1+(\alpha-1)/\alpha]}}{4^{\gamma-1}8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\gamma/\alpha}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{n=a}^{b-1} q(n)\left|\Delta^{2m}u(n)\right|^{\beta} \\
&\leq \frac{(b-a)^{(\gamma-1)[2m-1+(\alpha-1)/\alpha]}}{4^{\gamma-1}8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\gamma/\alpha} \\
&- q_0\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\beta},
\end{aligned}
\tag{43}
$$

where q_0 is defined as in (11). On the other hand, by using Hölder inequality (21) with $f(n) = 1$, $g(n) = |\Delta^{2m}u(n)|$, $\rho = \beta/(\beta-\alpha)$ and $\nu = \beta/\alpha$, we obtain

$$
\begin{aligned}
\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha} &\leq \left(\sum_{n=a}^{b-1}1^{\beta/(\beta-\alpha)}\right)^{(\beta-\alpha)/\beta}\left(\sum_{n=a}^{b-1}\left(\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\beta/\alpha}\right)^{\alpha/\beta} \\
&= (b-a)^{(\beta-\alpha)/\beta}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\beta}\right)^{\alpha/\beta}.
\end{aligned}
\tag{44}
$$

Therefore,

$$
(b-a)^{(\alpha-\beta)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\beta/\alpha} \leq \sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\beta}.
\tag{45}
$$

From (43) and (45), we get

$$
\begin{aligned}
&\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha} \\
&\leq \frac{(b-a)^{(\gamma-1)[2m-1+(\alpha-1)/\alpha]}}{4^{\gamma-1}8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\gamma/\alpha} \\
&- (b-a)^{(\alpha-\beta)/\alpha}q_0\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\beta/\alpha}.
\end{aligned}
\tag{46}
$$

For the right-hand of (46), from the inequality (7) in Lemma 2.1 with

$$
M = \frac{(b-a)^{(\gamma-1)[2m-1+(\alpha-1)/\alpha]}}{4^{\gamma-1}8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha},
$$

$x = \sum_{n=a}^{b-1}|\Delta^{2m}u(n)|^{\alpha}$, $N = (b-a)^{(\alpha-\beta)/\alpha}q_0$, $\lambda = \gamma/\alpha$, and $\theta = \beta/\alpha$, we have

$$
\begin{aligned}
&\frac{(b-a)^{(\gamma-1)[2m-1+(\alpha-1)/\alpha]}}{4^{\gamma-1}8^{(m-1)(\gamma-1)}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\gamma/\alpha} \\
&- (b-a)^{(\alpha-\beta)/\alpha}q_0\left(\sum_{n=a}^{b-1}\left|\Delta^{2m}u(n)\right|^{\alpha}\right)^{\beta/\alpha}
\end{aligned}
$$

$$\leq \Upsilon \left(\frac{\gamma - \alpha}{q_0(\beta - \alpha)} \right)^{(\gamma-1)/(\beta-\gamma)} \frac{\beta - \gamma}{\beta - \alpha} \cdot \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)(\beta-1)/[\alpha(\beta-\gamma)]} \sum_{n=a}^{b-1} |\Delta^{2m} u(n)|^{\alpha}, \tag{47}$$

where Υ is defined as in (32). From (46) and (47), we have

$$\sum_{n=a}^{b-1} |\Delta^{2m} u(n)|^{\alpha} \leq \Upsilon \left(\frac{\gamma - \alpha}{q_0(\beta - \alpha)} \right)^{(\gamma-1)/(\beta-\gamma)} \frac{\beta - \gamma}{\beta - \alpha} \cdot \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)(\beta-1)/[\alpha(\beta-\gamma)]} \sum_{n=a}^{b-1} |\Delta^{2m} u(n)|^{\alpha}. \tag{48}$$

Now, we claim that $\sum_{n=a}^{b-1} |\Delta^{2m} u(n)|^{\alpha} > 0$. In fact, if the above inequality is not true, we have $\sum_{n=a}^{b-1} |\Delta^{2m} u(n)|^{\alpha} = 0$, then $|\Delta^{2m} u(n)| = 0$ for $n \in \mathbb{Z}[a, b-1]$. So we get $\sum_{n=a}^{b-1} |\Delta^{2m} u(n)| = 0$. From (35), we obtain $u(c) = 0$, then we have $u(n) = 0$ for $n \in \mathbb{Z}[a, b]$, which contradicts $u(n) \not\equiv 0$, $n \in \mathbb{Z}[a, b]$. Thus dividing both sides of (48) by $\sum_{n=a}^{b-1} |\Delta^{2m} u(n)|^{\alpha}$, we obtain (31) holds. This completes the proof of Theorem 2.2. □

3 Applications

In this section, we present some examples and applications of our main results. First, we consider the nonexistence for solutions of the BVP consisting of Eq. (4) and the boundary conditions (2).

Theorem 3.1 *Assume*

$$\Theta \left(\frac{\gamma - \alpha}{q_0(\beta - \alpha)} \right)^{(\gamma-\alpha)/(\beta-\gamma)} \frac{(\beta - \gamma)}{\beta - \alpha} \left(\sum_{n=a}^{b-1} |r(n)|^{\alpha/(\alpha-1)} \right)^{(\alpha-1)(\beta-\alpha)/[\alpha(\beta-\gamma)]} < 1, \tag{49}$$

where q_0 and Θ are defined as in (11) and (12). Then BVP (4), (2) has no nontrivial solution.

Proof Assume the contrary. Then BVP (4), (2) has a nontrivial solution $u(n)$. By Theorem 2.1, inequality (10) holds. This contradicts assumption (49). This completes the proof of Theorem 3.1. □

Next, we give an application of the obtained Lyapunov-type inequality for the following eigenvalue problem:

$$|\Delta^{2m} u(n)|^{\alpha-2} \Delta^{2m} u(n) + q(n) |\Delta^{2m} u(n)|^{\beta-2} \Delta^{2m} u(n) - \lambda r(n) |u(n)|^{\gamma-2} u(n) = 0, \quad n \in \mathbb{Z}[a, b], \tag{50}$$

with the following boundary conditions:

$$\Delta^{2i} u(a) = \Delta^{2i} u(b) = 0, \quad i = 0, 1, \ldots, m-1; \qquad u(n) \not\equiv 0, \quad n \in \mathbb{Z}[a, b], \tag{51}$$

where $m \in \mathbb{N}$, $\lambda > 0$ is a parameter, $1 < \alpha < \gamma < \beta$ are constants, $r(n)$ and $q(n)$ are real-valued functions defined on $\mathbb{Z}$ with $q(n) > 0$. Thus, if there exists a nontrivial solution $u(n)$ of BVP (50), (51), from Theorem 2.2, we have

$$\lambda \geq \left(\sum_{n=a}^{b-1}\left|r(n)\right|^{\alpha/(\alpha-1)}\right)^{(1-\alpha)/\alpha}\left(\frac{\gamma-\alpha}{q_0(\beta-\alpha)}\right)^{(1-\gamma)/(\beta-1)} \cdot \left(\frac{\beta-\gamma}{\Upsilon(\beta-\alpha)}\right)^{(\beta-\gamma)/(\beta-1)}, \tag{52}$$

where q_0 and Υ are defined as in (11) and (32).

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Competing interests
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Authors' contributions
HL organized and wrote this paper. Further, he examined all the steps of the proofs in this research. The author read and approved the final manuscript.

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